MAT137 - Term 2, Week 1

- This lecture will assume you have watched the first seven videos on the definition of the integral (but will remind you about some things).
- Today we're talking about:
  - Sigma ($\sum$) notation.
  - Infima and suprema of sets and functions.
  - The definition of the integral.
- Before next week's lecture, please watch the remainder of the videos on the definition of the integral.

Sigma notation

Sigma notation is simply a way of making it easier to express certain long summations in a more compact form.

$\sum$ is the Greek letter *sigma*, which is Greek version of "S". S for "sum".

The index i ends at $i = n$

The summation symbol (capital Greek letter sigma)

$$\sum_{i=m}^{n} a_i$$

a_i is the formula for the ith term

i is the summation index (dummy variable)

The index i starts at $i = m$

You should spend a few minutes at some point practising how to write sigmas. Seriously.

Sigma notation

If you've ever done any programming, sigma notation can be thought of like a very simple *for* loop.

For example, the expression

$$\sum_{i=1}^{7} a_i$$

essentially executes the following pseudocode:

```
sum = 0
FOR i = 1 to 7
    sum = sum + a_i
    i = i + 1
RETURN sum
```

Sigma notation exercise

Consider the following sum written in sigma notation

$$\sum_{j=0}^{N} \frac{x^j}{2j+1}.$$

Does the value of this expression depend on...

1. ...x only?
2. ...N only?
3. ...j only?
4. ...x and j?
5. ...j and N?
6. ...x and N?

Sigma notation exercise

Write the following sums with sigma notation.

1. $2^7 + 3^7 + 4^7 + 5^7 + 6^7 + 7^7$
2. $3 + 5 + 7 + 9 + \cdots 75 + 77$
3. $\cos(0) - \cos(2) + \cos(4) - \cos(6) + \cos(8) - \cdots \pm \cos(2N)$
4. $2 + \frac{5}{2} + \frac{10}{3} + \frac{17}{4} + \frac{26}{5} + \frac{37}{6} + \frac{50}{7}$
5. $-\frac{2x^4}{3!} + \frac{3x^5}{4!} - \frac{4x^6}{5!} + \cdots - \frac{98x^{100}}{99!}$

Sigma notation exercise

Consider the following sum:

$$3 + 9 + 15 + 21 + 27 + 33 + \cdots + 297 + 303$$

Which of the following expressions represents the value of this sum (there may be more than one)?

1. $\displaystyle\sum_{n=1}^{51} 3(2n+1)$
2. $\displaystyle\sum_{n=1}^{51} 3(2n-1)$
3. $\displaystyle\sum_{i=0}^{50} 3(2i+1)$
4. $\displaystyle\sum_{i=0}^{50} 3(2n+1)$
5. $\displaystyle 3\sum_{n=0}^{50} (2n+1)$
6. $\displaystyle 3\sum_{n=7}^{57} (2n-13)$

Sigma notation exercises

Consider the expression:

$$\sum_{i=1}^{N} \sum_{k=1}^{i} A_{i,k}.$$

Here, $A_{i,k}$ is an expression that depends on *both* i and k in some way, such as for example $A_{i,k} = (7i)^k$.

Fill in the four question marks in the following expression so that it equals the one above.

$$\sum_{k=?}^{?} \sum_{i=?}^{?} A_{i,k}.$$

Sigma notation | Double Sums

$$\sum_{i=1}^{N} \sum_{k=1}^{i} A_{i,k}$$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$i=1$	$A_{1,1}$				
$i=2$	$A_{2,1}$	$A_{2,2}$			
$i=3$	$A_{3,1}$	$A_{3,2}$	$A_{3,3}$		
$i=4$	$A_{4,1}$	$A_{4,2}$	$A_{4,3}$	$A_{4,4}$	
$i=5$	$A_{5,1}$	$A_{5,2}$	$A_{5,3}$	$A_{5,4}$	$A_{5,5}$

$$\sum_{i=1}^{N}\sum_{k=1}^{i} A_{i,k} = \sum_{k=?}^{?}\sum_{i=?}^{?} A_{i,k}$$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$i=1$	$A_{1,1}$				
$i=2$	$A_{2,1}$	$A_{2,2}$			
$i=3$	$A_{3,1}$	$A_{3,2}$	$A_{3,3}$		
$i=4$	$A_{4,1}$	$A_{4,2}$	$A_{4,3}$	$A_{4,4}$	
$i=5$	$A_{5,1}$	$A_{5,2}$	$A_{5,3}$	$A_{5,4}$	$A_{5,5}$

Infima and suprema of sets

Just to remind you of some definitions...

Definition

Let A be a subset of $\mathbb{R}$.

- A number M is an upper bound of A if $\forall x \in A$, $x \leq M$.
- If A has at least one upper bound, we say it is bounded above.
- The supremum or least upper bound of A, denoted $\sup A$ is the smallest upper bound of A (if it exists).
- If $\sup A$ is an element of A, we say it is the maximum of A.

There are of course a set of analogous definitions for lower bounds, infima, and minimums.

Infima and suprema exercise

For each of the following sets of real numbers

- find its supremum or convince yourself it does not exist;
- do the same for the infimum;
- find the maximum and minimum, if they exist.

1. $A_1 = (0, 7]$
2. $A_2 = \{7, 8, 9\}$
3. $A_3 = \{\, x \in \mathbb{R} \ : \ x < 0 \text{ or } x \geq 7 \,\}$
4. $A_4 = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = \{\, \frac{1}{n} \ : \ n \in \mathbb{N} \,\}$
5. $A_5 = \{\dots, \frac{1}{343}, \frac{1}{49}, \frac{1}{7}, 1, 7, 49, 343, \dots\} = \{\, 7^n \ : \ n \in \mathbb{Z} \,\}$.

Infima and suprema exercise

In this exercise, we'll find useful alternative definitions of supremum.

Recall that M is the supremum of a set A if...

1. ...M is an upper bound of A;
2. ...and there are no smaller upper bounds of A.

In other words, if M is the supremum of A then any number smaller than M cannot be an upper bound of A.

Exercise

With that in mind, assume M is an upper bound for A. Which of the following statements mean "M is the supremum of A"?

1. $\exists x \in A$ such that $\forall \epsilon > 0$, $M - \epsilon < x \leq M$
2. $\forall \epsilon > 0$, $\exists x \in A$ such that $M - \epsilon < x \leq M$.
3. $\forall L < M$, $\exists x \in A$ such that $L < x \leq M$

Infima and suprema exercise

Suppose that A and B are subsets of $\mathbb{R}$. Which of the following statements is true? For any that are not true, find counterexamples.

You will find it helpful to draw pictures (of sets on a number line, for example).

1. If $B \subseteq A$ and A is bounded above, then B is bounded above.
2. If $B \subseteq A$ and B is bounded above, then A is bounded above.
3. If $B \subseteq A$ and A is bounded above, then $\sup B \leq \sup A$.
4. If $B \subseteq A$ and A is bounded below, then $\inf B \leq \inf A$.
5. If A and B are bounded above and $\sup B \leq \sup A$, then $B \subseteq A$.

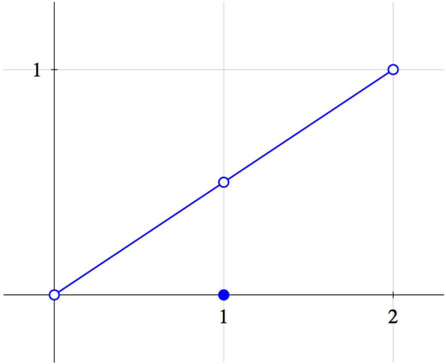
Infima and suprema of functions

As you saw in the videos, we can apply these ideas to a function via the *range* of the function.

We are doing this in order to develop a more robust definition of the definite integral than the one in the textbook, which only defines integrability for continuous functions.

Our definition is better than the textbook

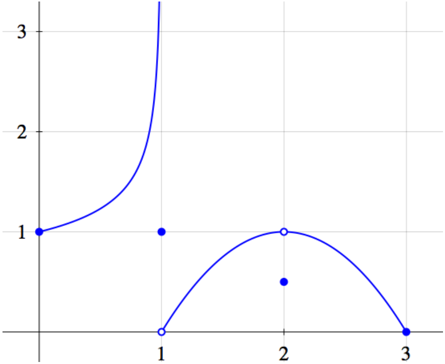
For example, the area under the following function should *obviously* be 1, but the textbook's definition would not apply to this function.



More importantly, our definition will be *much* more helpful once you get to MAT237.

Infima and suprema of functions exercise

Consider the following function f.



1. $[0, 3]$
2. $[1, 2]$
3. $(1, 2]$
4. $[2, 3]$

For each of the given domains determine whether f is bounded, its infimum and supremum (if they exist), and its maximum and minimum (if they exist).

Upper and lower sums

First let's remind ourselves of some notation from the videos.

Suppose that

- f is a bounded function on an interval $[a, b]$;
- $P = \{x_0, x_1, \ldots, x_N\}$ is a partition of $[a, b]$

We will usually use the following notation:

- $\Delta x_i := x_i - x_{i-1}$ is the length of the i^{th} subinterval created by P.
- M_i is the supremum of f on the i^{th} subinterval $[x_{i-1}, x_i]$.
- Similarly, m_i is the infimum of f on the i^{th} subinterval.

Upper and lower sums

We define:

- The P-upper sum for f

$$U_P(f) = \Delta x_1 M_1 + \Delta x_2 M_2 + \cdots + \Delta x_N M_N = \sum_{i=1}^{N} \Delta x_i M_i.$$

  This is always an *overestimate* of the area under the graph of f.

- The P-lower sum for f

$$L_P(f) = \Delta x_1 m_1 + \Delta x_2 m_2 + \cdots + \Delta x_N m_N = \sum_{i=1}^{N} \Delta x_i m_i.$$

  This is always an *underestimate* of the area under the graph of f.

These definitions are best understood with a picture, so make sure you can draw one.

Constant functions

Let's see that these definitions do what we expect in the simplest possible case.

Let f be the constant function 1, defined on $[0,7]$.

Clearly, by looking at a picture, we see that the area under the graph of f is 7.

$$\int_0^7 1\,dx = 7.$$

Exercise: Fix an *arbitrary* partition $P = \{x_0, x_1, \ldots, x_N\}$ of $[0,7]$, and *explicitly* compute $U_P(f)$ and $L_P(f)$.

Non-constant functions?

In the previous exercise we convinced ourselves that if f is a constant function defined on $[a, b]$, then for any partition P we have

$$L_P(f) = U_P(f) = \int_a^b f(x)\,dx.$$

Are there any other sorts of functions for which this is true?
What about "step functions"?

Exercise

Show that if f is a *non-constant* function defined on $[a, b]$, then there is a partition P such that $L_P(f) \neq U_P(f)$.

(Hint: This is *much* easier than it sounds. If you're doing anything tricky, you're overthinking it.)

Increasing functions

Suppose f is an *increasing* function on $[a, b]$, and let $P = \{x_0, x_1, \dots, x_N\}$ be a partition of $[a, b]$ as usual.

Which of the following sums equals $U_P(f)$? What about $L_P(f)$?

1. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, x_i$
2. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_i)$
3. $\displaystyle\sum_{i=0}^{N-1} \Delta x_i \, f(x_i)$
4. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, x_{i-1}$
5. $\displaystyle\sum_{i=1}^{N} \Delta x_i \, f(x_{i-1})$
6. $\displaystyle\sum_{i=1}^{N} \Delta x_{i-1} \, f(x_i)$

Upper and lower integrals

Let's remind ourselves of the definitions.

Suppose f is a bounded function defined on $[a, b]$.

Then the <u>upper integral</u> of f is:

$$\begin{aligned}\overline{I_a^b}(f) &:= \inf\{\text{upper sums of } f\} \\ &= \inf\left\{\, U_P(f) \;:\; P \text{ is a partition of } [a,b] \,\right\}.\end{aligned}$$

and similarly the <u>lower integral</u> is

$$\begin{aligned}\underline{I_a^b}(f) &:= \sup\{\text{lower sums of } f\} \\ &= \sup\left\{\, L_P(f) \;:\; P \text{ is a partition of } [a,b] \,\right\}.\end{aligned}$$

Upper and lower integrals

Remember that the upper sums are all *overestimates* of the area we're looking for.

Similarly, the lower sums are all *underestimates* of the area we're looking for.

This is just intuition, but it's good intuition.

A function is called integrable on $[a, b]$ if the "best" underestimate and "best" overestimate agree with one another.

Lower integral exercise

Recall the equivalent definition of supremum we found in an earlier exercise:

Definition

If M is an upper bound of a set A, then M is the supremum of A if it satisfies the following:

$$\forall \epsilon > 0, \exists x \in A \text{ such that } M - \epsilon < x \leq M.$$

Exercise: The lower integral is the supremum of all the lower sums. Try to write a definition of the lower integral that's similar to the alternative definition above.

Hint: "$\forall \epsilon > 0$ there is a partition..."