A proof that e is irrational

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The purpose of this note is to outline a proof that e is irrational that is accessible to anyone who knows some basic facts about series. I believe this proof is due to Joseph Fourier, from somewhere around 1790-1800.

We assume the reader is familiar with the following two identities, both of which should be known to a student who has studied Taylor series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \text{and} \quad 1 = \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

The former fact can be obtained simply by substituting $x = 1$ into the Taylor series of e^x centred at 0, and the latter fact can be obtained from the well-known geometric series formula.

First, we examine the series on the left and establish a bound for the difference between e and a partial sum of the series. We let S_n denote the nth partial sum of the series:

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!}.$$

Note that $S_n < e$ for all n, since every term in the infinite series is positive.

Claim. *For all $n > 0$, $e - S_n < \frac{1}{n!}$.*

Proof.

$$\begin{aligned}
e - S_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\
&= \frac{1}{n! \cdot (n+1)} + \frac{1}{n! \cdot (n+1)(n+2)} + \frac{1}{n! \cdot (n+1)(n+2)(n+3)} + \cdots \\
&= \frac{1}{n!}\left(\frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\right) \\
&< \frac{1}{n!}\left(\frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots\right) \\
&= \frac{1}{n!}\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) \\
&= \frac{1}{n!}
\end{aligned}$$

□

This claim is of independent interest, of course. The speed with which the factorial sequence grows tell us that the partial sums of the series for e approximate its value quite quickly. 10 is the smallest integer whose factorial is greater than one million, and therefore this number:

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{9!} + \frac{1}{10!}$$

differs from e by less than 10^{-6}. Adding just three more terms of the series can get the difference to smaller than 10^{-9}, and so on.

Theorem 1. *e is irrational.*

Proof. This is a proof by contradiction. So, assume $e = \frac{p}{q}$, where p and q are positive integers.

By the claim above, we must have that $\frac{p}{q} - S_q < \frac{1}{q!}$. However, we can take the quantity on the left and combine it into a single fraction with denominator q!

$$\begin{aligned} \frac{p}{q} - S_q &= \frac{p}{q} - \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right) \\ &= \frac{p(q-1)!}{q!} - \frac{q!}{q!} - \frac{q!}{q!} - \frac{(q)(q-1)\cdots(4)(3)}{q!} - \frac{(q)(q-1)\cdots(5)(4)}{q!} - \cdots - \frac{q}{q!} - \frac{1}{q!} \\ &= \frac{C}{q!} \end{aligned}$$

where

$$C = p(q-1)! - 2q! - (q)(q-1)\cdots(4)(3) - (q)(q-1)\cdots(5)(4) - \cdots - q - 1.$$

This number C is a bit daunting to compute, but all that matters for us is that it's an integer.

According to the claim, we must therefore have that $\frac{C}{q!} < \frac{1}{q!}$, or in other words that $C \leq 0$. However, if this is true, we would have:

$$e - S_q = \frac{p}{q} - S_q = \frac{C}{q!} \leq 0 \quad \Longrightarrow e \leq S_q$$

This contradicts the fact that $e > S_n$ for all n, finishing the proof. □