MAT137 - Week 12

- Today we're talking about:
  - Optimization.
  - Concavity.
  - Asymptotes.
  - Some curve sketching.

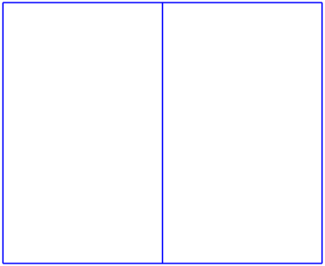
Optimization

General strategy for optimization problems:

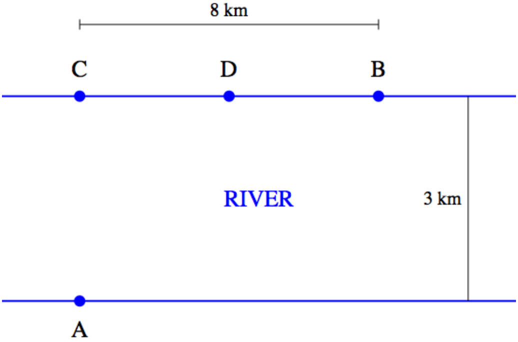
1. Draw a picture of the situation, if possible.
2. Label relevant quantities you know and assign variables to quantities you don't know.
3. Define a function whose maximal or minimal value is the answer to the question.
4. Make sure to note the domain of your function in the context of the problem.
5. Find all local critical points of the function on its domain.
6. Check whether these critical points represent minima or maxima of the function.
7. Check the endpoints of the domain, if any.

This is a classic optimization problem that every calculus student should do at least once.

A farmer has 700 metres of fencing and wants to fence off a rectangular field, along with extra fencing that divides the area into two equal parts down the middle. What dimensions should his field have?



You launch a rowboat from point A on a bank of a river, 3km wide, and want to reach point B, 8 km downstream on the opposite bank, as quickly as possible. You can row from A to the point C directly across the river and then run to B, or you can row directly from A to B, or you can row to some point D between C and B and then run to B. If you row at 6 km/h and run at 8 km/h, where should you land?



Optimization

Find the area of the smallest circle centred at the point $(1, 4)$ which intersects the parabola $y^2 = 2x$.

Concavity

We saw that a function is said to increase or decrease if its values literally go up or go down on an interval. We saw how if a function is differentiable, we can use the sign of its derivative to determine whether it increases or decreases.

We'll now study what happens when the derivative of a function increases or decreases, and see that we can use the sign of its *second* derivative to determine this, and how it affects the shape of its graph.

Definition

Let f be a function and let I be an interval on which it is differentiable.

We say that f is <u>concave up on I</u> if f' is increasing on I.

We say that f is <u>concave down on I</u> if f' is decreasing on I.

Inflection points

Analogously to local maxima/minima, we give names to the places where concavity changes.

Definition

Let f be continuous at a point a and differentiable near (but not necessarily at) a.

The point $(a, f(a))$ is called an inflection point if $\exists \delta > 0$ such that f is concave up (resp. concave down) on $(a - \delta, a)$ and concave down (resp. concave up) on $(a, a + \delta)$.

Example: Find the intervals on which $f(x) = \tan(x)$ is concave up and concave down, and find any inflection points.

Concavity and second derivatives

If f is a differentiable function, then f' is also a function. As we know already, we can determine if a differentiable function increases or decreases by examining its derivative.

So we should expect that if f' is differentiable (or in order words if f is twice differentiable), we can determine if f is concave up or concave down by examining f''.

Proposition

Let f be twice differentiable on an interval I. Then:

- *If $f''(x) > 0$ for all $x \in I$, then f' increases on I, and therefore f is concave up on I.*
- *If $f''(x) < 0$ for all $x \in I$, then f' decreases on I, and therefore f is concave down on I.*

Proof: Exercise.

Concavity and second derivatives

Proposition

Let f be differentiable on an open interval I, and suppose $(a, f(a))$ is an inflection point for some $a \in I$.

Then f' is a local maximum or local minimum at a, and therefore $f''(a) = 0$ or f is not twice differentiable at a.

Proof: Exercise.

NOTE: The converse is not true!

Example

Let $f(x) = \dfrac{x}{x^2+1}$.

On which intervals is f increasing or decreasing?
On which intervals is f concave up or concave down?
Sketch its graph.

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{2x(x^2-3)}{(x^2+1)^3}$$

Asymptotes

We basically know all the necessary theory to talk about asymptotes, we just have to give names to some things so they're clearly defined.

Definition

The line $x = a$ is called a vertical asymptote of a function f if at least one of the following is true:

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

In particular note that we only need an infinite limit on one side of a.

Caution

Note that it's not enough to find places where the denominator of a function equals zero. You <u>must</u> check that one of these limits is infinite.

For example the function

$$f(x) = \frac{x^3 + x^2 - 2x + 8}{x^2 - 4}$$

has a vertical asymptote at $x = -2$ but not at $x = 2$.

Asymptotes

Definition

The line $y = M$ is called a horizontal asymptote of a function f if at least one of the following is true:

$$\lim_{x\to\infty} f(x) = M \quad \text{or} \quad \lim_{x\to-\infty} f(x) = M.$$

Caution

Note, there's nothing in either of the previous definitions about "getting closer and closer to the line but never touching it".

A function can be defined at a vertical asymptote, or can cross a horizontal asymptote many times.

For example, we know that

$$\lim_{x \to \infty} \frac{\sin(x)}{x} = 0,$$

and therefore $y = 0$ is a horizontal asymptote of this function. But the function crosses the line infinitely many times.

Generalizing

It's easy to see that the limit in the previous definition can also be written in this form:

$$\lim_{x \to \pm\infty} f(x) - M = 0.$$

There's no reason to restrict ourselves to talking about functions approaching horizontal lines though.

Given another function g, we can say that f behaves asymptotically like g if

$$\lim_{x \to \pm\infty} f(x) - g(x) = 0.$$

In the special case where g is a non-horizontal (or *oblique*) line, g is called an oblique or slant asymptote. We'll see some examples of this later.

Curve sketching

We're now ready to sketch curves. Note that the important skill to learn here is combining a bunch of pieces of data into one graph that satisfies all the required properties.

To this end, we'll start by giving you all the properties, and asking you to sketch a graph that satisfies them.

Example

Sketch the graph of a function f that satisfies all the following properties.

- f is a polynomial with odd degree and positive leading coefficient.
- f is odd.
- $f(-3) = f(0) = f(3) = 0$ and there are no other axis intercepts.
- $f(-1) = 2$ and $f(1) = -2$.
- $f'(x) = 0$ only when $x = -1,\ 0,\ $ or 1.
- f is increasing on $(-\infty, -1]$ and $[1, \infty)$, and decreasing on $[-1, 1]$.
- $f''(x) = 0$ only when $x = -0.5,\ 0,\ $ or 0.5.
- f is concave up on $(-0.5, 0)$ and $(0.5, \infty)$, and concave down on all other intervals on which it is defined.

Example

Sketch the graph of a function f that satisfies all the following properties.

- f is defined for all real numbers *except* ± 3, and so is f'.
- f is odd.
- $f(0) = 0$, and there are no other axis intercepts.
- $f(4) = 7$, $f(-4) = -7$, and $f'(0) = f''(0) = 0$.
- f has vertical asymptotes at $x = \pm 3$, and all four one-sided limits are ∞ or $-\infty$.
- f has $y = x$ as an oblique asymptote
- f is increasing on $(-\infty, -4]$ and $[4, \infty)$, and decreasing on all other intervals on which it is defined.
- f is concave up on $(-3, 0)$ and $(3, \infty)$, and concave down on all other intervals on which it is defined.

Example

Sketch the function $f(x) = \frac{x-1}{\sqrt{4x^2-1}}$.

$$f'(x) = \frac{4x-1}{(4x^2-1)^{3/2}}.$$

$$f''(x) = -\frac{4(8x^2-3x+1)}{(4x^2-1)^{5/2}}.$$

Example

Sketch the function $f(x) = xe^{1/x}$.

$$f'(x) = \frac{e^{1/x}(x-1)}{x}$$

$$f''(x) = \frac{e^{1/x}}{x^3}$$