MAT137 - Week 11

- Remember that your second term test is tomorrow at 4pm.
- Today we're talking about:
  - Local and global extreme values.
  - Rolle's theorem and the Mean Value Theorem
  - Consequences of the Mean Value Theorem.
  - Optimization (if there's time).

Motivation

Broadly speaking, our goal here is to learn about functions by studying their derivatives.

Specifically we want to find local and global maxima and minima of functions, which we'll define in more detail shortly.

Global and local extreme values

We've already talked about the Extreme Value Theorem, which tells us that certain functions have global maximum and minimum values.

Theorem (Extreme Value Theorem)

If f is continuous on an interval $[a, b]$, then f attains a maximum and minimum value on $[a, b]$.

To be clear, in the case of a maximum this means that there is a $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$.

Local extreme values

Informally, a function f is said to have a *local* maximum at c if $f(c)$ is the largest value the function attains near c. The situation for a local minimum is analogous.

Definition

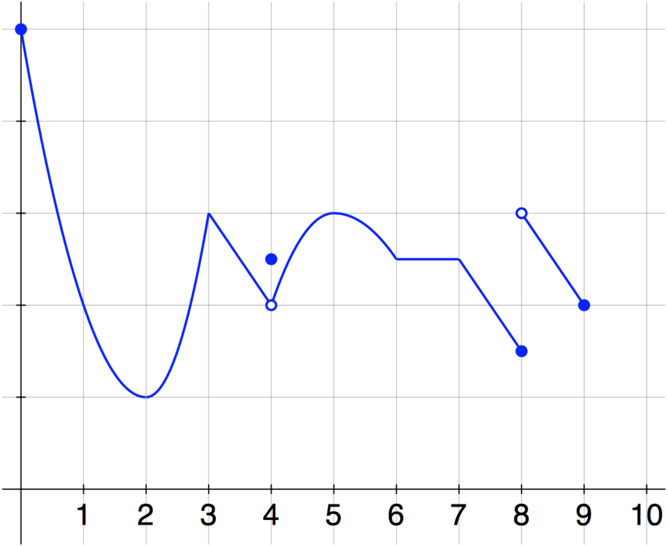
Let f be a function defined on an interval I, and let c be an interior point of I. (That is, c is not an endpoint of I.)

Then f is said to have a <u>local maximum at c</u> if there is a $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in (c - \delta, c + \delta)$.

The definition of a local minimum is analogous (with the inequality reversed).

Exercise

Find all local and global minima and maxima of this function:



Local vs. Global

Note that local extreme values must occur at interior points, while global extreme values can occur at endpoints.

We do know, for example, that if a function has a global maximum at an interior point of its domain, then it also has a local maximum there. (The same is true for minima.)

Our next result will tell us that there is an important connection between local extreme values and horizontal tangent lines of differentiable functions.

Local EVT

Theorem (Local EVT, or sometimes Fermat's Theorem)

Let f be a function defined on an interval I, and suppose f has a local maximum or local minimum at an interior point c of I.

Then either $f'(c) = 0$, or f is not differentiable at c.

We'll prove the case where f has a local minimum at c. The other case works analogously.

Critical points

The previous result leads us to make the following very useful definition.

Definition

Let f be defined on an interval I. An interior point c of I is called a critical point if $f'(c) = 0$ or f is not differentiable at c.

We see that in proving the local EVT, we have proved the following theorem as well:

Theorem

Let f be defined on an interval I, and let c be an interior point of I. If f has a local maximum/minimum at c, then c is a critical point.

Rolle's Theorem

From the local EVT follows a famous theorem called Rolle's Theorem, which we'll now state and prove.

This will feel like a baby version of the final Mean Value Theorem, but it's actually equivalent to it (as we'll soon see).

The theorem *basically* says that a differentiable function that hits the same value twice must turn around somewhere in between.

Rolle's Theorem

Theorem (Rolle's Theorem)

Let f be a function such that:

- *f is continuous on an interval $[a, b]$,*
- *f is differentiable on the corresponding open interval (a, b),*
- *$f(a) = f(b)$.*

Then, there is a $c \in (a, b)$ such that $f'(c) = 0$.

Make sure to draw yourself a picture of what this theorem says.

Applications

Rolle's Theorem alone has many applications. Here's a popular one:

Proposition

Let f is differentiable on $\mathbb{R}$ and $f'(x) > 0$ for all $x \in \mathbb{R}$.

Then f has at most one root.

(The result is also true if simply $f'(x) \neq 0$ for all $x \in \mathbb{R}$.)

Example: Show that $f(x) = e^x + x^3 + \sin(x) + 10x$ has exactly one root.

Applications

Another very popular application of Rolle's Theorem, which is almost exactly the same as the previous result.

Proposition

If f is differentiable on $\mathbb{R}$ and $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is injective.

Proof.

Homework! $\square$

The Mean Value Theorem

Finally, we arrive at the Mean Value Theorem (MVT).

It will look like a stronger theorem than Rolle's Theorem, but they're actually equivalent. We'll use Rolle's Theorem along with a simple trick to prove the MVT.

Again, as with Rolle's Theorem, make sure to draw yourself a picture of what this theorem says. The picture is easy to understand, even if the statement itself isn't easy to understand yet.

MVT

Theorem (Mean Value Theorem)

Let f be continuous on an interval $[a, b]$ and differentiable on the corresponding open interval (a, b).

Then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Observations

- The theorem is not the equation at the end. It's the entire statement. The hypotheses are critical.
- This theorem is an existence theorem. It doesn't tell you what the value of c is. It just tells you that such a c exists. Broadly speaking, this theorem says something like:

  If $\langle$some hypotheses are true$\rangle$, then an equation has a solution.

- If $f(a) = f(b)$, the MVT reduces to Rolle's Theorem.

MVT

To prove the MVT, we will simply apply Rolle's Theorem to the function

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a) \right].$$

This looks scary, but it's very natural if you draw a picture.

Applications

The "moral" of the story is that because of the MVT, your intuition about derivatives as slopes of tangent lines usually leads you to correct conclusions.

We'll start with two of the most important examples of this.

Corollary

Let f be differentiable on an open interval I.

Then f is a constant function if and only if $f'(x) = 0$ for all $x \in I$.

One direction of this proof is easy, but notice that we couldn't prove the other direction before.

Applications

A corollary of the previous result that is very important for the theory of integration:

Corollary

Suppose that f and g are differentiable on an open interval I, and $f'(x) = g'(x)$ for all $x \in I$.

Then f and g differ by a constant on I.

Applications

The next application is what connects derivatives to increasing and decreasing functions.

First, we need to actually define what it means for a function to be increasing or decreasing.

We'll do all of this for the case of increasing functions.

Increasing functions

Definition

Let f be a function and let I be an interval on which it is defined.

We say f is increasing on I if whenever $x_1, x_2 \in I$ and $x_1 < x_2$, then $f(x_1) < f(x_2)$.

Proposition

Let f be differentiable on an open interval I, and suppose that $f'(x) > 0$ for all $x \in I$.

Then f is increasing on I.

The analogous theorem about negative derivatives and decreasing functions is true, of course.

Example

On what intervals is the function $f(x) = \dfrac{3x^2}{x^3 - 4}$ increasing or decreasing?

We now know that we can determine whether a function is increasing or decreasing by finding the sign of its derivative. We can compute that

$$f'(x) = \frac{6x(x^3 - 4) - 3x^2(3x^2)}{(x^3 - 4)^2} = \frac{-3x(x^3 + 3)}{(x^3 - 4)^2}.$$