MAT137 - Week 10

- Remember that Problem Set 4 is due tomorrow at 3pm.
- Today we're talking about:
  - Limits at infinity.
  - L'Hôpital's Rule
  - Introduction to the Mean Value Theorem.

Limits at infinity

So far we've defined limits at a point, which look like

$$\lim_{x \to a} f(x) = L$$

if they exist. We've also defined

$$\lim_{x \to a} f(x) = \infty \quad \text{and} \quad \lim_{x \to a} f(x) = -\infty.$$

Today we'll talk about limits at infinity:

$$\lim_{x \to \infty} f(x) = L \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L.$$

Motivation

When discussing limits at a point, our intuition was that

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ can be made arbitrarily close to L by making x sufficiently close to a.

Now consider the function $g(x) = \frac{1}{x}$.

Exercise: Convince yourself that you can make $g(x)$ arbitrarily close to 0 by making x sufficiently large.

Limits at infinity

Exercise: Suppose f is a function defined on an interval of the form (p, ∞) for some $p \in \mathbb{R}$. Write down a definition for the statement

$$\lim_{x \to \infty} f(x) = L.$$

Definition

Suppose f is a function defined on an interval of the form (p, ∞) for some $p \in \mathbb{R}$. Then

$$\lim_{x \to \infty} f(x) = L.$$

means

$$\forall \epsilon > 0 \; \exists M \in \mathbb{R} \text{ such that } x > M \implies |f(x) - L| < \epsilon.$$

Exercise: Write down a similar definition for the statement $\lim_{x \to -\infty} f(x) = L$.

Examples

1. Prove that $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

2. Prove that $\lim_{x\to\infty} \frac{\sin(x)}{x^2} = 0$. (This reminds us of the Squeeze Theorem, which does generalize to this sort of situation.)

3. Prove that $\lim_{x\to\infty} \cos(x)$ does not exist.

Examples

1. Evaluate $\lim_{x\to\infty} x - \sqrt{x^2+7}$.

2. Evaluate $\lim_{x\to\infty} \frac{3x^2+7x+1}{8x^2+4}$.

3. Evaluate $\lim_{x\to\infty} \frac{3x^2+7x+1}{8x^3+4}$.

4. Evaluate $\lim_{x\to\infty} \frac{7e^{7x}+\sin(x)}{e^{7x}+7}$.

Indeterminate forms

Recall that if we know

$$\lim_{x\to a} f(x) = L \quad \text{and} \quad \lim_{x\to a} g(x) = M,$$

(and $M \neq 0$), then the limit law for quotients tells us that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Knowing the limits of f and g, we can determine the limit of $\frac{f}{g}$ from the form of the function alone.

Indeterminate forms

The same is not true if $L = M = 0$.

Exercise: For each part, find a pair of functions f and g such that

$$\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x),$$

but such that...

1. ... $\lim_{x\to 0} \frac{f(x)}{g(x)} = 7$.
2. ... $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$.
3. ... $\lim_{x\to 0} \frac{f(x)}{g(x)} = \infty$.
4. ... $\lim_{x\to 0} \frac{f(x)}{g(x)}$ doesn't exist, and doesn't equal $\pm\infty$.

Indeterminate forms

For this reason, if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

we say that $\lim_{x \to a} \dfrac{f(x)}{g(x)}$ is <u>indeterminate of type $\frac{0}{0}$</u>.

Indeterminate forms

The same is true if

$$\lim_{x\to 0} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to 0} g(x) = \pm\infty.$$

In this case, we say that $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)}$ is <u>indeterminate of type $\frac{\infty}{\infty}$</u>.

L'Hôpital's rule is a tool for dealing with limits of these two types.

Some intuition

Here's some intuition for the statement of L'Hôpital's rule.

Suppose L_1 and L_2 are lines with slopes m_1 and m_2, respectively. Also suppose they both have zeros at 7.

Then $\lim_{x \to 7} \frac{L_1(x)}{L_2(x)}$ is indeterminate of type $\frac{0}{0}$.

Of course, we can just write down their equations easily:

$$L_1(x) = m_1(x-7) \quad \text{and} \quad L_2(x) = m_2(x-7).$$

and evaluate the limit easily:

$$\lim_{x \to 7} \frac{L_1(x)}{L_2(x)} = \lim_{x \to 7} \frac{m_1}{m_2} = \frac{m_1}{m_2} = \lim_{x \to 7} \frac{L_1'(x)}{L_2'(x)}$$

Some intuition

For more general functions (not all functions though), if

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

and in addition f and g are differentiable near (and at) a, and $g'(a) \neq 0$, then f and g are closely-approximated by their tangent lines at a:

$$f'(a)(x-a) \quad \text{and} \quad g'(a)(x-a).$$

So we might expect to get:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(a)(x-a)}{g'(a)(x-a)} = \frac{f'(a)}{g'(a)}.$$

!!!THIS IS NOT A PROOF!!! ...I secretly assumed many things.

L'Hôpital's rule

This theorem is tricky to state, because there are many cases.

Theorem

Suppose that

- $\lim_{x\to a} \frac{f(x)}{g(x)}$ *is indeterminate of type* $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$.
- f *and* g *are differentiable near* a *(except possibly at* a*).*
- g *is never 0 near* a *(except possibly at* a*).*
- g' *is never 0 near* a *(except possibly at* a*).*
- $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ *exists, or is* $\pm\infty$.

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

L'Hôpital's rule

In the previous theorem, "near a" means "on an open interval containing a".

The theorem also holds for limits as $x \to \infty$ or $x \to -\infty$, in which case "near a" is replaced with "on an interval of the form (p, ∞) or $(-\infty, p)$ for some $p \in \mathbb{R}$, respectively.

Examples

1. Use L'Hôpital's rule to compute $\lim_{x \to 0} \frac{x^2 - 7x}{e^x - 1}$.

2. Compute $\lim_{x \to \infty} \frac{x^2}{e^x}$.

3. Compute $\lim_{x \to 0} \frac{2x - \sin(2x)}{x \sin(x)}$.

Homework: Show that for any natural number N, $\lim_{x \to \infty} \frac{x^N}{e^x} = 0$.

Warnings

L'Hôpital's rule is very powerful, but with great power comes great responsibility.

Warning 1: The hypotheses are important.

Example: Evaluate $\lim\limits_{x\to\infty} \dfrac{x+\sin(x)}{x}$.

INCORRECT SOLUTION: The top and bottom both $\to \infty$, so this is indeterminate of type $\frac{\infty}{\infty}$. So:

$$\lim_{x\to\infty} \frac{x+\sin(x)}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1+\cos(x)}{1} = \lim_{x\to\infty} \cos(x).$$

The last limit doesn't exist, so $\lim\limits_{x\to\infty} \dfrac{x+\sin(x)}{x}$ doesn't exist.

Warnings

Warning 2: It doesn't always help.

Example: Evaluate $\lim\limits_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

The top and bottom both $\to \infty$, so this is indeterminate of type $\frac{\infty}{\infty}$. So:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \overset{\text{L'H}}{=} \cdots\cdots$$

These equalities are all true, they just don't go anywhere.

Warnings

Warning 3: Don't blindly apply it without simplifying things if you can.

Example: Compute $\lim_{x \to 0^+} \frac{\log(x)}{\frac{1}{x}}$.

$$\lim_{x \to 0^+} \frac{\log(x)}{\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \overset{\text{L'H}}{=} \lim_{x \to 0^+} \frac{-\frac{1}{x^2}}{\frac{2}{x^3}} \overset{\text{L'H}}{=} \lim_{x \to 0^+} \frac{\frac{2}{x^3}}{\frac{-6}{x^4}} \overset{\text{L'H}}{=} \cdots\cdots$$

Indeterminate products

There are other indeterminate forms, all of which eventually reduce to one of the ones we've seen.

1. **Products of the form $0 \cdot \infty$.**

If

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty,$$

then

$$\lim_{x \to a} f(x)g(x) = ???$$

This is called indeterminate of type $0 \cdot \infty$.

Indeterminate products

Solution: If $\lim_{x \to a} f(x)g(x)$ is indeterminate of type $0 \cdot \infty$, then

- $\lim_{x \to a} \frac{f(x)}{\frac{1}{g(x)}}$ is indeterminate of type $\frac{0}{0}$.
- $\lim_{x \to a} \frac{g(x)}{\frac{1}{f(x)}}$ is indeterminate of type $\frac{\infty}{\infty}$.

Example: Compute $\lim_{x \to 0} x \log(x)$.

Indeterminate exponents

2. **Exponents of the form 0^0.** If

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x),$$

then

$$\lim_{x \to a} f(x)^{g(x)} = ???$$

This is called indeterminate of type 0^0.

Indeterminate exponents

Solution: If $\lim_{x \to a} f(x)^{g(x)}$ is indeterminate of type 0^0, then

$$\lim_{x \to a} \log\left(f(x)^{g(x)}\right) = \lim_{x \to a} g(x) \log\left(f(x)\right)$$

is indeterminate of type $0 \cdot \infty$.

Other indeterminate exponents

3. **Exponents of the form 1^∞ and ∞^0.**

These are dealt with similarly to those of form 0^0.

4. **Limits of the form $\infty - \infty$.**

There is no special trick to these.

Examples

1. Compute $\lim_{x\to\infty}(1+x)^{1/x}$.

2. Compute $\lim_{x\to 0^+}(\tan(x))^x$.

3. Compute $\lim_{x\to 0^+}\csc(x)-\cot(x)$.

4. Compute $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log(x)}\right)$.

Be careful

What's wrong with this proof? Compute: $\lim_{x \to -\infty} x - \sqrt{x^2 + x}$.

Proof.

$$\begin{aligned}
\lim_{x \to -\infty} x - \sqrt{x^2 + x} &= \lim_{x \to -\infty} x - \sqrt{x^2 + x} \cdot \frac{x + \sqrt{x^2 + x}}{x + \sqrt{x^2 + x}} \\
&= \lim_{x \to -\infty} \frac{x^2 - (x^2 + x)}{x + \sqrt{x^2 + x}} \\
&= \lim_{x \to -\infty} \frac{-x}{x\left(1 + \sqrt{1 + \frac{1}{x}}\right)} \\
&= \lim_{x \to -\infty} \frac{-1}{1 + \sqrt{1 + \frac{1}{x}}} = -\frac{1}{2}
\end{aligned}$$

□