MAT137 - Week 8

- Problem set 3 is due tomorrow at 3pm.
- Today we're talking about:
  - Implicit differentiation.
  - Related rates.
  - Inverse functions.
- **Watch the videos on exponentials and logarithms before next week's class.**

Explicit functions

An equation like $y = x^2 + \sin(x)$ expresses a relationship between values of x and y. More specifically, it says that y an explicit function of x. The function is $f(x) = x^2 + \sin(x)$.

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos(x).$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y. In this case though, the values of y cannot be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates all the information in the earlier equation.

But we still might want to ask how y varies when x varies.

Implicit functions

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases - when y is non-negative or non-positive - the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

Implicit functions

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{y} \right)$.
When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1-x^2}} \left(= \dfrac{-x}{-\sqrt{1-x^2}} = \dfrac{-x}{y} \right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$. This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\,\frac{dy}{dx}.$$

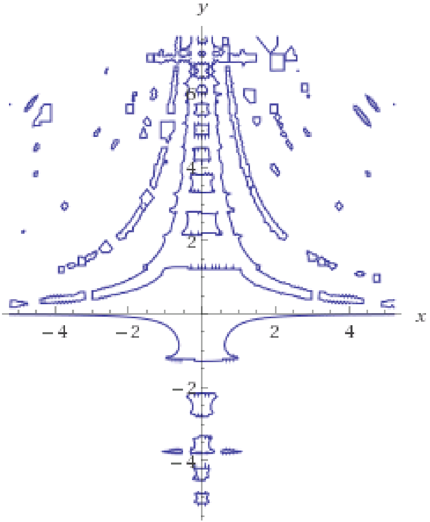
Implicit differentiation

In general, given some complex relationship between x and y, like this...

$$\tan^2(xy) = 3x^2y + \sec(y^2)$$

...it's just impossible to split into cases in which you can express y as an explicit function of x, so implicit differentiation is our only tool that always works.

By the way, here's what that curve looks like:



Compute $\frac{dy}{dx}$ if x and y satisfy the equation $\sin(x + y^2) = xy$.

Power rule improvement: Prove that $\frac{d}{dx}(x^a) = a\,x^{a-1}$ for any nonzero rational number a.

Related rates

Idea of these problems: If you know a relationship between two quantities, you can derive a relationship between rates of change of those two quantities.

For example: If you know how the area A of a circle relates to its radius R ($A = \pi R^2$), and you know the area is changing at some rate $\frac{dA}{dt}$, then you can figure out the rate $\frac{dr}{dt}$ at which the radius must be changing.

Example

Here's the related rates problem everyone sees first:

A 10 foot ladder leans against a wall. The bottom of the ladder starts slipping away at a rate of 0.5 feet per second. How quickly is the top of the ladder dropping when the bottom is 4 feet from the wall?

Example

A spherical balloon is being inflated with 1 cubic metre of air per hour. How quickly is its diameter increasing when it is 2 metres in diameter?

Example

You are walking in a straight line at 2m/s, and you pass a cat.

The cat sits 5m away from your path, and eerily watches you as you walk away.

How fast is the cat's head turning when you're 20m away from her?

Another classic example

Two ants are taking a nap. The first one is resting at the tip of the minute hand of a cuckoo clock, which is 25 cm long. The second one is resting at the tip of the hour hand, which is half the length. At what rate is the distance between the two ants changing at 3:30?

Inverses

A *binary operation* is an operation that takes two of the same sort of thing, and outputs one thing of the same sort.

Examples:

- Addition (takes two numbers, outputs one number)
- Multiplication
- Matrix multiplication (takes two matrices, outputs one matrix)
- function composition (takes to functions, outputs one function)

We'll denote a general binary operation with $\star$, and talk about "objects" as the things they act on.

Inverses

Given a binary operation $\star$, an object is called the identity for $\star$, and denoted by $\mathrm{id}_\star$, if operating with it doesn't do anything to the other input.

That is, for any object A, we have

$$A \star \mathrm{id}_\star = \mathrm{id}_\star \star A = A.$$

What is the identity of addition? Multiplication? Matrix multiplication? Function composition?

Inverses

Given an object A, an <u>inverse for A</u> is an object which when paired with A outputs the identity.

For example with addition, -7 is the inverse for 7, since

$$7 + (-7) = 0 = \mathrm{id}_+ \, .$$

What is the inverse for 7 under multiplication?

Note that not every number has an inverse under multiplication!

Inverse functions

For function composition, things work the same but are trickier to think about.

We saw that the identity for function composition is $\mathrm{id}_\circ(x) = x$.

So given a function f, its <u>inverse function</u> is another function, usually denoted by f^{-1}, such that

$$f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}_\circ$$

Or in other words:

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Inverse functions

Just as with multiplication, not every function has an inverse.

For example, consider the function $f(x) = x^2$.

There is no *function* that "undoes" squaring x, so this function has no inverse.

If you let g be the same function, but defined for only positive x, then g does have an inverse: $g^{-1}(x) = \sqrt{x}$.

Which functions have inverses?

A function has an inverse if and only if it is injective (also known as "one-to-one").

Recall this definition from the beginning of the course:

Definition

Suppose f is a function defined on a domain D. f is called *injective* if for all $x_1, x_2 \in D$, $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.

In other words, f is injective if it "passes the horizontal line test".

Inverse trig functions

The sine function is very far from being injective, so $f(x) = \sin(x)$ has no inverse function.

However, if we let $g(x) = \sin(x)$ *only for* $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then this function is injective and therefore does have an inverse.

Definition

$\arcsin(x)$ is defined to be the inverse of $g(x) = \sin(x)$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

In other words, $\arcsin(x) = y$ if and only if $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\sin(y) = x$.