A proof that $e$ is irrational

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The purpose of this note is to outline a proof that $e$ is irrational that is accessible to anyone who knows some basic facts about series. I believe this proof is due to Joseph Fourier, from somewhere around 1790-1800.

We assume the reader is familiar with the following two identities, both of which should be known to a student who has studied Taylor series:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} \quad \text{and} \quad 1 = \sum_{n=1}^{\infty} \frac{1}{2^n}.$$

The former fact can be obtained simply by substituting $x = 1$ into the Taylor series of $e^x$ centred at 0, and the latter fact can be obtained from the well-known geometric series formula.

First, we examine the series on the left and establish a bound for the difference between $e$ and a partial sum of the series. We let $S_n$ denote the $n$th partial sum of the series:

$$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(n-1)!} + \frac{1}{n!}.$$

Note that $S_n < e$ for all $n$, since every term in the infinite series is positive.

Claim. For all $n > 0$, $e - S_n < \frac{1}{n!}$.

Proof.

$$e - S_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots$$

$$= \frac{1}{n! \cdot (n+1)} + \frac{1}{n! \cdot (n+1)(n+2)} + \frac{1}{n! \cdot (n+1)(n+2)(n+3)} + \cdots$$

$$= \frac{1}{n!} \left( \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \right)$$

$$< \frac{1}{n!} \left( \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots \right)$$

$$= \frac{1}{n!} \left( \sum_{n=1}^{\infty} \frac{1}{2^n} \right)$$

$$= \frac{1}{n!} \blacksquare$$
This claim is of independent interest, of course. The speed with which the factorial sequence grows tells us that the partial sums of the series for $e$ approximate its value quite quickly. 10 is the smallest integer whose factorial is greater than one million, and therefore this number:

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{9!} + \frac{1}{10!}$$

differs from $e$ by less than $10^{-6}$. Adding just three more terms of the series can get the difference to smaller than $10^{-9}$, and so on.

**Theorem 1.** $e$ is irrational.

**Proof.** This is a proof by contradiction. So, assume $e = \frac{p}{q}$, where $p$ and $q$ are positive integers.

By the claim above, we must have that $\frac{p}{q} - S_q < \frac{1}{q^7}$. However, we can take the quantity on the left and combine it into a single fraction with denominator $q!$:

$$\frac{p}{q} - S_q = \frac{p}{q} - \left(1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{q!}\right)$$

$$= \frac{p(q - 1)!}{q!} - \frac{q!}{q!} - \frac{(q - 1)!}{q!} - \frac{(q)(q - 1)\cdots(4)(3)}{q!} - \frac{(q)(q - 1)\cdots(5)(4)}{q!} - \cdots - \frac{q}{q!} - \frac{1}{q!}$$

$$= \frac{C}{q!}$$

where

$$C = p(q - 1)! - 2q! - (q)(q - 1)\cdots(4)(3) - (q)(q - 1)\cdots(5)(4) - \cdots - q - 1.$$  

This number $C$ is a bit daunting to compute, but all that matters for us is that it’s an integer.

According to the claim, we must therefore have that $\frac{C}{q!} < \frac{1}{q^7}$, or in other words that $C \leq 0$. However, if this is true, we would have:

$$e - S_q = \frac{p}{q} - S_q = \frac{C}{q!} \leq 0 \implies e \leq S_q$$

This contradicts the fact that $e > S_n$ for all $n$, finishing the proof. $\square$