MAT137 - Week 9

- Today we're talking about:
  - Inverse trig functions (continued from last time).
  - Derivatives of inverse functions.
  - Exponentials and logarithms.
  - Logarithmic differentiation.
  - (If there's time) the Mean Value Theorem.

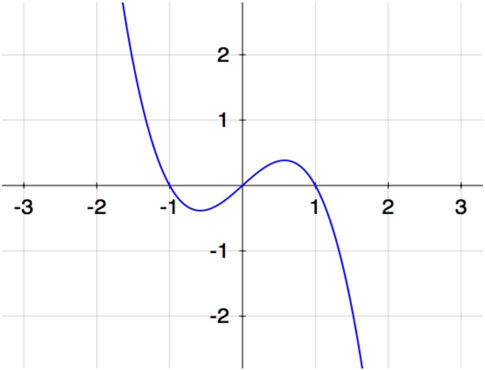
Inverse functions

Last class we determined that only injective functions have inverses.

Many functions, like $f(x) = x^2$, are not injective. However we can restrict them to intervals like $(-\infty, 0]$ or $[0, \infty)$ on which they are injective, and which can't be expanded any further while maintaining injectivity.

Exercise

Let f be the following function:



1. What is the largest interval containing -1 on which f has an inverse?
2. What is the largest interval containing 0 on which f has an inverse?

Sketch the graphs of these inverses.

arcsin

Last time, we defined the function $\arcsin(x)$ as the inverse of the function

$$g(x) = \sin(x) \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

In other words,

$$\arcsin(x) = \theta \quad \Leftrightarrow \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \sin(\theta) = x.$$

arcsin

Remember that the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is simply a choice we make in order to obtain an injective function.

This is just like how we choose to define $\sqrt{x}$ as the positive number whose square is x.

We can restricte $\sin(x)$ to another interval, like

$$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \text{ or } \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right].$$

and define an inverse for those restrictions as well. Those wouldn't be arcsin though.

arccos and arctan

$\cos(x)$ and $\tan(x)$ are also not injective, so we also need to restrict their domains to be able to define inverses.

By convention, we...

- ...restrict $\cos(x)$ to the interval $[0, \pi]$;
- ...restrict $\tan(x)$ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

There is a convenient table with this information for all six trig functions in Tyler's notes, page 75.

Exercise

Using this definition from above...

$$\arcsin(x) = \theta \quad \Leftrightarrow \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \sin(\theta) = x.$$

...compute the following:

1. $\sin(\arcsin(\frac{1}{2}))$, $\cos(\arcsin(\frac{1}{2}))$, $\tan(\arcsin(\frac{1}{2}))$. (Do these *without computing* $\arcsin(\frac{1}{2})$.)
2. $\sin(\arcsin(2))$
3. $\arcsin(\sin(1))$
4. $\arcsin(\sin(7))$ (Hint: The answer is not 7.)
5. $\arcsin(\sin(6))$

The derivative of *arcsin*(x)

To compute $\frac{d}{dx}\arcsin(x)$, we use implicit differentiation.

By definition, we know that

$$\sin(\arcsin(x)) = x$$

for all $x \in [-1, 1]$.

Differentiate both sides of this expression with respect to x. Then rearrange to obtain:

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1-x^2}}.$$

Exercise

Repeat this analysis but for arctan. That is, find...

- ...its domain.
- ...its range.
- ...a sketch of its graph.
- ...its derivative.

The rest of the inverse trig functions

Here are the derivatives of the remaining inverse trig functions.

- $\dfrac{d}{dx} arccos(x) = \dfrac{-1}{\sqrt{1-x^2}}$.
- $\dfrac{d}{dx}\,\text{arcsec}(x) = \dfrac{1}{|x|\sqrt{x^2-1}}$.
- $\dfrac{d}{dx}\,\text{arccsc}(x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$.
- $\dfrac{d}{dx}\,\text{arccot}(x) = \dfrac{-1}{1+x^2}$

(Don't memorize these.)

Derivatives of inverse functions in general

The same analysis we did for $\arcsin(x)$ can work for any inverse function.

Let f be an injective function. We'd like to compute $(f^{-1})'$.

By definition, we know that

$$f(f^{-1}(x)) = x.$$

Differentiate both sides of this expression with respect to x. Then rearrange to obtain:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

(I think Tyler's notes call this the "Inverse Function Theorem".)

Exercise

Suspose f is an injective function such that:

- $f(0) = 7$
- $f'(0) = \pi$

Compute $(f^{-1})'(7)$.

Exercise

Let $f(x) = -2x + 7$.

1. Sketch the graph of f, and convince yourself that f is injective on its whole domain.
2. Sketch the graph of f^{-1} by reflecting your previous graph.
3. Using the general formula from earlier, derive a formula for $(f^{-1})'(x)$.
4. Derive a formula for $f^{-1}(x)$.
5. Differentiate the formula you obtained, and confirm that the result agrees with your earlier answer.

Recall from the videos:

An exponential function is a function of the form $f(x) = a^x$ for some fixed $a > 0$.

Given an $a > 0$, the base a logarithm $\log_a$ is defined to be the inverse of a^x. That is,

$$y = \log_a(x) \quad \Leftrightarrow \quad a^y = x$$

or in other words

$$a^{\log_a(x)} = x = \log_a(a^x).$$

Exponentials and Logarithms

e is a special number which is *defined to be* the unique number such that

$$\frac{d}{dx} e^x = e^x.$$

Because e is so important, we call the base e logarithm the "natural logarithm", and denote it simply by log or ln. (I tend to use log.)

For a general $a > 0$, we have:

$$\frac{d}{dx} a^x = \log(a)\, a^x$$

Exercises

1. Solve the following equation for x.

$$\log_x(x+6) = 2.$$

2. Compute $f'(x)$ if $f(x) = \tan(7^{x^7})$.

3. Let $a > 0$. Compute $\frac{d}{dx}\log_a(x)$.

(Hint: Remember that $\log_a(x)$ is the inverse of a^x.)

4. Compute $g'(x)$ if $g(x) = \log|x|$.

Logarithmic differentiation

Idea: Logarithms simplify things.

Multiplication $\to$ addition: $\log(xy) = \log(x) + \log(y)$.

Division $\to$ subtraction: $\log(x/y) = \log(x) - \log(y)$.

Exponentiation $\to$ multiplication: $\log(x^y) = y\log(x)$.

We can use this to make complicated functions look simpler, so we can differentiate them.

Examples

Example: Compute $f'(x)$ if $f(x) = x^x$.

Answer: $f'(x) = x^x(\log(x) + 1)$.

Example: Let f, g, and h all be differentiable functions. Use logarithmic differentiation to differentiate $f(x)g(x)h(x)$.

Example: Compute $g'(x)$ if $g(x) = \log_x(x + 2)$.

Logarithmic differentiation

Logarithmic differentiation allowed us to extend the power rule to its final form:

$$\frac{d}{dx}x^a = ax^{a-1} \text{ for any real number } a.$$

With this new tool, you can essentially differentiate any function you write down.