MAT137 - Term 2, Week 2

- This lecture will assume you have watched all of the videos on the definition of the integral (but will remind you about some things).
- Today we're talking about:
  - More on the definition of the integral.
  - Riemann sums.
  - Elementary properties of the integral.
  - Antiderivatives.
  - The Fundamental Theorem of Calculus.

Upper and lower integrals

Let's remind ourselves of the definitions.

Suppose f is a bounded function defined on $[a, b]$.

Then the <u>upper integral</u> of f is:

$$\begin{aligned}\overline{I_a^b}(f) &:= \inf\{\text{upper sums of } f\} \\ &= \inf\left\{\, U_P(f) \;:\; P \text{ is a partition of } [a,b] \,\right\}.\end{aligned}$$

and similarly the <u>lower integral</u> is

$$\begin{aligned}\underline{I_a^b}(f) &:= \sup\{\text{lower sums of } f\} \\ &= \sup\left\{\, L_P(f) \;:\; P \text{ is a partition of } [a,b] \,\right\}.\end{aligned}$$

Upper and lower integrals

Remember that the upper sums are all *overestimates* of the area we're looking for. The upper integral acts like the "smallest" or "best" overestimate.

Similarly, the lower sums are all *underestimates* of the area we're looking for. The lower integral acts like the "largest" or "best" underestimate.

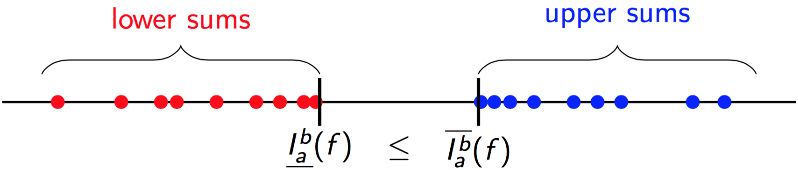
This is just intuition, but it's good intuition.

A function is called integrable on $[a, b]$ if the "best" underestimate and "best" overestimate agree with one another. In this case, their common value is denoted by

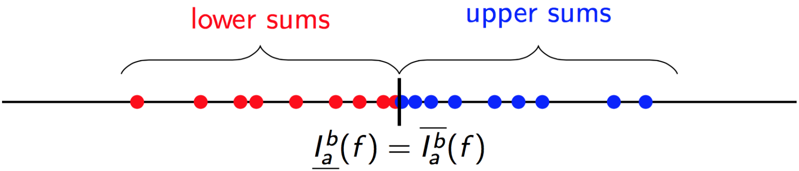
$$\int_a^b f(x)\, dx.$$

Upper and lower integrals

You should keep these picture in mind.
A general (not necessarily integrable) function will give you this:



While an integrable function will give you this:



Make sure you understand the definition

Let f be a bounded function on $[a, b]$. Which of the following two statements must be true?

1. There exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_P(f).$$

2. There exist paritions P and Q of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \quad \text{and} \quad \overline{I_a^b}(f) = U_Q(f).$$

Lower integral exercise

Recall the equivalent definition of supremum we found in an exercise last class:

Definition

If M is an upper bound of a set A, then M is the supremum of A if it satisfies the following:

$$\forall \epsilon > 0, \exists x \in A \text{ such that } M - \epsilon < x \leq M.$$

Exercise: The lower integral is the supremum of all the lower sums. Try to write a definition of the lower integral that's similar to the alternative definition above.

Hint: "$\forall \epsilon > 0$ there is a partition..."

Computing an integral, the absurdly hard way

In this exercise, you're going to do an outrageous amount of work to prove something very obvious. It's important that you do this once or twice.

Consider the function $f(x) = \begin{cases} 0 & x = 0 \\ 7 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$. Draw a picture of it in your notes.

1. Fix an arbitrary partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0, 1]$. What is $U_P(f)$?
2. For the same partition, what is $L_P(f)$?
   (Be sure to draw a picture of the rectangles!)
3. What is the upper integral, $\overline{I_0^1}(f)$?
4. What is the lower integral, $\underline{I_0^1}(f)$?
5. What can you conclude about f on $[a, b]$?

Riemann sums

Reminder of the definition (from the videos):

Let f be a bounded function on $[a, b]$. Let $P = \{x_0, x_1, \ldots, x_N\}$ be a partition of $[a, b]$.

For each subinterval $[x_{i-1}, x_i]$ created by the partition, choose any number in that subinterval and call it x_i^*.

The number

$$S_P^*(f) = \sum_{i=1}^{N} f(x_i^*)\Delta x_i$$

is called a Riemann sum for f and P.

Note: There are many possible Riemann sums for the same f and P, since we can choose different x_i^*'s.

Also recall that given a partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[a, b]$, we denote by $\|P\|$ the norm of the partition.

This is the length of the *longest* subinterval created by P.

Here's the point: If we *already know* f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S_P^*(f).$$

This limit is confusing to calculate in general, so we use simple sequences of partitions instead.

Partition exercises

Consider the interval $[0, 7]$. What are the norms of the following partitions?

1. $P = \{0, 1, 2, 3, 4, 5, 6, 7\}$.
2. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 5, 6, 7\}$.
3. $P = \{0, 1, 2, 3, 7\}$
4. $P = \{0, 2, 4, 6, 7\}$
5. $P = \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7\}$

Describe a sequence of partitions $P_1, P_2, P_3, \ldots$ such that $\|P_n\| \to 0$ as $n \to \infty$.

Using Riemann sums

Theorem

Let f be integrable on $[a, b]$.

Let $P_1, P_2, P_3, \dots$ be a sequence of partitions of $[a, b]$ such that

$$\lim_{n\to\infty} \|P_n\| = 0.$$

Then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} S^*_{P_n}(f).$$

Example

Let's compute an integral using Riemann sums.

Consider the function $f(x) = x^2$ defined on $[0, 1]$. Is this function integrable?

(Recall: It is a theorem that if f is continuous on $[a, b]$, then it is integrable on $[a, b]$.)

For each $n \geq 1$, let P_n be the partition that splits $[0, 1]$ up into n equal subintervals. Write down this partition.

For each i, what is Δx_i? What is $\|P_n\|$? What is $\lim_{n \to \infty} \|P_n\|$?

Example, continued

The fact that $\lim_{n\to\infty} \|P_n\| = 0$ means we can use this sequence of partitions to compute the integral.

Next we have to compute Riemann sums for these partitions. To do that, we need a point x_i^* in each subinterval of each partition.

Let's use the right endpoints: $x_i^* = \frac{i}{n}$.

Then $f(x_i^*) = \frac{i^2}{n^2}$.

Write down the Riemann sum $S_{P_n}^*(f)$ and simplify it as much as possible.

Example, continued

We now need to compute:

$$\lim_{n\to\infty} S^*_{P_n}(f) = \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2.$$

Do to this, recall the following formula:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Use this formula to evaluate the the limit and obtain the value of the integral.

Homework exercise: Do the same thing, but instead choose the *left* endpoints for x_i^* rather than the right endpoints. Verify that you get the same answer.

Elementary properties of integrals

Reminder of some things from the videos:

- If $b < a$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

- Definite integrals are *linear*:

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx,$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for any real number } c.$$

Elementary properties of integrals

This property is sometimes called “linearity of domain” or something similar:

For any numbers a, b, c,

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

In the case where $a < b < c$ at least, this is very intuitive from a picture.

Example

Suppose f and g are integrable on $[0, 7]$, and you know the following things:

- $\int_0^3 f(x)\,dx = 4$
- $\int_3^5 f(x)\,dx = 1$
- $\int_5^7 f(x)\,dx = 2$
- $\int_2^5 f(x)\,dx = 3$
- $\int_0^7 f(x) + g(x)\,dx = 10$
- $\int_7^6 f(x)\,dx = -1$

Compute the following quantities:

- $\int_0^7 7f(x)\,dx$
- $\int_0^2 f(x)\,dx$
- $\int_5^6 f(x)\,dx$
- $\int_0^7 g(x)\,dx$

Our next goal

Our next goal is to state the Fundamental Theorem of Calculus, or FTC.

The FTC provides a connection between two very different-looking things. We've already seen the first: definite integrals.

The second is related to our work in the first term, and that's what we'll talk about next.

Antidervatives

This definition is sort of like the "opposite" of a derivative.

Definition

Let f be a function defined on $[a, b]$.

Another function F is called an antiderivative of f on $[a, b]$ if

- F is continuous on $[a, b]$, and differentiable on (a, b)
- $F'(x) = f(x)$ for all $x \in (a, b)$.

So an antiderivative of f is essentially another function F whose derivative is f.

Properties of antiderivatives

We don't know much about antiderivatives yet, but from the definition alone there are many things we can say.

Proposition

Let F and G be antiderivatives of f and g on $[a, b]$, respectively, and let $c \in \mathbb{R}$. Then:

- *cF is an antiderivative of cf.*
- *$F + G$ is an antiderivative of $f + g$*

These follow immediately from the linearity of differentiation (ie. from the limit laws, basically).

Properties of antiderivatives

Here's another very important one. We laid the groundwork for this fact while talking about the MVT last term.

Proposition

Suppose F and G are both antiderivatives of f on $[a, b]$.

Then F and G differ by a constant.

Proof.

Exercise! □

This exercise says that antiderivatives are not unique.

Computing antiderivatives

Okay, so how do we actually compute antiderivatives? This is something we'll talk about **a lot** this term, but let's get a feel for it now.

Compute antiderivatives of the following functions, by guessing and checking:

1. $f(x) = 2x$
2. $f(x) = 7x^3 + 1$
3. $f(x) = \sqrt{x}$
4. $f(x) = e^x$
5. $f(x) = \sin(x)$
6. $f(x) = 2\cos(x) + 2x$

Compute the antiderivative of $f(x) = 7x^3 + e^x$ that satisfies $F(0) = 0$.

The FTC

The FTC describes a remarkable link between definite integrals and antiderivatives.

The theorem has two parts.

- The first part uses definite integrals to define an antiderivative of any continuous function.
- The second part says you can use antiderivatives to compute definite integrals.

Each part can be proved from the other part, so different couses and textbooks do them in different orders.

An important concept

In order to understand the FTC, we need to understand a tricky way of defining a function.

Suppose f is an integrable function on $[a, b]$. Define a function $g(x)$ in the following way:

$$g(x) = \int_a^x f(t)\,dt \quad \text{for } x \in [a, b].$$

This is much easier to understand as a picture, so be sure to draw one in your notes.

An important concept

Let's explore this idea a bit.

First, convince yourself that the t inside the integral is just a "dummy variable". It only lives inside the integral. g is a function of x only.

Next, think about what role a plays here. Let c be any other number in $[a, b]$. What can you say about the relationship between $g(x)$ and

$$h(x) = \int_c^x f(t)\,dt?$$

We find that $h(x)$ differs from $g(x)$ by a constant, whose value is precisely

$$\int_a^c f(t)\,dt$$

An important concept

The function h we just defined depended on a choice of $c \in [a, b]$.

Any c we pick will work, so we actually have a whole family of functions here: one for every $c \in [a, b]$.

So given any $c \in [a, b]$, we'll define

$$F_c(x) = \int_c^x f(t)\, dt.$$

An important concept

Then as we've seen, any two of these functions F_c differ by a constant:

$$\begin{aligned} F_c(x) - F_d(x) &= \int_c^x f(t)\,dt - \int_d^x f(t)\,dt \\ &= \int_c^d f(t)\,dt + \int_d^x f(t)\,dt - \int_d^x f(t)\,dt \\ &= \int_c^d f(t)\,dt. \end{aligned}$$

Does this remind you of something we mentioned recently?

These functions are the key objects that the first part of the FTC talks about.

The FTC

Theorem (Fundamental Theorem of Calculus)

1. *Suppose f is integrable on $[a, b]$, and $c \in [a, b]$. Then*

$$F_c(x) = \int_c^x f(t)\,dt$$

   is continuous on $[a, b]$.

   F_c is differentiable at any point x where f is continuous, and at such a point $F_c'(x) = f(x)$.

2. *Let f be continuous on $[a, b]$. If F is <u>any</u> antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$