Parametrization of ideal classes in rings associated to binary forms

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Abstract. We parametrize the ideal classes of rings associated to integral binary forms by classes of tensors in $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$. This generalizes Bhargava's work on Higher Composition Laws, which gives such parametrizations in the cases $n = 2, 3$. We also obtain parametrizations of 2-torsion ideal classes by symmetric tensors. Further, we give versions of these theorems when $\mathbb{Z}$ is replaced by an arbitrary base scheme S, and geometric constructions of the modules from the tensors in the parametrization.

1. Introduction

The goal of this paper is to give a parametrization of ideal classes in the rings associated to binary n-ic forms. Every integral binary form of degree n has a ring of rank n (a ring isomorphic to $\mathbb{Z}^n$ as a $\mathbb{Z}$-module) associated to it (see, for example, [16, 17, 20]). When the form is irreducible, the associated ring is an order in a degree n number field. For $n = 2$, the ring associated to a binary quadratic form is just the quadratic ring of the same discriminant. For $n = 3$, binary cubic forms parametrize cubic rings exactly (see [8, 11]). For $n > 3$ not all rank n rings (or orders in degree n number fields) are associated to a binary n-ic form. The special orders which are associated to binary n-ic forms have been studied as a natural generalization of monogenic orders in [17], and in [7] it is found that prime splitting in these orders can be understood simply in terms of the factorization of the form modulo the prime. In [20] it is shown that binary n-ic forms parametrize rank n rings that have an ideal class with certain special structure. When $n = 4$, there is another particularly simple description given in [19]; that is, binary quartic forms parametrize quartic rings with monogenic cubic resolvents. The cubic resolvent of a quartic ring is an integral model of the classical cubic resolvent field, and was introduced in [3] to parametrize quartic rings. Thus the orders associated to binary forms are interesting examples of orders in number fields with various nice properties. In this paper, we parametrize the ideal classes of these orders by classes of tensors or trilinear forms.

It is a classical result that ideal classes of quadratic rings are parametrized by binary quadratic forms, originally by Dedekind and Dirichlet [6] (see also [1] for another parametrization of ideal classes of quadratic rings). This parametrization has had important applications in number theory, from genus theory to the computation of class groups of quadratic fields. Bhargava [2] has found a parametrization of ideal classes of cubic rings. This is his space of 2

by 3 by 3 boxes of integers, or classes of elements of $\mathbb{Z}^2 \otimes \mathbb{Z}^3 \otimes \mathbb{Z}^3$. Bhargava's parametrization of ideal classes of cubic rings is much more recent, but already it can be applied to find the average size of the 2-torsion part of the class group of cubic orders, both for maximal and non-maximal orders (see forthcoming work of Bhargava).

In this paper, we prove that classes of elements of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ parametrize ideal classes of the rings associated to binary n-ic forms for all n. This gives an explicit parametrization of class groups of a natural infinite family of orders in rank n number fields. When $n = 2, 3$, these are the results of Bhargava in [1, 2], though the more general results in this paper require entirely new methods for their proofs. The proofs of the Dedekind–Dirichlet correspondence and Bhargava's parametrizations fundamentally rely on the fact that they are in small finite cases. For example, in [2] the proof of the main result relies on verifying (by hand or computer) that a system of 18 linear and 9 quadratic equations has a unique solution. When 3 is replaced by n in that problem, such a computation is infeasible. The new methods in this paper rely on understanding the theory of certain special ideal classes associated to binary forms, as developed in [20].

One can also study symmetric elements of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$, that is elements of $\mathbb{Z}^2 \otimes \operatorname{Sym}_2 \mathbb{Z}^n$. We prove that these symmetric tensors parametrize 2-parts of class groups, just as in the cases $n = 2, 3$ in [1, 2]. In the case $n = 4$, the space $\mathbb{Z}^2 \otimes \operatorname{Sym}_2 \mathbb{Z}^4$ is exactly the space used in the forthcoming work of Bhargava and Shankar to determine the average size of the 4-Selmer group of elliptic curves. Morales [14, 15] has also studied elements of $\mathbb{Z}^2 \otimes \operatorname{Sym}_2 \mathbb{Z}^n$ and associated modules to them that are related to the 2-part of certain class groups, though he associates modules for a slightly different ring than that in our work.

We also prove analogous parametrizations when the integers are replaced by an arbitrary base scheme S (or base ring when $S = \operatorname{Spec} R$), and so we also generalize the results from [1, 2] from the integers to an arbitrary base. In Morales's work [15], he has replaced $\mathbb{Z}$ by an arbitrary maximal order in a number field in his constructions of modules from symmetric tensors.

In this paper, we give both algebraic and geometric constructions for the modules associated to an element of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$. The algebraic construction is given by explicit formulas for the action of the ring elements on a $\mathbb{Z}$-basis for an ideal. The geometric construction gives the modules as sections of line bundles of schemes naturally given by the tensor, and is quite simple for nice tensors.

As this paper generalizes the results of [2] from $\mathbb{Z}$ to an arbitrary base, it could be applied to counting problems in number theory over arbitrary orders in number fields or function fields, specifically to finding the average of the 2-part of the class group of orders that are cubic over a fixed order. For $n > 3$, the associated counting problems for classes of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ are much harder, and involve reduction theory problems at the boundary of current research. Given how little is known about the average size of 2-parts of class groups in any natural infinite family of degree n number fields, the application of this paper and new work for the reduction theory of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ has potentially very interesting results.

This work can also be interpreted geometrically, because a rank n algebra over a scheme S is equivalent to a finite, flat degree n cover of S. In those terms, this work explicitly constructs the moduli space of line bundles (and degenerations) on certain finite covers. In the case $n = 3$, this includes all cubic covers. This paper could be applied over $S = \mathbb{P}^1$ to study explicitly the universal Jacobian of the Hurwitz stack of trigonal curves, analogously to the work of the author and Erman [10] on the universal Jacobian of hyperelliptic curves. The explicit description of the moduli space provided by this paper gives a proof of unirationality of

this universal Jacobian as well as a framework for computing its Picard group and possibly its Chow ring.

1.1. Outline of results. We can represent an element of the space $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ as a pair $A = (A_1, A_2)$ of n by n matrices. Let $\mathrm{Det}(A)$ be the binary n-ic form $\mathrm{Det}(A_1 x + A_2 y)$. For a non-zero integral binary form f, let R_f be the ring of global functions of the subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ cut out by f. (The same ring, via other constructions, is associated to f in [7, 16, 17, 20] and reviewed in Section 2 of the current paper.) A binary form f is *non-degenerate* if it has non-zero discriminant. Let G be the subgroup of elements (g_1, g_2) of $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{GL}_n(\mathbb{Z})$ such that $\mathrm{Det}(g_1)\,\mathrm{Det}(g_2) = 1$. Over the integers, we have the following version of our main theorem which makes some assumptions on f for simplicity of statement. (This version follows from Theorem 3.1 using Propositions 5.1, 5.4, and 5.8.)

Theorem 1.1. *For a primitive non-degenerate binary n-ic form f with coefficients in $\mathbb{Z}$, there is a bijection between G-classes of $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ such that $\mathrm{Det}(A) = f$ and* (*not necessarily invertible*) *ideal classes of R_f.*

If f is monic in x and $\mathrm{Det}(A) = f$, then $\mathrm{Det}(A_1) = 1$. We can then act by an element of G so as to assume that A_1 is the identity matrix. Further G action fixing A_1 (the identity matrix) is just conjugation of A_2. If $f = F(x_1, x_2)$ is monic, then $R_f = \mathbb{Z}[\theta]/F(\theta, 1)$, and Theorem 1.1 generalizes the classical result that ideal classes of monogenic orders correspond to conjugacy classes in $\mathbb{Z}^n \otimes \mathbb{Z}^n$ whose characteristic polynomial is $F(t, -1)$. So, we can view Theorem 1.1 as placing rings associated to binary forms in analogy with monogenic rings, as in [7, 17].

In the case $n = 3$, Theorem 1.1 is slightly stronger than the corresponding version in [2], which gives a correspondence between A associated to invertible ideals and invertible ideal classes of R_f.

As in [1,2], we must define a notion of balanced to state a more general theorem. We will show in Section 5 that there are several equivalent ways to formulate the notion of balanced, but we first give the one closest to Bhargava's notion in [1, 2]. For a non-zero form f, there is a naturally associated ideal I_f of R_f, given as an R_f-module by the global sections of $\mathcal{O}(n-3)$ on $\mathbb{P}^1_{\mathbb{Z}}$ pulled back to the subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ cut out by f. Moreover, there is a natural map $\delta : I_f \to H^1(\mathbb{P}^1_{\mathbb{Z}}, \mathcal{O}(-3)) = \mathbb{Z}^2$. The ideal class I_f was constructed in [20] and the constructions of I_f and δ are reviewed in Section 2 of the current paper. A *fractional ideal* for R_f is a finitely generated R_f-submodule of $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$.

A *balanced* pair of ideals (for f) is a pair (M, N) of fractional R_f-ideals such that $MN \subset I_f$ and $|M||N| = |I_f|$, where $|P|$ denotes the norm of P. Two such pairs (M, N) and (M', N') are in the same class if $M' = \lambda M$ and $N' = \lambda^{-1} N$ for some invertible element $\lambda \in R_f \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that an invertible ideal has a unique balancing partner, and thus classes of balanced pairs of invertible ideals are the same as invertible ideal classes.

Theorem 1.2. *For a non-degenerate binary n-ic form f with coefficients in $\mathbb{Z}$, the map that sends a balanced pair (M, N) to the tensor in $\mathbb{Z}^2 \otimes \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \otimes \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ associated to the map*

$$M \otimes_{\mathbb{Z}} N \to MN \to I_f \overset{\delta}{\to} \mathbb{Z}^2$$

gives a bijection

$$\{ \textit{ classes of balanced pairs } (M, N) \textit{ for } f \} \longleftrightarrow \left\{ \begin{array}{l} G\text{-classes of} \\ A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n \text{ with} \\ \mathrm{Det}(A) = f \end{array} \right\}.$$

Theorem 1.2 follows from Theorems 3.1 and 5.5. In [2], Bhargava asks for an appropriate formula of balanced for degenerate forms so as to obtain a theorem such as the above. We will give such a notion, which can be seen for example in Theorem 1.4.

We also relate symmetric elements of $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ to the 2-part of class groups of rings associated to binary forms. For nice forms f we get the following, which follows from Theorems 4.1 and 5.9.

Theorem 1.3. *For a primitive non-degenerate binary n-ic form f with coefficients in* $\mathbb{Z}$, *the map that sends M to the tensor in $\mathbb{Z}^2 \otimes \mathrm{Sym}_2 \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ associated to the map*

$$M \otimes_{\mathbb{Z}} M \xrightarrow{k^{-1}} k^{-1}M^2 \rightarrow I_f \xrightarrow{\delta} \mathbb{Z}^2$$

gives a bijection

$$\left\{ \begin{array}{l} \textit{classes of } (M, k) \textit{ where } M \textit{ is a fractional} \\ R_f\text{-ideal, } k \textit{ is an invertible element of} \\ R_f \otimes_{\mathbb{Z}} \mathbb{Q}, \textit{ and } M = (I_f k : M) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathrm{GL}_n(\mathbb{Z}) \textit{ classes of} \\ A \in \mathbb{Z}^2 \otimes \mathrm{Sym}_2 \mathbb{Z}^n \text{ with} \\ \mathrm{Det}(A) = \pm f \end{array} \right\},$$

where $\mathrm{Sym}_2 \mathbb{Z}^n$ are symmetric n by n matrices, the action of $g \in \mathrm{GL}_n(\mathbb{Z})$ is by multiplication on the left by g and right by g^t, and (M, k) and (M_1, k_1) are in the same class if $M_1 = \lambda M$ and $k_1 = \lambda^2 k$ for some invertible element $\lambda \in R_f \otimes_{\mathbb{Z}} \mathbb{Q}$, and $(I_f k : M)$ is the fractional ideal of elements x such that $xM \subset I_f k$.

We also give a version of the above in Theorem 4.1 for all non-zero forms f, which again uses a more delicate notion of balanced. If we restrict Theorem 1.3 to invertible modules M, then the condition $M = (I_f k : M)$ is replaced by $M^2 = I_f k$, and the restricted set is an extension of a torsor of the 2-part of the class group of R_f by R_f^*/R_f^2. (We say a torsor instead of a principal homogeneous space because I_f might not be a square in the class group and there would be no such M in that case.)

This paper also gives analogous results over an arbitrary base scheme S. We consider V, U, W, locally free $\mathcal{O}_S$-modules of ranks 2, n, and n, respectively. We then study global sections $p \in V \otimes U \otimes W$. We can construct $\mathrm{Det}(p) \in \mathrm{Sym}^n V \otimes \wedge^n U \otimes \wedge^n W$, which is a binary n-ic form. Fix any f in $\mathrm{Sym}^n V \otimes L$, where L is a locally free rank 1 $\mathcal{O}_S$-module. There is a natural associated rank n $\mathcal{O}_S$-algebra R_f and an R_f module I_f with a natural map $\delta : I_f \rightarrow V$ of $\mathcal{O}_S$-modules (see [20], or Section 6 of the current paper). For an $\mathcal{O}_S$-module P, we write P^* for $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(P, \mathcal{O}_S)$, even when P is a module for another $\mathcal{O}_S$-algebra.

A *balanced pair of modules* for a non-zero-divisor f is a pair of R_f-modules M and N, each a locally free rank n $\mathcal{O}_S$-module such that $\wedge^n_{\mathcal{O}_S} M \otimes \wedge^n_{\mathcal{O}_S} N \cong L^*$, and a map of R_f-modules $M \otimes_{R_f} N \rightarrow I_f$, such that when the composition

$$M \otimes_{\mathcal{O}_S} N \rightarrow M \otimes_{R_f} N \rightarrow I_f \rightarrow V$$

is written as $A \in M^* \otimes N^* \otimes V$ we have $\mathrm{Det}(A) = fu$, where u is a unit in $\mathcal{O}_S$. (We can then choose a unique isomorphism $\wedge^n_{\mathcal{O}_S} M \otimes \wedge^n_{\mathcal{O}_S} N \cong L^*$ such that $\mathrm{Det}(A) = f$.) We have the following, proven in Theorem 6.2.

Theorem 1.4. *For every non-zero-divisor binary n-ic form $f \in \text{Sym}^n V \otimes L$, the map that sends (M, N) to the tensor in $A \in M^* \otimes N^* \otimes V$ associated to the map*

$$M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \overset{\delta}{\to} V$$

gives a bijection

$$\left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{balanced pairs } (M, N) \textit{ of} \\ \textit{modules for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of } A \in V \otimes U \otimes W, \\ \textit{where } U \textit{ and } W \textit{ are locally free rank } n \\ \mathcal{O}_S\textit{-modules with an isomorphism} \\ \wedge^n U \otimes \wedge^n W \cong L \textit{ such that } \text{Det}(A) = f \end{array} \right\}.$$

The condition that f is not a zero-divisor is awkward because it is not invariant under base change in S. Unfortunately, the methods of this paper require this condition.

From a $p \in V \otimes U \otimes W$ we give two constructions of the corresponding ideal classes or modules. The first construction (in Section 6) is algebraic and explicit and the second (in Section 8) is geometric and more intuitive. We give a heuristic description of the geometric construction here. If we have locally free $\mathcal{O}_S$-modules F and G, and $s \in F \otimes G$, then we can construct the k-minor $\wedge^k s \in \wedge^k F \otimes \wedge^k G$. If H is also a locally free $\mathcal{O}_S$-module, and we have $s \in F \otimes G \otimes H$, then we have a k-minor $\wedge_H^k s$ with H-coefficients in $\wedge^k F \otimes \wedge^k G \otimes \text{Sym}^k H$ (see [21, Section 8.2] for the details of this construction). For $p \in V \otimes U \otimes W$, the n-minor with coefficients in V defines a subscheme $T_p(V)$ in $\mathbb{P}(V)$, the 2-minor with coefficients in U defines a subscheme $T_p(U)$ in $\mathbb{P}(U)$, and the 2-minor with coefficients in W defines a subscheme $T_p(W)$ in $\mathbb{P}(W)$. Abusing notation, we let π denote the map from all of these schemes to S. The *heuristic* definition of R_f is to take $\pi_* \mathcal{O}_{T_p(V)}$ (or $\pi_* \mathcal{O}_{T_p(U)}$ or $\pi_* \mathcal{O}_{T_p(W)}$ – all the $\mathcal{O}_S$-algebras turn out to be the same in the nicest cases), and $M = \pi_* \mathcal{O}_{T_p(U)}(1)$ and $N = \pi_* \mathcal{O}_{T_p(W)}(1)$ (where $\mathcal{O}(1)$ is as pulled back from the corresponding projective bundle). This construction does not work for all p (e.g., the three algebras given for R_f are not necessarily the same) and it is not functorial in S. As in the case of binary n-ic forms in [20], we use hypercohomology to extend our heuristic geometric construction to a construction that works in all cases and is functorial.

1.2. Outline of the paper. In Sections 2–5, we work over the integers, though in fact the constructions given there are the constructions that work for a general base S. This part of the paper is written to be accessible to those mainly interested in class groups of number fields.

In Section 2 we review the rings and ideals associated to binary n-ic forms, and give some computations with these rings and ideals that are needed for the work in this paper. In Section 3, we prove our main theorem (Theorem 3.1) over the integers. We first give the algebraic constructions of a pair of modules from an element of $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ in Section 3.1. In Section 3.2 we prove Theorem 3.1 when the leading coefficient of f is not zero. In Section 3.3, we study the $\text{GL}_2(\mathbb{Z})$ invariance of our construction of modules, and use this to finish the proof of Theorem 3.1. In Section 4 we give the general analogs of Theorem 3.1 for symmetric tensors.

In Section 5, we further study the notion of balanced pairs of modules, and show it is equivalent to a characteristic polynomial condition and an index condition (Proposition 5.1). In Section 5.1, we show that for non-degenerate binary n-ic forms all balanceable modules are fractional ideals (Proposition 5.4), and prove that in this case the definition of balanced modules is equivalent to the definition of balanced ideals given above (Theorem 5.5). In Section 5.2,

we specialize to the case of primitive non-degenerate forms, where we see that every fractional ideal has a unique balancing partner. This is the final step in the proof of Theorem 1.1.

In Section 6, we prove versions of these main theorems over an arbitrary base. In particular, we prove Theorem 1.4 (as Theorem 6.2) and a symmetric version. In Section 7, we give a geometric construction of the modules from the universal tensor and prove it is the same as the algebraic construction in Section 3.1. The main obstacle is that we give multiple ring constructions and we must show that they agree. The rings are given by global sections of different schemes, but the schemes themselves are not isomorphic. Finally, in Section 8, we give a geometric construction over arbitrary base of the modules from a triple tensor and prove that it commutes with base change (Corollary 8.3).

2. Binary forms, rings, and ideals

Given a binary n-ic form $f_0 x_1^n + f_1 x_1^{n-1} x_2 + \cdots + f_n x_2^n$ with $f_i \in \mathbb{Z}$, there is a naturally associated rank n ring R_f (see [16, 17, 20]) and a sequence of R_f-modules (see [18, 20]). Here we review the facts from [20] about R_f and these modules that are necessary in this paper, as well as make some computations that will be critical in this work. We will eventually need these results over more general rings than $\mathbb{Z}$, so we will now work over an arbitrary ring B in place of $\mathbb{Z}$.

Let $f = f_0 x_1^n + f_1 x_1^{n-1} x_2 + \cdots + f_n x_2^n$ a binary n-ic form with coefficients $f_i \in B$ such that f is not a zero-divisor in $B[x, y]$. We first give geometric constructions of a ring and ideals from f, as given in [20]. We define R_f as the B-algebra of global sections of the regular functions of T_f, the subscheme of $\mathbb{P}^1_B$ defined by f. We have line bundles $\mathcal{O}_{T_f}(k)$ on T_f pulled back from $\mathcal{O}(k)$ on $\mathbb{P}^1_B$. We define I_f to be $\Gamma(\mathcal{O}_{T_f}(n-3))$ (i.e. the global sections of $\mathcal{O}_{T_f}(n-3)$), and J_f to be $\Gamma(\mathcal{O}_{T_f}(n-2))$. This gives I_f and J_f the structure of R_f-modules. Note that our I_f is the I_f^{n-3} or $\mathcal{I}_{f\,n-3}$ of [20], and our J_f is the I_f^{n-2} or $\mathcal{I}_{f\,n-2}$ of [20].

Equivalent, but more concrete, constructions of R_f, I_f, and J_f are also given in [20], and we give those now, as they will be easier to work with. For these constructions, we assume f_0 is not a zero-divisor in B. Write $f = F(x, y)$. Let $B' = B_{f_0}$ (the ring B with f_0 inverted). We can also define the B-algebra R_f as the subring of $B'[\theta]/F(\theta, 1)$ generated by $\zeta_0, \ldots, \zeta_{n-1}$ with

$$\zeta_0 = 1, \quad \zeta_k = f_0 \theta^k + \cdots + f_{k-1}\theta \text{ for } k > 0.$$

The ζ_k give a B-module basis of R_f, and it is shown in [20, Theorem 2.4] that this definition of R_f agrees with the geometric one given above, and in particular that the B-module generated by the ζ_i is closed under multiplication. Note that if f_0 is a unit in B, then $R_f = B[\theta]/F(\theta, 1)$. We can define I_f and J_f as sub-B-modules of $B'[\theta]/F(\theta, 1)$, such that

I_f is the B-module generated by $1, \theta, \theta^2, \ldots, \theta^{n-3}, \zeta_{n-2}, \zeta_{n-1}$,

   or equivalently generated by $1, \theta, \theta^2, \ldots, \theta^{n-3}, f_0\theta^{n-2}, f_0\theta^{n-1} + f_1\theta^{n-2}$,

J_f is the B-module generated by $1, \theta, \theta^2, \ldots, \theta^{n-3}, \theta^{n-2}, \zeta_{n-1}$.

When $n = 2$, we use only the second description of I_f given above. In [20, Theorem 2.4], it is shown that these definitions of I_f and J_f agree with the geometric ones given above, and in particular that I_f and J_f are closed under multiplication by elements of R_f. We have a map of R_f-modules $I_f \to J_f$ given by inclusion. This map is not canonical (i.e. not GL_2 equivariant

for the natural action of GL_2 on binary forms) and does not arise geometrically, yet it will be important in our proofs.

The elements $f_0, \zeta_1, \ldots, \zeta_{n-1}$ are a B'-module basis of $B'[\theta]/F(\theta, 1)$. Let $\check{\zeta}_i$ be the B'-module basis of $\operatorname{Hom}_{B'}(B'[\theta]/F(\theta, 1), B')$ dual to the ζ_i. So $\check{\zeta}_i(\zeta_j) = \delta_{ij}$ for $j > 0$. Also, let $\check{\theta}_i$ be the B'-module basis of $\operatorname{Hom}_{B'}(B'[\theta]/F(\theta, 1), B')$ dual to $1, \theta, \theta^2, \ldots, \theta^{n-1}$. We can apply these $\check{\zeta}_i$ and $\check{\theta}_i$ to elements in I_f and J_f since they lie in $B'[\theta]/F(\theta, 1)$, but they are not necessarily dual to a B-module basis of I_f or J_f. The following are the key computations we will need.

Proposition 2.1. *For $r \in B'[\theta]/F(\theta, 1)$ and $1 \leq k \leq n - 1$,*

$$\check{\zeta}_{n-1}(\zeta_k r) = \check{\theta}_{n-1-k}(r) - f_k \check{\zeta}_{n-1}(r).$$

Proof. We will write out $\zeta_k r$ in terms of powers of θ and then read off the coefficient of θ^{n-1}. First, we write $r = \sum_{j=0}^{n-1} r_j \theta^j$ and so

$$\zeta_k r = (f_0 \theta^k + \cdots + f_{k-1} \theta)(r_{n-1} \theta^{n-1} + \cdots + r_0).$$

To find the θ^{n-1} coefficient, we only have to look at terms of r with $j \geq n - 1 - k$. From the $r_{n-1-k} \theta^{n-1-k}$ term we get a θ^{n-1} coefficient of $r_{n-1-k} f_0$. From the remaining terms, we get the sum

$$\sum_{j=n-k}^{n-1} r_j \theta^j (f_0 \theta^k + \cdots + f_{k-1} \theta) = \sum_{j=n-k}^{n-1} r_j \theta^{j-(n-k)} (f_0 \theta^n + \cdots + f_{k-1} \theta^{n-k+1})$$

$$= \sum_{j=n-k}^{n-1} -r_j \theta^{j-(n-k)} (f_k \theta^{n-k} + \cdots + f_n)$$

and the only term of the final sum with a non-zero θ^{n-1} coefficient is the $j = n-1$ term which has a θ^{n-1} coefficient of $-r_{n-1} f_k$. So $\check{\theta}_{n-1}(r) = f_0 \check{\theta}_{n-1-k}(r) - f_k \check{\theta}_{n-1}(r)$, and dividing by f_0 proves the proposition. $\square$

Corollary 2.2. *For $r \in B'[\theta]/f(\theta, 1)$,*

$$\check{\zeta}_{n-1}(\theta r) = \check{\zeta}_{n-2}(r).$$

Proof. We have

$$\check{\zeta}_{n-1} := \frac{\check{\theta}_{n-1}}{f_0} \quad \text{and} \quad \check{\zeta}_{n-2} := \frac{\check{\theta}_{n-2} - \frac{f_1 \check{\theta}_{n-1}}{f_0}}{f_0},$$

and thus this follows from the above proposition when $k = 1$. $\square$

Lemma 2.3. *If we have a homomorphism ϕ of B-modules from some B-module P to J_f, then the image of ϕ is in I_f if and only if the image of $\check{\zeta}_{n-2}\phi$ is in B.*

Proof. The elements of I_f are just the elements $j \in J_f$ for which $\check{\zeta}_{n-2}(j) \in B$. $\square$

Thus $\check{\zeta}_{n-1}$ and $-\check{\zeta}_{n-2}$ give two B-module maps from I_f to B, or a B-module map we denote $\delta : I_f \to V = B^2$, where $(-1)^{i+1}\zeta_{n-i}$ gives the map to the ith coordinate of V. We have chosen the maps in such a way that the map $\delta : I_f \to V$ is the canonical map given in [20, Equation (3.8)] (where the k in [20, Equation (3.8)] is $n-3$ for our purposes). This map is useful because it doesn't lose information about R_f-module maps. More formally, we have the following.

Proposition 2.4.   *For any binary n-ic form f and any R_f-module P, composition with the map $\delta : I_f \to V$ gives an injection of R_f modules*

$$\mathrm{Hom}_{R_f}(P, I_f) \to \mathrm{Hom}_B(P, V).$$

Proof.   We suppose, for the sake of contradiction, that we have a non-zero map $\phi \in \mathrm{Hom}_{R_f}(P, I_f)$ such that the image of ϕ is in the kernel of $I_f \to V$. Let r be a non-zero element of $\mathrm{im}(\phi)$. Then, by Proposition 2.1 we have

$$\check{\theta}_{n-1-k}(r) = \check{\zeta}_{n-1}(\zeta_k r) = 0$$

for $2 \le k \le n-1$. Thus, we see that $r = 0$.   □

In [20, Corollary 3.7], it is shown that as R_f modules, $J_f \cong \mathrm{Hom}_B(R_f, B)$, and thus we have the following proposition.

Proposition 2.5.   *For any binary n-ic form f and any R_f-module P, composition with the map $\check{\zeta}_{n-1}: J_f \to B$ gives an isomorphism of R_f modules*

$$\mathrm{Hom}_{R_f}(P, J_f) \xrightarrow{\check{\zeta}_{n-1}} \mathrm{Hom}_B(P, B).$$

3. Main theorems

We write an element $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ as pair A_1, A_2 of $n \times n$ matrices. The *determinant* of A is the binary n-ic form $\mathrm{Det}(A_1 x_1 + A_2 x_2)$. For non-zero binary form f with integer coefficients, we defined in Section 2 a rank n ring R_f and two modules I_f and J_f for that ring. Recall that we have a map $\delta : I_f \to V$ of abelian groups, where $V = \mathbb{Z}^2$. We will next define a notion of a balanced pair of R_f-modules. The idea is that the product of the pair should map to I_f, but that map should be constrained by the form f itself. Note the definition we now give is different from the one given in the Introduction, but we will see in Theorem 5.5 that the definitions agree for non-degenerate f.

Definition.   A *based balanced pair of modules* for f is a pair of R_f-modules M and N, a choice of basis $M \cong \mathbb{Z}^n$ and $N \cong \mathbb{Z}^n$, and a map of R_f-modules $M \otimes_{R_f} N \to I_f$, such that when the composition $M \otimes_{\mathbb{Z}} N \to M \otimes_{R_f} N \to I_f \to V$ is written as a pair of matrices A_1 and A_2 (viewing elements of M as row vectors and elements of N as column vectors), we have $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$. If v_i, m_j, and n_k are the bases of V, M, and N respectively indicated above, then the j, k entry of A_i is the coefficient of v_i in the image of $m_j \otimes n_k$, i.e. $(-1)^{i+1}\check{\zeta}_{n-i}(m_j \otimes n_k)$. We will often refer to the based balanced pair as M, N, with the bases and balancing map understood.

Definition. A *balanced pair of modules* for a non-zero form f is a pair of R_f-modules M and N, each a free rank n $\mathbb{Z}$-module, and a map of R_f-modules $M \otimes_{R_f} N \to I_f$, such that when the composition $M \otimes_{\mathbb{Z}} N \to M \otimes_{R_f} N \to I_f \to V$ is written as a pair of matrices A_1 and A_2, we have $\mathrm{Det}(A_1 x_1 + A_2 x_2) = \pm f$. Given a balanced pair of modules for a non-zero form f, there is a unique choice of generator χ of $\wedge^n M \otimes \wedge^n N$ such that $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$ when constructing A with bases of M and N that give $\chi \in \wedge^n M \otimes \wedge^n N$. If we have based balanced pairs (M, N) and (M', N') such that the modules and balancing maps are the same and only the bases differ, then the change of bases must preserve χ since both based balanced pairs give $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$.

In this section we prove the following theorem.

Theorem 3.1. *For every non-zero binary n-ic form f with coefficients in $\mathbb{Z}$, the map that sends a balanced pair (M, N) to the tensor $\phi(M, N) \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ associated to the map*

$$M \otimes_{\mathbb{Z}} N \to M \otimes_{R_f} N \to I_f \xrightarrow{\delta} V$$

gives a bijection

$$\left\{ \begin{array}{l} \textit{based balanced pairs } (M, N) \\ \textit{of modules for } f \end{array} \right\} \longleftrightarrow \left\{ A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n \textit{ with } \mathrm{Det}(A) = f \right\}.$$

We define G to be the subgroup of $\mathrm{GL}_n(\mathbb{Z}) \times \mathrm{GL}_n(\mathbb{Z})$ of elements (g_1, g_2) such that $\mathrm{Det}(g_1) \mathrm{Det}(g_2) = 1$. Then, G acts equivariantly in the above bijection (acting of the bases of M and N), and we obtain a bijection

$$\left\{ \begin{array}{l} \textit{isomorphism classes of balanced} \\ \textit{pairs } (M, N) \textit{ of modules for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} G\textit{-classes of } A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n \\ \textit{with } \mathrm{Det}(A) = f \end{array} \right\}.$$

In Section 3.1 we construct a based balanced pair of modules from an $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$. Our construction is completely concrete, and we give formulas for the action of R_f on M and N. In Section 3.2, we prove that this construction gives an inverse to the map ϕ when $f_0 \neq 0$. In Section 3.3, we use the GL_2 equivariance of our construction to reduce to the case that $f_0 \neq 0$, which will prove Theorem 3.1.

3.1. Construction of balanced pair of modules. We are given $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$, which we can write as a pair A_1, A_2 of $n \times n$ matrices. Let f be the determinant of A. In this section, we will construct a based balanced pair (M, N) of modules for f. We begin by letting $M = \mathbb{Z}^n$ and $N = \mathbb{Z}^n$ as abelian groups. It remains to specify the R_f action on M and N and the map of R_f-modules $M \otimes_{R_f} N \to I_f$. We can write the elements of M as row vectors with entries in $\mathbb{Z}$ and the elements of N as column vectors with entries in $\mathbb{Z}$. Heuristically, the action of R_f will by given by θ acting on M on the right by $-A_2 A_1^{-1}$ and on N on the left by $-A_1^{-1} A_2$. The trouble with this construction is that θ is not an element of R_f (unless f_0, the x^n coefficient of f, is ± 1) and that A_1 is not necessarily invertible (it could be the zero matrix!). We could solve both of these problems by inverting f_0, but it is possible that $f_0 = 0$. So we will pass to a universal situation, where we can always invert f_0.

We replace $\mathbb{Z}$ by the ring $\Lambda = \mathbb{Z}[\{u_{ijk}\}_{1 \leq i \leq 2, 1 \leq j \leq n, 1 \leq k \leq n}]$ (the free polynomial algebra on $2n^2$ variables over $\mathbb{Z}$), and we replace A with the universal tensor $\mathcal{C}$ in $\Lambda^2 \otimes_\Lambda \Lambda^n \otimes_\Lambda \Lambda^n$,

where $\mathcal{C}_i$ has j, k entry $u_{i,j,k}$. We have a binary n-ic form $c = \mathrm{Det}(\mathcal{C}_1 x_1 + \mathcal{C}_2 x_2)$ with coefficients in Λ. We now let $M_{\mathcal{C}} = \Lambda^n$ and $N_{\mathcal{C}} = \Lambda^n$ as Λ-modules. We will give an action of the Λ-algebra R_c on $M_{\mathcal{C}}$ and $N_{\mathcal{C}}$ and then we will give a map of R_c-modules $M_{\mathcal{C}} \otimes_{R_c} N_{\mathcal{C}} \to I_c$. This construction will be equivariant for the G_Λ actions, where G_Λ is the subgroup of $\mathrm{GL}_n(\Lambda) \times \mathrm{GL}_n(\Lambda)$ of (g_1, g_2) such that $\mathrm{Det}(g_1) \mathrm{Det}(g_2) = 1$. To recover a construction over $\mathbb{Z}$, we can just specialize by letting the $u_{i,j,k} = a_{i,j,k}$ in our formulas.

3.1.1. R_c action. We will write elements of $M_{\mathcal{C}}$ as row vectors with entries in Λ and elements of $N_{\mathcal{C}}$ as column vectors with entries in Λ. We can write

$$c = c_0 x_1^n + c_1 x_1^{n-1} x_2 + \cdots + c_n x_2^n.$$

We will invert c_0 and denote all of the corresponding objects with a prime. For example, we have $\Lambda' = \Lambda_{c_0}$, the ring Λ with c_0 inverted. We also have $R'_c = R_c \otimes_\Lambda \Lambda'$, which is just the result of inverting c_0 in R_c. If we write $c = C(x_1, x_2)$, we know from Section 2 that $R'_c = \Lambda'[\theta]/C(\theta, 1)$. We have $M'_{\mathcal{C}} = M_{\mathcal{C}} \otimes_\Lambda \Lambda'$ and $N'_{\mathcal{C}} = N_{\mathcal{C}} \otimes_\Lambda \Lambda'$.

We define an action of R'_c on $M'_{\mathcal{C}}$ and $N'_{\mathcal{C}}$ (which we still view as row vectors and column vectors respectively, just now with entries in Λ') by having θ act like $-\mathcal{C}_2 \mathcal{C}_1^{-1}$ (on the right) on the row vectors and $-\mathcal{C}_1^{-1}\mathcal{C}_2$ (on the left) on the column vectors. Since $\mathrm{Det}(\mathcal{C}_1 x_1 + \mathcal{C}_2 x_2) = c$, the matrices $-\mathcal{C}_2 \mathcal{C}_1^{-1}$ and $-\mathcal{C}_1^{-1}\mathcal{C}_2$ satisfy their (common) characteristic polynomial $C(t, 1)$. Thus we have given a well-defined action of $\Lambda'[\theta]/u(\theta, 1)$ on $M'_{\mathcal{C}}$ and $N'_{\mathcal{C}}$. This restricts to an action of R_c on $M'_{\mathcal{C}}$ and $N'_{\mathcal{C}}$, which we will now show descends to an action of R_c on $M_{\mathcal{C}}$ and $N_{\mathcal{C}}$.

Lemma 3.2. *For $1 \le k \le n - 1$, the entries of the matrix*

$$(1) \qquad c_0(-\mathcal{C}_1^{-1}\mathcal{C}_2)^k + c_1(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{k-1} + \cdots + c_{k-1}(-\mathcal{C}_1^{-1}\mathcal{C}_2),$$

which a priori are in $\mathbb{Q}(u_{i,j,k})$ (the fraction field of Λ), are actually in Λ.

Proof. Over the field $\mathbb{Q}(u_{i,j,k})$, since $-\mathcal{C}_1^{-1}\mathcal{C}_2$ satisfies its characteristic polynomial, we have

$$c_0(-\mathcal{C}_1^{-1}\mathcal{C}_2)^n + c_1(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{n-1} + \cdots + c_{k-1}(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{n-k+1}$$
$$+ c_k(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{n-k} + c_{k+1}(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{n-k-1} + \cdots + c_n(-\mathcal{C}_1^{-1}\mathcal{C}_2)^0 = 0.$$

Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are invertible over the field $\mathbb{Q}(u_{i,j,k})$, the last equation is equivalent to

$$(2) \quad c_0(-\mathcal{C}_1^{-1}\mathcal{C}_2)^k + c_1(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{k-1} + \cdots + c_{k-1}(-\mathcal{C}_1^{-1}\mathcal{C}_2)$$
$$= -\big(c_{k+1}(-\mathcal{C}_2^{-1}\mathcal{C}_1)^0 + \cdots + c_n(-\mathcal{C}_2^{-1}\mathcal{C}_1)^{n-k}\big).$$

If we view the matrix entries of both sides of equation (2) as reduced ratios of elements of the UFD Λ, the denominator of the left-hand side can only involve u_{1jk} and the denominator of the right-hand side can only involve u_{2jk}. Thus, the matrices

$$c_0(-\mathcal{C}_1^{-1}\mathcal{C}_2)^k + c_1(-\mathcal{C}_1^{-1}\mathcal{C}_2)^{k-1} + \cdots + c_{k-1}(-\mathcal{C}_1^{-1}\mathcal{C}_2)$$

must have all their entries in Λ. $\qquad \square$

By definition, the Λ-algebra R_c has a basis as a Λ-module given by $1, \zeta_1, \ldots, \zeta_{n-1}$, where $\zeta_k = c_0 \theta^k + \cdots + c_{k-1}\theta \in \Lambda'[\theta]/c(\theta, 1)$. Thus the action of ζ_k on $N'_\mathscr{C}$ is given by a matrix whose coefficients are in Λ, and so it restricts to an action on $N_\mathscr{C}$. An analogous argument can be made for $M_\mathscr{C}$. This construction is clearly equivariant for the G_Λ actions.

3.1.2. Balancing map. Now we will construct a map of R_c-modules $M_\mathscr{C} \otimes_{R_c} N_\mathscr{C} \to I_c$. The matrix $\mathscr{C}_1$ gives us an Λ-module pairing on $M_\mathscr{C}$ and $N_\mathscr{C}$ into Λ by $\alpha \star \beta = \alpha \mathscr{C}_1 \beta$ for $\alpha \in M_\mathscr{C}$ and $\beta \in N_\mathscr{C}$. In other words, the matrix $\mathscr{C}_1$ acts on $N_\mathscr{C}$ on the left and gives n Λ-module homomorphisms from $N_\mathscr{C}$ into Λ, one for each row of $\mathscr{C}_1$, and we map the ith basis element m_i of M to the Λ-module homomorphism of $N_\mathscr{C}$ into Λ given by the ith row of $\mathscr{C}_1$. This gives us a Λ-module map from $M_\mathscr{C}$ into $\mathrm{Hom}_\Lambda(N_\mathscr{C}, \Lambda)$ and we have $\mathrm{Hom}_\Lambda(N_\mathscr{C}, \Lambda) \cong \mathrm{Hom}_{R_c}(N, J_c)$, by Proposition 2.5. We use the symbol $\circ$ to denote the resulting pairing of $M_\mathscr{C}$ and $N_\mathscr{C}$ into J_c. Thus, $\check{\zeta}_{n-1}(\alpha \circ \beta) = \alpha \star \beta$. This pairing in J_c is clearly equivariant for the G_Λ actions.

Now we will show that $\circ$ gives a map of R_c-modules $M_\mathscr{C} \otimes_{R_c} N_\mathscr{C} \to I_c$. To see this, we extend our pairing $\circ$ to $M'_\mathscr{C} \otimes_{\Lambda'} N'_\mathscr{C} \to J'_c$, which is an R'_c module map for the R'_c action on $N_\mathscr{C}$. In fact, we can show that $\circ$ factors through $M'_\mathscr{C} \otimes_{R'_c} N'_\mathscr{C}$, i.e. $(\theta \alpha) \circ \beta = \alpha \circ (\theta \beta)$ for $\alpha \in M'_\mathscr{C}$ and $\beta \in N'_\mathscr{C}$. If we fix an α and let β vary over the elements of $N'_\mathscr{C}$, the expressions $(\theta \alpha) \circ \beta = \alpha \circ (\theta \beta)$ give two homomorphisms from $N'_\mathscr{C}$ into J'_c and we can check if they are the same by taking $\check{\zeta}_{n-1}$. Now,

$$\check{\zeta}_{n-1}((\theta \alpha) \circ \beta) = (\theta \alpha) \star \beta = \alpha(-\mathscr{C}_2 \mathscr{C}_1^{-1})\mathscr{C}_1 \beta$$

and

$$\check{\zeta}_{n-1}(\alpha \circ (\theta \beta)) = \alpha \mathscr{C}_1(-\mathscr{C}_1^{-1}\mathscr{C}_2)\beta,$$

so we see that $\circ$ gives a map of R'_c modules $M'_\mathscr{C} \otimes_{R'_c} N'_\mathscr{C} \to J'_c$ and thus our original $\circ$ is a map of R_c-modules $M_\mathscr{C} \otimes_{R_c} N_\mathscr{C} \to J_c$.

We have an inclusion of R_c modules $I_c \subset J_c$ (given in Section 2), and we will use Lemma 2.3 to see that for all $\alpha \in M_\mathscr{C}$ and $\beta \in N_\mathscr{C}$, the element $\alpha \circ \beta$ is in I_c. Fix an $\alpha \in M_\mathscr{C}$. Then by Lemma 2.3, $\alpha \circ N_\mathscr{C} \subset I_c$ if and only if $\check{\zeta}_{n-2}(\alpha \circ N_\mathscr{C}) \subset \Lambda$. By Corollary 2.2 we see that $\check{\zeta}_{n-2}(\alpha \circ N_\mathscr{C}) = \check{\zeta}_{n-1}(\alpha \circ (\theta N_\mathscr{C})) = \alpha \star (\theta N_\mathscr{C})$. However, we have seen that the pairing $\alpha \star (\theta \beta)$ is given by the matrix $-\mathscr{C}_2$, and thus $\check{\zeta}_{n-2}(\alpha \circ N_\mathscr{C}) \subset \Lambda$. Thus we have given a map of R_c-modules $M_\mathscr{C} \otimes_{R_c} N_\mathscr{C} \to I_c$. Note that we have defined $\circ$ so that if m_j and n_k are the chosen bases of $M_\mathscr{C}$ and $N_\mathscr{C}$ respectively,

$$(3) \qquad \check{\zeta}_{n-1}(m_j \circ n_k) = u_{1jk} \quad \text{and} \quad -\check{\zeta}_{n-2}(m_j \circ n_k) = u_{2jk},$$

which makes $M_\mathscr{C}$ and $N_\mathscr{C}$ a based balanced pair of modules for c.

3.1.3. Back to $\mathbb{Z}$. Now given $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$, to find the action of R_f on M, we take the matrix by which ζ_k acted on $M_\mathscr{C}$ above, and substitute $a_{i,j,k}$ for the $u_{i,j,k}$, and similarly for N. Of course, the conditions for this to be a ring action will be satisfied since they are satisfied formally. Also, we have a map of $\mathbb{Z}$-modules $M \otimes_\mathbb{Z} N \to I_f$ given by specializing the formulas from the last section, and we can see that this factors through a map of R_f-modules $M \otimes_{R_f} N \to I_f$ because the conditions for the factorization and for the map to respect the R_f-module structure are satisfied formally. Let $\psi(A) = (M, N)$.

3.2. Proof of Theorem 3.1 when $f_0 \neq 0$. Now we prove Theorem 3.1 by showing that ϕ and ψ are inverse constructions. Suppose we have $A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$. Let $\psi(A) = (M, N)$, and let $A' = \phi(M, N)$. By equation (3), we have that $A' = A$. Now, suppose we have (M, N), a based balanced pair of modules for f, and $\phi(M, N) = A$ and $\psi(A) = (M', N')$. We first check that the action of R_f is the same on M and M' (and N and N'), and then we will check that the balancing maps agree. We assume that $f_0 \neq 0$. In this case, we may invert f_0 as in Section 3.1, and obtain $\mathbb{Z}[\theta]/F(\theta, 1)$-modules M_{f_0} and N_{f_0}.

Proposition 3.3. *If we write elements of M_{f_0} as row vectors and elements of N_{f_0} as column vectors, then θ acts by $-A_2 A_1^{-1}$ on the right on M_{f_0} and θ acts by $-A_1^{-1} A_2$ on the left on N_{f_0}.*

Proof. We denote the map $M \otimes_{\mathbb{Z}} N \to I_f$ by $\circ$ and define $\alpha \star \beta$ to be $\check{\zeta}_{n-1}(\alpha \circ \beta)$. We fix a non-zero $\alpha \in M_{f_0}$ and suppose for the sake of contradiction that $\alpha \star N_{f_0} = 0$. Then $\alpha \circ N_{f_0} = 0$ by Proposition 2.5, and thus $\alpha A_1 = \alpha A_2 = 0$. Thus α is in the left kernel of $A_1 x_1 + A_2 x_2$ for formal x_i and so we obtain $f = \mathrm{Det}(A_1 x_1 + A_2 x_2) = 0$, a contradiction. Therefore, if $\alpha \circ N_{f_0} = 0$, then $\alpha = 0$. We have $(\theta \alpha) \star \beta = \check{\zeta}_{n-2}(\alpha \circ \beta)$ by Corollary 2.2 and $\check{\zeta}_{n-2}(\alpha \circ \beta) = \alpha(-A_2)\beta = (\alpha(-A_2 A_1^{-1}))A_1 \beta$. We conclude that $\theta \alpha = \alpha(-A_2 A_1^{-1})$. A similar argument can be made for N_{f_0}. $\square$

This proposition shows that the pairs of modules (M, N) and (M', N') have the same R_f action. We know that the map $\mathrm{Hom}_{R_f}(M \otimes_{R_f} N, I) \to \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} N, V)$ is injective (from Proposition 2.4), and thus since $\phi(M, N) = A$ and $\phi(M', N') = \phi(\psi(A)) = A$, we see that (M, N) and (M', N') have the same balancing map. Therefore, we have proven Theorem 3.1 when $f_0 \neq 0$. We will finish the proof at the beginning of the next section, by reducing to this case.

3.3. GL(V) invariance. Let $V = \mathbb{Z}^2$. Then $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{GL}(V)$ acts on tensors $V \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ and also on binary n-ic forms in $\mathrm{Sym}^n V$. The determinant map is equivariant for these actions. Let $g \in \mathrm{GL}(V)$, so that $g(A) = A'$ and $g(f) = f'$. Then we have isomorphisms $R_f \cong R'_f$, and $I_f \cong I'_f$ (e.g., by the geometric construction of [20, Section 2.3]). In fact, g also gives a map $V \to V$ such that the following diagram commutes:

$$
\begin{array}{ccc}
I_f & \xrightarrow{\ g\ } & I'_f \\
\downarrow & & \downarrow \\
V & \xrightarrow{\ g\ } & V.
\end{array}
$$

We can also understand the isomorphisms given above more concretely. Consider $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z})$. If $A = (A_1, A_2)$, then

$$g(A) = A' = (A'_1, A'_2) = (aA_1 + bA_2, cA_2 + dA_2).$$

If $f = F(x, y)$, then

$$g(f) = f' = F(ax + cy, bx + dy).$$

Write $f' = F'(x, y)$. If θ is a root of $F(x, 1)$, then $\theta' = \frac{d\theta - c}{-b\theta + a}$ is a root of $F'(x, 1)$. This induces the map $R'_f \cong R_f$. Note that $\theta = \frac{a\theta' + c}{b\theta' + d}$. We can view I_f and I'_f as fractional ideals

in the same $\mathbb{Q}$-algebra. They are given as fractional ideals of different $\mathbb{Q}$-algebras, Q_f and Q'_f respectively, in Section 2, but the map $\theta' \mapsto \frac{d\theta-c}{-b\theta+a}$ gives an isomorphism of those $\mathbb{Q}$-algebras. Then the map $I_f \cong I'_f$ is given by

$$I_f \to Q_f \cong Q'_f \xrightarrow{\times(b\theta'+d)^{n-3}} Qf' \supset I'_f.$$

Note that $\frac{1}{b\theta'+d} = \frac{a-b\theta}{ad-bc}$.

Theorem 3.4. *The construction ϕ given in the statement of Theorem* 3.1 *is* $\mathrm{GL}_2(\mathbb{Z})$-*equivariant under the identifications $R_f \cong R'_f$ and $I_f \cong I'_f$ and the map $g : V \to V$ given above (which give a $\mathrm{GL}_2(\mathbb{Z})$ action on balanced pairs) and with the natural action on tensors in $\mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$.*

Proof. This can easily be checked on a basis of $\mathrm{GL}_2(\mathbb{Z})$, or alternatively, it follows from the geometric versions of the constructions in Section 7. For example, if θ acts like $-A_2 A_1^{-1}$ then $\theta' = \frac{d\theta-c}{-b\theta+a}$ will act like

$$(-dA_2 A_1^{-1} - c)(bA_2 A_1^{-1} + a)^{-1} = -(A'_2)(A'_1)^{-1}.$$

Viewing $\check{\zeta}_{n-1}, \check{\zeta}_{n-2}$ and $\check{\zeta}'_{n-1}, \check{\zeta}'_{n-2}$ as maps of I_f and I'_f, respectively, we have that $I_f \cong I'_f$ induces

$$\check{\zeta}'_{n-1} \mapsto a\check{\zeta}_{n-1} - b\check{\zeta}_{n-2} \quad \text{and} \quad -\check{\zeta}'_{n-2} \mapsto \check{\zeta}_{n-1} - d\check{\zeta}_{n-2},$$

which exactly gives that our construction of (A_1, A_2) from $\check{\zeta}_{n-1}, -\check{\zeta}_{n-2}$ is equivariant. We can check this on a generating set of $\mathrm{GL}_2(\mathbb{Z})$, though it also follows from [20, Proposition 3.3]. If we write an element v of V as a column vector, then g acts on V by the standard left action. In the map from $I_f \to V$, an element $\alpha \in I_f$ maps to

$$\begin{bmatrix} \check{\zeta}'_{n-1}(\alpha) \\ -\check{\zeta}'_{n-2}(\alpha) \end{bmatrix}. \qquad \square$$

Thus, to check that the R_f action and balancing map on pairs M, N and M', N' agree, we can check after a $\mathrm{GL}_2(\mathbb{Z})$ action on f so that $f_0 \neq 0$ (as long as $f \neq 0$). This proves Theorem 3.1.

4. Symmetric tensors

In the map

$$\left\{ \begin{array}{l} \text{based balanced pairs } (M, N) \\ \text{of modules for } f \end{array} \right\} \longleftrightarrow \left\{ A \in \mathbb{Z}^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n \text{ with } \mathrm{Det}(A) = f \right\}.$$

of Theorem 3.1, it is easy to see from the construction that pairs in which M and N are the same based module exactly correspond to A such that A_1 and A_2 are symmetric matrices. For an $\mathbb{Z}$-module V, we use $\mathrm{Sym}_2 V$ to denote the submodule of $V \otimes V$ invariant under the natural involution.

Definition. A *self balanced module* for a non-zero form f is an R_f-module M, that is a free rank n $\mathbb{Z}$-module, and a map of R_f-modules $M \otimes_{R_f} M \to I_f$, such that when the composition $M \otimes_{\mathbb{Z}} M \to M \otimes_{R_f} M \to I_f \to V$ is written as a pair of matrices A_1 and A_2 (using a single choice of basis for M), we have $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$.

We easily conclude the following.

Theorem 4.1. *For every non-zero binary n-ic form f with coefficients in $\mathbb{Z}$, the map ϕ given in Theorem 3.1 induces a bijection*

$$\left\{ \begin{array}{l} \textit{isomorphism classes of self} \\ \textit{balanced modules M for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \mathrm{GL}_n(\mathbb{Z}) \textit{ classes of } A \in \mathbb{Z}^2 \otimes \mathrm{Sym}_2 \mathbb{Z}^n \\ \textit{with } \mathrm{Det}(A) = f \end{array} \right\},$$

where $\mathrm{Sym}_2 \mathbb{Z}^n$ are symmetric n by n matrices, and the action of $g \in \mathrm{GL}_n(\mathbb{Z})$ is by multiplication on the left by g and right by g^t.

5. Equivalent formulations of the balancing condition

In order to prove Theorems 1.1 and 1.3 in the Introduction, we will show that for primitive forms, modules that appear in balanced pairs have unique balance partners, and that for non-degenerate forms, modules that appear in balanced pairs can be realized as fractional ideals. First, we will see an equivalent formulation of the definition of balanced.

We define a *characteristic R_f-module M* to be an R_f-module M which is a free rank n $\mathbb{Z}$-module such that for any element $\zeta \in R_f$ the action of ζ on M viewed as a $\mathbb{Z}$-module has the same characteristic polynomial as the action of ζ on R_f (by multiplication) viewed as a $\mathbb{Z}$-module. Fractional ideals of R_f are characteristic modules of R_f.

Given two based $\mathbb{Z}$-modules M and N with bases α_i and β_i and an isomorphism $M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$, the *index $[N : M]$* is the absolute value of the determinant of the matrix Q with entries in $\mathbb{Q}$ such that

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \end{bmatrix} \cdot Q.$$

Now we will see that our condition of balanced is equivalent to M characteristic and an index condition on the map $M \otimes_{R_f} N \to I_f$.

Proposition 5.1. *Consider a non-zero binary form f over $\mathbb{Z}$, and two R_f modules M and N, with an R_f-module map $M \otimes_{R_f} N \to I_f$, such that M and N are both free rank n $\mathbb{Z}$-modules. Then this data gives a balanced pair of modules for f if and only if the map $M \to \mathrm{Hom}_{R_f}(N, I_f)$ is injective, and $[\mathrm{Hom}_{R_f}(N, J_f) : M] = [J_f : I_f]$ (with any inclusion of I_f in J_f as R_f-modules), and either M or N is characteristic.*

The equality of indexes does not depend on the choice of inclusion of I_f in J_f, because any two inclusions differ by multiplication by a non-zero-divisor in $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$. This multiplies both $[\mathrm{Hom}_{R_f}(N, J_f) : M]$ and $[J_f : I_f]$ by the absolute value of the norm of that element.

Proof. We can act by $\mathrm{GL}_2(\mathbb{Z})$ so as to assume $f_0 \neq 0$. Then, we use the inclusion of I_f in J_f given in Section 2 and see that $[J_f : I_f] = f_0$.

Lemma 5.2. *Suppose we have two R_f modules M and N, with a map $M \otimes_{R_f} N \to I_f$, such that M and N are both free rank n $\mathbb{Z}$-modules, either M or N is characteristic, the map $M \to \mathrm{Hom}_{R_f}(N, I_f)$ is injective, and $[\mathrm{Hom}_{R_f}(N, J_f) : M] = f_0$. Let A, as usual, denote the map $M \otimes_{\mathbb{Z}} N \to V$. We write elements of M as row vectors with entries in $\mathbb{Z}$. Then θ acts on $M' = M_{f_0}$ by $-A_2 A_1^{-1}$ on the right, and θ acts on $N' = N_{f_0}$ by $-A_1^{-1} A_2$ on the left. Also, $\mathrm{Det}(A_1 x_1 + A_2 x_2) = \pm f$.*

Proof. We define $m \star n$ to be $\check{\xi}_{n-1}(m \circ n)$. By Proposition 2.5, we see that $\star$ is faithful for both M and N if and only if $\circ$ is. We see that $\circ$ is faithful on M since the index of M in $\mathrm{Hom}_{R_f}(N, J_f)$ is not zero. If there were an $n \in N$ such that $M \circ n = 0$ for all n, then inverting f_0 we would find that $M_{f_0} \star n = 0$ but $M_{f_0} = \mathrm{Hom}_{\mathbb{Z}_{f_0}}(N_{f_0}, \mathbb{Z}_{f_0})$, and we obtain a contradiction since N_{f_0} is a free $\mathbb{Z}_{f_0}$-module and thus there is some homomorphism from N_{f_0} to $\mathbb{Z}_{f_0}$ which is non-zero on n. So, we fix an $\alpha \in M$ and let β vary in N. Then

$$\alpha\theta \star \beta = \check{\xi}_{n-2}(\alpha \circ \beta) = -\alpha A_2 \beta = -\alpha A_2 A_1^{-1} A_1 \beta = (-\alpha A_2 A_1^{-1}) \star \beta.$$

We similarly obtain the action of θ on N.

We have that $|\mathrm{Det}(A_1)|$ is the index of M in $\mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ or $\mathrm{Hom}_{R_f}(N, J_f)$, which is the index of I_f in J_f, i.e. $|f_0|$. Since M is characteristic, we know that θ and thus $-A_2 A_1^{-1}$ acts with characteristic polynomial $F(t, 1)/f_0$ on M_{f_0}. It follows that $\mathrm{Det}(A_1 x_1 + A_2 x_2) = \pm f$. We can argue similarly if N is characteristic. □

Now, suppose that M, N are balanced. Then we know that M, N are constructed from an element $A \in Z^2 \otimes \mathbb{Z}^n \otimes \mathbb{Z}^n$ such that $\mathrm{Det}(A) = f$. We can see from the construction of the action of θ on M_{f_0} that M is characteristic. We have a map $M \to \mathrm{Hom}_{R_f}(N, I_f)$, and composition with $I_f \subset J_f$ gives $M \to \mathrm{Hom}_{R_f}(N, J_f) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The map $M \to \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is given by A_1, and thus $[\mathrm{Hom}_{R_f}(N, J) : M] = |f_0|$, which implies $M \subset \mathrm{Hom}_{R_f}(N, J_f)$, and thus the map $M \to \mathrm{Hom}_{R_f}(N, I_f)$ is injective as well. □

Corollary 5.3. *For a non-zero form f, if N is a finitely generated invertible module for R_f, then there exists a unique balancing partner M for N.*

Proof. If N is a finitely generated invertible module, then N can be realized as an invertible fractional ideal of R_f (see [4, II.5.7, Proposition 12]). Then $\mathrm{Hom}_{R_f}(N, J_f) = N^{-1} J_f$ and $\mathrm{Hom}_{R_f}(N, I_f) = N^{-1} I_f$. In that case, $[\mathrm{Hom}_{R_f}(N, J_f) : \mathrm{Hom}_{R_f}(N, I_f)] = [J_f : I_f]$ and for M to be balanced with N it is necessary and sufficient that $M = \mathrm{Hom}_{R_f}(N, I_f)$. □

5.1. Non-degenerate forms. When f is a non-degenerate binary n-ic form over $\mathbb{Z}$, we have the following.

Proposition 5.4. *If f is a non-degenerate binary n-ic form over $\mathbb{Z}$ (i.e. $\mathrm{disc}(f) \neq 0$), then all characteristic modules can be realized as fractional ideals. This gives a bijection between isomorphism classes of characteristic R_f-modules and fractional ideal classes of R_f.*

Proof. We assume $f_0 \neq 0$ by an action of $\mathrm{GL}_2(\mathbb{Z})$. Then, we see that we can put the action of θ on a characteristic module M in rational normal form over $\mathbb{Q}$, and since it acts with the same separable characteristic polynomial as the action of θ on $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$, in rational

normal form these actions must be the same. Thus, we can view M as a $\mathbb{Z}$-submodule of $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$, or a fractional ideal. Clearly two fractional ideals in the same class give isomorphic modules. Moreover, a module homomorphism between two fractional ideals $I_1 \to I_2$ sends $q \in \mathbb{Q} \cap I_1$ to some element $k \in I_2$, and since the map is an R_f-module map, we see that it is multiplication by k/q. $\qquad\square$

For a fractional ideal M, let $|M|$ denote the norm of M, given by the index $[R_f : M]$, which can be defined even if M is not a submodule of R_f, since they sit in a common $\mathbb{Q}$-vector space. Then, we can reformulate the balancing condition in terms of norms. This is the version of balanced used in [1, 2].

Theorem 5.5. *For non-degenerate f, the map that takes a pair (M, N) of R_f-ideals to the same pair of R_f-modules with the map*

$$M \otimes_{R_f} N \to MN \to I_f$$

gives a bijection

$$\left\{ \begin{array}{l} \textit{classes of } (M, N) \textit{ where } M \textit{ and} \\ N \textit{ are fractional } R_f\textit{-ideals,} \\ MN \subset I_f \textit{ and } |M||N| = |I_f| \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of balanced} \\ \textit{pairs } (M, N) \textit{ of modules for } f \end{array} \right\},$$

where (M, N) and (M_1, N_1) are in the same class if $M_1 = \lambda M$ and $N_1 = \lambda^{-1} N$ for some invertible element $\lambda \in R_f \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. All modules that appear in balancing pairs are characteristic by Proposition 5.1, and thus can be realized as fractional ideals. For a balanced pair M, N of modules, we can take any fractional ideal representative of M. For now we choose any fractional ideal representative of N, but we will modify this choice later so that the map $M \otimes N \to I_f$ is just given by $M \otimes N \to MN \subset R_f \otimes_{\mathbb{Z}} \mathbb{Q}$ with image landing in R_f. The map $M \otimes N \to I_f$ factors through MN.

We now argue that $\tau : MN \to I_f$ is injective. As usual, we assume $f_0 \neq 0$ by a $\mathrm{GL}_2(\mathbb{Z})$ action if necessary. We can detect the injectivity after tensoring with $\mathbb{Q}$ because $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. Over $\mathbb{Q}$ we have that MN is at least rank n because it contains N and thus is rank n. We can take θ^k as a basis of MN, and we see where they map to in $I_f = J_f = \mathrm{Hom}(R, \mathbb{Q})$. Then $\tau(\theta^k)$ is the map in $\mathrm{Hom}(R, \mathbb{Q})$ that sends ζ_i to $\check{\zeta}_{n-1}(\zeta_i \theta^k)$. By Proposition 2.1, we have

$$\check{\zeta}_{n-1}(\zeta_i \theta^k) = \begin{cases} 1 & \text{if } k + i = n - 1 \text{ and } i > 0, \\ 1/f_0 & \text{if } k = n - 1 \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we see that $\tau(MN) = \mathrm{Hom}(R, \mathbb{Q})$, when working over $\mathbb{Q}$, and therefore over $\mathbb{Z}$ we have that $MN \to I_f$ is injective.

A map $MN \to I_f$ is just multiplication by some element of $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$. The element is not a zero-divisor since $MN \to I_f$ is injective, and thus it is invertible in $R_f \otimes_{\mathbb{Z}} \mathbb{Q}$. We can choose that element to be 1 by taking a different representative for N in its ideal class. If we had chosen a different representative for M, this would change the class of (M, N).

Suppose we have a balanced pair (M, N) realized as ideal classes with $MN \subset I$. We will show that the index condition for balanced is equivalent to the norm condition in the above theorem.

Proposition 5.6. *Let f be a non-zero form. If M and N are fractional ideals of R_f with $MN \subset I_f$, then $[\operatorname{Hom}_{R_f}(N, J_f) : M] = [J_f : I_f]$ if and only if $|M||N| = |I_f|$.*

Proof. We can act by $\operatorname{GL}_2(\mathbb{Z})$ so as to assume $f_0 \neq 0$. We claim that $|M||N|$ is the product of $|J_f|$ with the determinant of the pairing $\zeta_{n-1}(mn)$. When $M = R_f$ and $N = J_f$, we see from Proposition 2.1 that the determinant of the pairing is 1, and thus the claim is true. If we change $\mathbb{Q}$-bases from R_f, J_f to M, N, we change the determinant of the pairing by $N(M)N(N)/N(J_f)$ and thus the determinant of the pairing $\zeta_{n-1}(mn)$ is $|M||N|/|J_f|$.

The index of M in $\operatorname{Hom}_{R_f}(N, J_f)$ is just the index of M in $\operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, which is giving by the pairing $\zeta_{n-1}(mn)$. Thus $[\operatorname{Hom}_{R_f}(N, J_f) : M] = |M||N|/|J_f|$. We see that $[\operatorname{Hom}_{R_f}(N, J_f) : M] = [J_f : I_f]$ if and only if $|M||N| = |I_f|$. $\qquad\square$

Theorem 5.5 now follows from the above proposition. $\qquad\square$

For symmetric tensors, we can make a similar argument to prove the following.

Theorem 5.7. *For non-degenerate f, the map that takes a pair (M, k) to the R_f-module M and the map*

$$M \otimes_{R_f} M \xrightarrow{k^{-1}} k^{-1}M^2 \to I_f$$

give a bijection

$$\left\{ \begin{array}{l} \text{classes of } (M, k) \text{ where } M \text{ is a} \\ \text{fractional } R_f\text{-ideal, } k \text{ is an invertible} \\ \text{element of } R_f \otimes_{\mathbb{Z}} \mathbb{Q}, \text{ and } M^2 k \subset I_f \\ \text{and } |M|^2|(k)| = |I_f| \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of self} \\ \text{balanced modules } M \text{ for } \pm f \end{array} \right\},$$

where (M, k) and (M_1, k_1) are in the same class if $M_1 = \lambda M$ and $k_1 = \lambda^{-2}k$ for some invertible element $\lambda \in R_f \otimes_{\mathbb{Z}} \mathbb{Q}$.

5.2. Primitive forms. If f is primitive, then I_f and J_f are invertible R_f modules [20, Proposition 2.1]. For general non-zero forms, we saw in Corollary 5.3 that invertible ideals have unique balancing partners. For primitive f we have the following.

Proposition 5.8. *If f is a primitive non-degenerate form, and N is a characteristic R_f-module, then $\operatorname{Hom}_{R_f}(N, I_f)$ is the unique balancing partner for N (i.e. an R_f-module M and map $M \otimes_{R_f} N \to I_f$ that gives a balanced pair).*

Proof. In this case, we see that $[\operatorname{Hom}_{R_f}(N, J_f) : \operatorname{Hom}_{R_f}(N, I_f)] = [J_f : I_f]$. This is because $\operatorname{Hom}_{R_f}(N, J_f)$ and $\operatorname{Hom}_{R_f}(N, I_f)$ are naturally realized as fractional R_f ideals. Then we see that

$$\operatorname{Hom}_{R_f}(N, J_f)J_f^{-1}I_f \subset \operatorname{Hom}_{R_f}(N, I_f), \quad \operatorname{Hom}_{R_f}(N, I_f)I_f^{-1}J_f \subset \operatorname{Hom}_{R_f}(N, J_f).$$

Thus $\mathrm{Hom}_{R_f}(N, J_f) = \mathrm{Hom}_{R_f}(N, I_f)I_f^{-1}J_f$, and $[\mathrm{Hom}_{R_f}(N, J_f): \mathrm{Hom}_{R_f}(N, I_f)]$ is the norm of $J_f I_f^{-1}$, which is $[J_f : I_f]$. Then, for M to be balanced with N it is necessary and sufficient that $M = \mathrm{Hom}_{R_f}(N, I_f)$. $\qquad\square$

Theorem 1.1 now follows from Propositions 5.8 and 5.4 and Theorem 3.1. We can also apply Proposition 5.8 to symmetric tensors.

Theorem 5.9. *For non-degenerate primitive f, the map that takes a pair (M, k) to the R_f-module M and the map*

$$M \otimes_{R_f} M \xrightarrow{k^{-1}} k^{-1}M^2 \to I_f$$

give a bijection

$$\left\{ \begin{array}{l} \textit{classes of } (M, k) \textit{ where } M \textit{ is a} \\ \textit{fractional } R_f\textit{-ideal, } k \textit{ is an invertible} \\ \textit{element of } R_f \otimes_{\mathbb{Z}} \mathbb{Q}, \textit{ and} \\ M = (I_f k : M) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of self} \\ \textit{balanced modules } M \textit{ for } \pm f \end{array} \right\},$$

where (M, k) and (M_1, k_1) are in the same class if $M_1 = \lambda M$ and $k_1 = \lambda^2 k$ for some invertible element $\lambda \in R_f \otimes_{\mathbb{Z}} \mathbb{Q}$, and $(I_f k : M)$ is the fractional ideal of elements x such that $xM \subset I_f k$.

6. Main theorem over an arbitrary base

The proof of Theorem 3.1 works over an arbitrary base with some modifications. Let S be a scheme. For a sheaf U on S, we use $f \in U$ to denote that f is a global section of U. We consider binary n-ic forms with coefficients in $\mathcal{O}_S$, i.e. $f_0 x_1^n + f_1 x_1^{n-1} x_2 + \cdots + f_n x_2^n$ with $f_i \in \mathcal{O}_S$. We say such a form is a *zero-divisor* if it is a zero-divisor in the $\mathcal{O}_S$-algebra $\mathcal{O}_S[x, y]$, which means that for some open $\mathcal{U}$ of S, that f is a zero-divisor in $\mathcal{O}_S[x, y](\mathcal{U})$. We can construct an $\mathcal{O}_S$-module R_f and an R_f-module I_f by using the construction of Section 2 for the universal form over $\mathbb{Z}[f_0, \ldots, f_n]$ and then pulling back to S with the map given by our desired form. From this construction, we inherit a map of $\mathcal{O}_S$-modules $\delta : I_f \to V = \mathcal{O}_S^2$.

Definition. A *based balanced pair of modules* for f is a pair of R_f-modules M and N, a choice of basis $M \cong \mathcal{O}_S^n$ and $N \cong \mathcal{O}_S^n$, and a map of R_f-modules $M \otimes_{R_f} N \to I_f$, such that when the composition $M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \to V$ is written as a pair of matrices A_1 and A_2, we have $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$. A *balanced free pair of modules* for f is a pair of R_f-modules M and N, each a free rank n $\mathcal{O}_S$-module, and a map of R_f-modules $M \otimes_{R_f} N \to I_f$, such that when the composition $M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \to V$ is written as a pair of matrices A_1 and A_2, we have $\mathrm{Det}(A) = fu$, where u is a unit in $\mathcal{O}_S$. Given a balanced pair of modules for a non-zero-divisor form f, there is a unique choice of generator χ of $\wedge^n M \otimes \wedge^n N$ such that $\mathrm{Det}(A_1 x_1 + A_2 x_2) = f$ when constructing A with bases of M and N that give $\chi \in \wedge^n M \otimes \wedge^n N$, we obtain $\mathrm{Det}(A) = f$.

Theorem 6.1. *For every non-zero-divisor binary n-ic form f with coefficients in $\mathcal{O}_S$, the map that sends a balanced pair (M, N) to the tensor $\phi(M, N) \in \mathcal{O}_S^2 \otimes \mathcal{O}_S^n \otimes \mathcal{O}_S^n$ associated to the map*

$$M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \to V = \mathcal{O}_S^2$$

gives a bijection

$$\left\{ \begin{array}{l} \text{based balanced pairs } (M, N) \\ \text{of modules for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} A \in \mathcal{O}_S^2 \otimes \mathcal{O}_S^n \otimes \mathcal{O}_S^n \\ \text{with } \mathrm{Det}(A) = f \end{array} \right\}.$$

We define G to be the subgroup of $\mathrm{GL}_n(\mathcal{O}_S) \times \mathrm{GL}_n(\mathcal{O}_S)$ of elements (g_1, g_2) such that $\mathrm{Det}(g_1) \mathrm{Det}(g_2) = 1$. Then, G acts equivariantly in the above bijection (acting of the bases of M and N), and we obtain a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of balanced} \\ \text{free pairs } (M, N) \text{ of modules for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} G\text{-classes of } A \in \mathcal{O}_S^2 \otimes \mathcal{O}_S^n \otimes \mathcal{O}_S^n \\ \text{with } \mathrm{Det}(A) = f \end{array} \right\}.$$

Proof. To construct a based balanced pair of modules from $A \in \mathcal{O}_S^2 \otimes \mathcal{O}_S^n \otimes \mathcal{O}_S^n$, we can simply pullback the construction from the universal tensor (and we call this construction ψ). Again, the balancing map composed with $I_f \to V$ gives the construction ϕ of an element of $\mathcal{O}_S^2 \otimes \mathcal{O}_S^n \otimes \mathcal{O}_S^n$ from a based balanced pair. Now, suppose we have (M, N), a based balanced pair of modules for f, and $\phi(M, N) = A$ and $\psi(A) = (M', N')$. We need to check that the action of R_f is the same on M and M' (and N and N'), and that the balancing maps agree. It suffices to check this everywhere locally over S, and so we can assume that S is affine, and $S = \mathrm{Spec}\, B$. Then, if suffices to check in a larger ring, so we let E be the ring obtained from inverting all the non-zero-divisors in $B[x, y]$.

We have that $B[x, y] \subset E$. We see that x is not a zero-divisor in $B[x, y]$, because $xg = 0$ implies that the leading coefficient of g is 0. We consider $G(t_1, t_2) = F(xt_1, yt_1 + \frac{1}{x}t_2)$. This is a binary n-ic form in variables t_i with coefficients in E. Over E we see it is a $\mathrm{GL}_2(E)$ transformation of f. We have that $G(1, 0) = F(x, y)$, and thus f is the leading coefficient of the new form. However, f has an inverse in E and thus is not a zero-divisor. By the GL_2 invariance of our constructions, we can reduce to checking in the case where f_0 is not a zero-divisor. In this case we can prove Proposition 3.3 just as in the case of $\mathbb{Z}$. $\qquad\square$

In fact, we can consider a completely general binary n-ic form over S given by a locally free rank 2 $\mathcal{O}_S$-module V, a locally free rank 1 $\mathcal{O}_S$-module L, and a global section $f \in \mathrm{Sym}^n V \otimes L$. We say a form f is a *zero-divisor* if it is a zero-divisor on any open $\mathcal{U}$ of S on which V and L are free (and in this case the notion of zero-divisor is defined above). When f is not a zero-divisor, we have an associated $\mathcal{O}_S$-algebra $R_f := \pi_*(\mathcal{O}_{T_f})$, where T_f is the subscheme of $\mathbb{P}(V)$ cut out by f, and $\pi : T_f \to S$. We have line bundles $\mathcal{O}_{T_f}(k)$ on T_f pulled back from $\mathcal{O}(k)$ on $\mathbb{P}(V)$, and we define

$$I_f := \pi_*(\mathcal{O}_{T_f}(n - 3)) \otimes (\wedge^2 V)^{\otimes 2} \otimes L,$$

which has an action of R_f through the first factor. From this construction, we have a natural map $\delta : I_f \to V^* \otimes \wedge^2 V \cong V$ as in [20, Equations (3.8), (3.9)]. Locally on S where V and L are free, the constructions of R_f and I_f pullback from the constructions of R_f and I_f for the universal form over $\mathbb{Z}[f_0, \ldots, f_n]$ as given in Section 2 and used at the start of this section. See [20, Section 3] for more details of these constructions.

We now consider $A \in V \otimes U \otimes W$, where U and W are locally free rank n $\mathcal{O}_S$-modules with an orientation isomorphism $\wedge^n U \otimes \wedge^n W \cong L$. An isomorphism between $A \in V \otimes U \otimes W$ and $A' \in V \otimes U' \otimes W'$ is given by isomorphisms $U \cong U'$ and $W \cong W'$ that take A to A' and respect the orientations. We have the determinant

$$\mathrm{Det}(A) \in \mathrm{Sym}^n V \otimes \wedge^n U \otimes \wedge^n V \cong \mathrm{Sym}^n V \otimes L$$

(see [21, Section 8.2] for the details of this kind of construction). Given a scheme S and a locally free $\mathcal{O}_S$-module U, we let U^* denote the $\mathcal{O}_S$-module $\mathcal{H}\mathrm{om}_{\mathcal{O}_S}(U, \mathcal{O}_S)$, even if U is also a module for another sheaf of algebras.

Definition. A *balanced pair of modules* for a non-zero-divisor f is a pair of R_f-modules M and N, each a locally free rank n $\mathcal{O}_S$-module such that $\wedge^n M \otimes \wedge^n N \cong L^*$, and additionally a map of R_f-modules $M \otimes_{R_f} N \to I_f$, such that when the composition $M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \to V$ is written as $A \in M^* \otimes N^* \otimes V$ the image of $\mathrm{Det}(A)$ in $\mathrm{Sym}^n V \otimes L$ (via an isomorphism $\wedge^n M^* \otimes \wedge^n N^* \cong L$) is fu, where u is a unit in $\mathcal{O}_S$. When f is a non-zero-divisor, given that $\wedge^n M \otimes \wedge^n N \cong L^*$, there is a unique choice of isomorphism so that the image of $\mathrm{Det}(A)$ is f.

Theorem 6.2. *For every non-zero-divisor binary n-ic form $f \in \mathrm{Sym}^n V \otimes L$, the map that sends (M, N) to the tensor in $A \in M^* \otimes N^* \otimes V$ associated to the map*

$$M \otimes_{\mathcal{O}_S} N \to M \otimes_{R_f} N \to I_f \to V$$

gives a bijection

$$\left\{ \begin{array}{l} \textit{isomorphism classes of} \\ \textit{balanced pairs } (M, N) \textit{ of} \\ \textit{modules for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \textit{isomorphism classes of } A \in V \otimes U \otimes W, \\ \textit{where } U \textit{ and } W \textit{ are locally free rank } n \\ \mathcal{O}_S\textit{-modules with an isomorphism} \\ \wedge^n U \otimes \wedge^n W \cong L \textit{ such that } \mathrm{Det}(A) = f \end{array} \right\}.$$

Proof. From A, we can construct R_f modules from U^* and W^* by giving the R_f action locally where U and W are free and we can choose bases, and then seeing that it is invariant under change of basis by elements of $\mathrm{GL}_n \times \mathrm{GL}_n$ that preserve the orientation. Similarly, we can construct the balancing map. Again, the construction of A from a balanced pair of modules just combines the balancing map with $I_f \to V$. To see that these constructions are inverse, it suffices to check locally on S, where we can assume V, U, and W are free.   □

As when working over $\mathbb{Z}$, we can also get a version on the theorem for symmetric tensors. Recall, that for an $\mathcal{O}_S$-module V, we use $\mathrm{Sym}_2 V$ to denote the submodule of $V \otimes V$ invariant under the natural involution.

Definition. A *self balanced module* for a non-zero-divisor f is an R_f-module M, that is a locally free rank n $\mathcal{O}_S$-module and such that $(\wedge^n M)^{\otimes 2}$ is isomorphic to L^*, and additionally a map of R_f-modules $M \otimes_{R_f} M \to I_f$, such that when the composition $M \otimes_{\mathcal{O}_S} M \to M \otimes_{R_f} N \to I_f \to V$ is written as $A \in M^* \otimes M^* \otimes V$ we have $\mathrm{Det}(A) = fu$, where u is a unit in $\mathcal{O}_S$.

Theorem 6.3. *For every non-zero-divisor binary n-ic form $f \in \operatorname{Sym}^n V \otimes L$, the map that sends M to the tensor in $A \in M^* \otimes M^* \otimes V$ associated to the map*

$$M \otimes_{\mathcal{O}_S} M \to M \otimes_{R_f} M \to I_f \to V$$

gives a bijection

$$\left\{ \begin{array}{l} \text{isomorphism classes of self} \\ \text{balanced modules } M \text{ for } f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of } A \in V \otimes \operatorname{Sym}_2 U, \\ \text{where } U \text{ is a locally free rank } n \text{ } \mathcal{O}_S\text{-module} \\ \text{with an orientation isomorphism} \\ (\wedge^n U)^{\otimes 2} \cong L, \text{ and such that } \operatorname{Det}(A) = f \end{array} \right\} .$$

7. Geometric construction of modules from the universal tensor

Just as we have given both concrete and geometric constructions of R_f, I_f, and J_f, in this section we will give geometric constructions of the R_f-modules M and N that were constructed explicitly above for the universal tensor.

Notation. Let B be a ring. When we have a matrix $M \in B^m \otimes B^n$, we can multiply M by vectors in two ways. We can multiply M by a length m vector on the left, and we can multiply M by a length n column vector on the right. When we have an element $A \in B^\ell \otimes B^m \otimes B^n$, we can multiply it by vectors in three different ways, and we realize that the "on the left" and "on the right" descriptions do not generalize appropriately for three dimensional tensors. We will need a new language. An element $A \in B^\ell \otimes B^m \otimes B^n$ is comprised of entries a_{ijk}, with $1 \le i \le \ell$, $1 \le j \le m$, and $1 \le k \le n$. We say that a_{ijk} is the entry in the ith *aisle*, jth *row*, and kth *column*. Note that aisle, row, and column denote two dimensional submatrices, i.e. codimension one slices of A. For $A \in B^2 \otimes B^n \otimes B^n$, the n by n matrix we called A_i above is the ith aisle of A.

If we have a sequence $x_1, \ldots, x_\ell$, we can form it into a vector and combine it with $A \in B^\ell \otimes B^m \otimes B^n$ to get the m by n matrix we call $A(x, \cdot, \cdot)$ with j, k entry $\sum_i a_{ijk} x_i$. The dots indicate that we have not also multiplied by vectors in the other situations. For example, for $A \in B^2 \otimes B^n \otimes B^n$, the matrix $A(x, \cdot, \cdot)$ is what we have previously referred to as $A_1 x_1 + A_2 x_2$. Similarly, if we have a sequence $y_1, \ldots, y_m$, we can form a $2 \times n$ array $A(\cdot, y, \cdot)$ with i, k entry $\sum_j a_{ijk} y_j$. We could call this array a matrix, but it is more convenient to continue to refer to its aisles and columns. If we have a sequence $z_1, \ldots, z_n$, we can form a $2 \times m$ array $A(\cdot, \cdot, z)$ with i, j entry (in the ith aisle and jth row) $\sum_\ell a_{ijk} z_k$. In fact, we will always use an x variable in the first place, y in the second place, and a z in the third place, and thus we will use the short hand $A(x)$ for $A(x, \cdot, \cdot)$ and $A(y)$ for $A(\cdot, y, \cdot)$. We may refer to the j, k entry of $A(x)$ by $A(x)_{j,k}$ and the i, k entry of $A(y)$ by $A(y)_{i,k}$. We can also multiply A by more than one vector at a time. For example, $A(x, y, \cdot)$ (denoted by $A(x, y)$ for short) is a length n vector with ℓth entry $\sum_i \sum_j a_{ijk} x_i y_j$.

Given a 2-dimensional array A with entries in some ring, we can form the ideal $\mathcal{M}(A)$ of the determinants of its maximal minors. For example, for $A \in B^2 \otimes B^n \otimes B^n$, we have that $\mathcal{M}(A(x)) = (\operatorname{Det}(A_1 x_1 + A_2 x_2))$. We have previously called the subscheme of $\mathbb{P}^1_B$ defined by this ideal $T_{\operatorname{Det}(A_1 x_1 + A_2 x_2)}$. Now, in order to emphasize certain symmetries, we say that $\mathcal{M}(A(x))$ defines a subscheme $T_{A(x)} \subset \mathbb{P}^1_B$. Analogously, we have a subscheme $T_{A(y)} \subset \mathbb{P}^{n-1}_B$ cut out by the determinants of 2×2 minors of $A(y)$.

The scheme $T_{A(y)}$ has line bundles $\mathcal{O}_{T_{A(y)}}(k)$ pulled back from $\mathcal{O}(k)$ on $\mathbb{P}_B^{n-1}$. Heuristically, we would like to say that $R_{\mathrm{Det}(A)}$ is the B-algebra of global functions of $T_{A(y)}$ and then we would have an $R_{\mathrm{Det}(A)}$-module $\Gamma(\mathcal{O}_{T_{A(y)}}(1))$. However, we have already defined $R_{\mathrm{Det}(A)}$ as the B-algebra of global functions of $T_{A(x)}$ which is not always isomorphic to $\Gamma(\mathcal{O}_{T_{A(y)}})$. Furthermore, $\Gamma(\mathcal{O}_{T_{A(y)}})$ is not always a free rank n B-algebra, even when $\mathrm{Det}(A)$ is a non-zero-divisor. We can, however, use this geometric construction of modules for sufficiently general $2 \times n \times n$ tensors, including for the universal tensor of these dimensions. As in Section 3.1, we work over the ring $\Lambda = \mathbb{Z}[u_{ijk}]$ and with the universal tensor $\mathcal{C}$ in $\Lambda^2 \otimes_\Lambda \Lambda^n \otimes_\Lambda \Lambda^n$ with i, j, k entry $u_{i,j,k}$. We have a binary n-ic form $c = \mathrm{Det}(\mathcal{C}(x))$ with coefficients in Λ.

Theorem 7.1. *The Λ-algebra $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}})$ is isomorphic to R_c.*

Proof. Recall that $R_c = \Gamma(\mathcal{O}_{T_{\mathcal{C}(x)}})$ by definition. We would like to show that we have an isomorphism of Λ-algebras $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}) \cong \Gamma(\mathcal{O}_{T_{\mathcal{C}(x)}})$. We might expect that this would follow because we had an isomorphism of Λ-schemes $T_{\mathcal{C}(y)} \cong T_{\mathcal{C}(x)}$, but that is not the case. However, restricted to a large open subscheme S' (defined below) of $\mathrm{Spec}\,\Lambda$, we do get such an isomorphism. Let $T'_{\mathcal{C}(y)} = T_{\mathcal{C}(y)} \otimes_S S'$, which is an open subscheme of $T_{\mathcal{C}(y)}$. Similarly, let $T'_{\mathcal{C}(x)} = T_{\mathcal{C}(x)} \otimes_S S'$, which is an open subscheme of $T_{\mathcal{C}(x)}$. In Lemma 7.6, we give an explicit isomorphism of S'-schemes $T'_{\mathcal{C}(y)} \cong T'_{\mathcal{C}(x)}$.

We will now define a subscheme Z of $S = \mathrm{Spec}\,\Lambda$ to correspond to tensors that are very degenerate. Thus we can think of the tensor $\mathcal{C} \otimes_S S'$ over S' as the universal "not too degenerate" tensor. The $(n-1)$-minors of $\mathcal{C}(x)$ form a matrix $W(\{x_1^{n-1}, x_1^{n-2}x_2, \ldots, x_2^{n-1}\}, \cdot, \cdot)$. The i, j, k entry of $W \in \Lambda^n \otimes \Lambda^n \otimes \Lambda^n$ is $(-1)^{j+k}$ times the $x_1^{n-i}x_2^{i-1}$ coefficient of the determinant of the submatrix of $\mathcal{C}(x)$ obtained by deleting the jth row and kth column. We can form $\mathrm{Det}(W(y))$, a degree n polynomial in the y_i, and form the ideal $\mathfrak{w}$ of Λ of its coefficients, with d_i the coefficient of y_i^n. If we do the analogous construction, starting with $\mathcal{C}(y)$, we see that the next-to-maximal minors of $\mathcal{C}(y)$ are the entries themselves. Then we can form $\mathrm{Det}(\mathcal{C}(x)) = c$, and form the ideal $(c_0, \ldots, c_n)$ of its coefficients. Let Z be the subscheme of S defined by the ideal $(c_0, \ldots, c_n)\mathfrak{w}$. Let S' be the open subscheme $S \setminus Z$ of S.

We will prove the theorem with the following lemmas.

Lemma 7.2. *The codimension of Z in S is at least 2.*

Proof. Suppose for the sake of contradiction that Z is codimension 1. Then either the subscheme of S defined by $(c_0, \ldots, c_n)$ or the subscheme defined by $\mathfrak{w}$ must be codimension 1 and thus given by a principal ideal. However, we note that c_0 and c_n are expressions in disjoint sets of the u_{ijk}. In fact, c_0 only involves the u_{1jk} and c_n the u_{2jk}. Thus c_0 and c_n have no nontrivial divisor in the UFD Λ, and the subscheme cut out by $(c_0, \ldots, c_n)$ cannot be codimension 1. Similarly, d_j does not involve any u_{**j}. Thus a common divisor of $d_1, \ldots, d_n$ must be trivial, and so the subscheme cut out by $(d_1, \ldots, d_n)$ cannot be codimension 1. We conclude the subscheme cut out by $\mathfrak{w}$, which is contained in the subscheme cut out by $(d_1, \ldots, d_n)$, cannot be codimension 1. $\qquad\square$

Lemma 7.3. *We have that the restriction map $\Gamma(\mathcal{O}_S) \to \Gamma(\mathcal{O}_{S'})$ is an isomorphism and thus $\Gamma(\mathcal{O}_{S'})$ is naturally isomorphic to Λ.*

Proof. This follows because Z is codimension at least 2 (e.g., by [13, Chapter 4, Theorem 1.14]). □

From this lemma we conclude that $\Gamma(\mathcal{O}_{T'_{\mathcal{C}(x)}})$ and $\Gamma(\mathcal{O}_{T'_{\mathcal{C}(y)}})$ are Λ-algebras.

Lemma 7.4. *The restriction map $\Gamma(\mathcal{O}_{T_{\mathcal{C}(x)}}) \to \Gamma(\mathcal{O}_{T'_{\mathcal{C}(x)}})$ is an isomorphism of Λ-algebras.*

Proof. Let $\pi : T_{\mathcal{C}(x)} \to S$. We will see in Theorem 8.4 that the $\mathcal{O}_S$-module $\pi_*(\mathcal{O}_{T_{\mathcal{C}(x)}})$ is locally free on S. We will show that $\pi_*(\mathcal{O}_{T_{\mathcal{C}(x)}})$ is isomorphic to the pushforward of $\pi_*(\mathcal{O}_{T'_{\mathcal{C}(x)}})$ to S, and then taking global sections will prove the lemma. We cover S with opens $\mathcal{U}_i$ that trivialize $\pi_*(\mathcal{O}_{T_{\mathcal{C}(x)}})$. Since S is irreducible, $\mathcal{U}_i$ is the same dimension as S. On each $\mathcal{U}_i$, we then have that $\mathcal{U}_i \cap Z$ is at least codimension 2 in $\mathcal{U}_i$. Thus, on $\mathcal{U}_i$, the sheaf $\pi_*(\mathcal{O}_{T_{\mathcal{C}(x)}})$ is isomorphic to $\mathcal{O}_S^{\oplus n}$, for which the restriction map to S' is an isomorphism by Lemma 7.3. This means that restricted to $\mathcal{U}_i$, we have that $\pi_*(\mathcal{O}_{T_{\mathcal{C}(x)}})$ is isomorphic to the pushforward of $\pi_*(\mathcal{O}_{T'_{\mathcal{C}(x)}})$ to S. □

Lemma 7.5. *The restriction map $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}) \to \Gamma(\mathcal{O}_{T'_{\mathcal{C}(y)}})$ is an isomorphism of Λ-algebras.*

Proof. The sheaf $\pi_*(\mathcal{O}_{T_{\mathcal{C}(y)}})$ is locally free on S by Theorem 8.4. We then use the same argument as in Lemma 7.4. □

Lemma 7.6. *We have an isomorphism of S'-schemes $T'_{\mathcal{C}(y)} \cong T'_{\mathcal{C}(x)}$.*

Proof. First we will give the main idea of the proof. The idea is that we can define a correspondence between points of $T'_{\mathcal{C}(y)}$ and $T'_{\mathcal{C}(x)}$ by $\mathcal{C}(x, y) = 0$. For points in $T'_{\mathcal{C}(x)}$, we have $\mathrm{Det}(\mathcal{C}(x)) = 0$, and thus there should be some values of y_j such that $\mathcal{C}(x, y) = 0$. For these y_j, we have that $\mathcal{C}(y)$ sends a non-trivial vector x to 0, and thus is rank 1. Conversely, for points in $T'_{\mathcal{C}(y)}$, we have that $\mathcal{C}(y)$ is rank 1, and thus should send a non-trivial vector x to zero, and $\mathcal{C}(x, y) = 0$ implies that $\mathcal{C}(x)$ has determinant zero. The first difficulty in making this idea rigorous is that the correspondence is only a bijection when $\mathcal{C}$ is sufficiently non-degenerate, which is why we have had to restrict to the base S'. Over S', we could prove that the tensors are sufficiently non-degenerate to give a bijection of field valued points of $T'_{\mathcal{C}(y)}$ and $T'_{\mathcal{C}(x)}$. Below, we must work a bit harder to prove an isomorphism of schemes.

First we give a map $T'_{\mathcal{C}(y)} \to T'_{\mathcal{C}(x)}$. We will give maps from open sets of $T'_{\mathcal{C}(y)}$ to $\mathbb{P}^1_{S'}$. We will then show that the open sets cover $T'_{\mathcal{C}(y)}$. We will then show that these maps agree on overlaps, and finally we will show that the image lands in $T'_{\mathcal{C}(x)}$. Given an $1 \leq i \leq n$, we map $T'_{\mathcal{C}(y)}$ to $\mathbb{P}^1_{S'}$ via $x_1 = -\sum_j u_{2jk} y_j$ and $x_2 = \sum_j u_{1jk} y_j$, which we can do on the open set $E_k \subset T'_{\mathcal{C}(y)}$ defined as the complement of the ideal $(\sum_j u_{1jk} y_j, \sum_j u_{2jk} y_j)$. Note that $\sum_j u_{ijk} y_j$ is the i, k entry of $\mathcal{C}(y)$, or of the next-to-maximal minor of $\mathcal{C}(y)$. Suppose there was a point of $T'_{\mathcal{C}(y)}$ not in any E_k. If we write y for the vector of the y_j, then at this point we have $\mathcal{C}(y) = 0$, i.e. $\mathcal{C}_1(y) = 0$ and $\mathcal{C}_2(y) = 0$, and thus for formal x_i, we have $\mathcal{C}(x, y) = 0$, and thus $\mathrm{Det}(\mathcal{C}(x)) = 0$ at this point, which contradicts our choice of S' to be in the complement of $(c_0, \ldots, c_n)$.

The fact that these maps agree on the intersection of E_k and E_ℓ is exactly given by the fact that on $T'_{\mathcal{C}(y)}$ the 2×2 minor of $\mathcal{C}(y)$ including rows k and ℓ is 0. To see that the image of our map lands in $T'_{\mathcal{C}(x)}$, we check on open P_i of $T'_{\mathcal{C}(y)}$, where y_i is non-zero. We have that $\mathcal{C}(x)$ on E_ℓ has j, k entry

$$-u_{1,j,k} \sum_a a_{2a\ell} y_a + u_{2,j,k} \sum_a a_{1a\ell} y_a,$$

and thus $\mathcal{C}(x, y)$ has kth entry

$$\sum_j y_j \left(-u_{1,j,k} \sum_a u_{2a\ell} y_a + u_{2,j,k} \sum_a u_{1a\ell} y_a \right)$$
$$= \sum_j -u_{1,j,k} y_j \sum_a u_{2a\ell} y_a + \sum_j u_{2,j,k} y_j \sum_a u_{1a\ell} y_a$$
$$= -\mathcal{C}(y)_{1,k} \mathcal{C}(y)_{2,\ell} + \mathcal{C}(y)_{2,k} \mathcal{C}(y)_{1,\ell},$$

which is zero by the definition of $T'_{\mathcal{C}(y)}$. On P_i we form the column vector y/y_i of regular functions, with jth entry y_j/y_i, and we see that $\mathcal{C}(x, y/y_i) = 0$. Thus we can write the ith row of $\mathcal{C}(x)$ as a linear combination of the other rows, and conclude that $\mathrm{Det}(\mathcal{C}(x)) = 0$.

Next, we will give a map $T'_{\mathcal{C}(x)} \to T'_{\mathcal{C}(y)}$, which should be seen in analogy to the map $T'_{\mathcal{C}(y)} \to T'_{\mathcal{C}(x)}$. We will give maps from open sets of $T'_{\mathcal{C}(x)}$ to $\mathbb{P}^{n-1}_{S'}$, and show that the open sets cover $T'_{\mathcal{C}(y)}$. We will then show that these maps agree on overlaps, and finally we will show that the image lands in $T'_{\mathcal{C}(x)}$. Given an $1 \leq i \leq n$, we map $T'_{\mathcal{C}(x)}$ to $\mathbb{P}^{n-1}_{S'}$ by letting y_j equal the j, k minor of $\mathcal{C}(x)$, that is, y_j equals $(-1)^{j+k}$ times the determinant of the submatrix of $\mathcal{C}(x)$ obtained by deleting the jth row and kth column. We have defined the y_j to be a column of minors of $\mathcal{C}(x)$. This is a well-defined map to $\mathbb{P}^{n-1}$ on the open set $F_k \subset T'_{\mathcal{C}(x)}$ defined as the complement of the ideal of $(n-1)$-minors of $\mathcal{C}(x)$ for the kth column. Suppose there was a point of $T'_{\mathcal{C}(x)}$ not in any F_k, then at this point we have all minors of $\mathcal{C}(x)$ are 0. This means that $W(\{x_1^{n-1}, x_1^{n-2} x_2, \dots, x_2^{n-1}\}, \cdot, \cdot) = 0$. Thus for formal y, we have $W(\cdot, y, \cdot)$ has a non-trivial kernel and thus $\mathrm{Det}(W(y)) = 0$, which contradicts our choice of S' to be in the complement of $\mathfrak{w}$.

The fact that these maps agree on the intersection of F_k and F_ℓ is exactly given by the fact that the 2×2 minors of the classical adjoint matrix are divisible by the determinant of the original matrix. To see that the image of our map lands in $T'_{\mathcal{C}(y)}$, we check on opens P_i of $T'_{\mathcal{C}(x)}$, where x_i is non-zero. On P_i we form the column vector x/x_i of regular functions, with jth entry x_j/x_i. Computing $\mathcal{C}(x, y)_\ell$ with the y_i we have defined on F_k is the same as computing the determinant of $\mathcal{C}(x)$ with the kth column replaced by the ℓth column. Whether or not $k = \ell$, since $\mathrm{Det}(\mathcal{C}(x)) = 0$, we obtain $\mathcal{C}(x, y)_\ell = 0$. Thus, $\mathcal{C}(x/x_i, y) = 0$ and we can write the ith aisle of $\mathcal{C}(y)$ as multiple of the other aisle. We conclude that $\mathcal{C}(y)$ has all 2 by 2 minors 0.

Now we need to check that the maps we have just given are inverses to one another. We first check on the inverse image of E_k in F_ℓ. Here we start with x_i, we define new y_j, and then from the y_j we define new x'_i. We will compute $-x'_1 x_2 + x'_2 x_1$. Since we have $x'_1 = -\sum_j u_{2jk} y_j$ and $x'_2 = \sum_j u_{1jk} y_j$, we have that

$$-x'_1 x_2 + x'_2 x_1 = \sum_j (u_{1jk} x_1 + u_{2jk} x_2) y_j.$$

We note that $u_{1jk}x_1 + u_{2jk}x_2 = \mathcal{C}(x)_{j,k}$, and that y_j is defined to be the j, ℓ minor of $\mathcal{C}(x)$. Thus $-x_1'x_2 + x_2'x_1$ is the determinant of the matrix obtained from $\mathcal{C}(x)$ by replacing the ℓth column by the kth column, and is zero in any case on $T_{\mathcal{C}(x)}$. This shows that our maps compose to the identity on the inverse image of E_k in F_ℓ for all k and ℓ, and thus on $T_{\mathcal{C}(x)}$.

We now check on the inverse image of F_k in E_ℓ. Here we start with y_j, we define new x_i, and then from the x_i we define new y_j'. At first, we will use formal y_j (i.e. not assuming the relation in $T_{\mathcal{C}(y)}$). Then we will compute $y_j'y_m - y_m'y_j$ is in the ideal of relations $\mathcal{M}(\mathcal{C}(y))$ that cut out $T_{\mathcal{C}(y)}$. We can form an $n \times n$ matrix M with a, b entry $(-1)^a\mathcal{C}(x)_{a,b}y_a$ if $a = j, m$ and $\mathcal{C}(x)_{a,b}$ otherwise. Note that $(-1)^m y_j'y_m$ is the j, k minor of M and $(-1)^j y_j'y_m$ is the m, k minor of M. For any matrix N the difference of $(-1)^m$ times the j, k minor of N and $(-1)^j$ times the m, k minor of N is in the ideal generated by maximal minors of $\bar{N}$, which is obtained from N by deleting rows j and m and adding a row that is the jth row of N plus $(-1)^{j+m}$ times the mth row of N. The maximal minors of $\bar{M}$ are not changed if we add multiples of the original (non-deleted) rows of M to the new row of $\bar{M}$. We add $(-1)^j y_a$ times the original ath row of M to the new row of $\bar{M}$ to obtain $\bar{M}'$. The maximal minors of $\bar{M}'$ certainly lie in the ideal generated by the elements of its "new" row, and we claim these elements are in $\mathcal{M}(\mathcal{C}(y))$. In the bth column, the element in the new row of $\bar{M}'$ is

$$\sum_{i,c} u_{icb}x_i y_c (-1)^j = \sum_{i,c,a}(-1)^{i+j}u_{icb}u_{(3-i)a\ell}y_a y_c = \sum_{i}(-1)^{i+j}\mathcal{C}(y)_{i,b}\mathcal{C}(y)_{3-i,\ell},$$

which is the b, ℓ minor of $\mathcal{C}(y)$ and thus in $\mathcal{M}(\mathcal{C}(y))$. This shows that our maps compose to the identity on $T_{\mathcal{C}(x)}$. $\qquad\square$

Thus it follows that we have an isomorphism of Λ-algebras

$$R_c = \Gamma(\mathcal{O}_{T_{\mathcal{C}(x)}}) \cong \Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}),$$

which completes the proof of Theorem 7.1. $\qquad\square$

From Theorem 7.1, we have an R_c-module structure on $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1))$. We now see that it is related to the module $M_{\mathcal{C}}$ we constructed in Section 3.1.1 from the universal tensor.

Theorem 7.7. *We have an isomorphism of R_c-modules*

$$\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1)) \cong \mathrm{Hom}_\Lambda(M_{\mathcal{C}}, \Lambda),$$

where $M_{\mathcal{C}}$ is as in the construction ψ of two R_c modules $M_{\mathcal{C}}$ and $N_{\mathcal{C}}$ given in Section 3.1.1.

Proof. We will see in Theorem 8.4 that $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1))$ is a free Λ-module with basis $y_1, \ldots, y_n$. Thus, it just suffices to check that the ζ_i act on the y_j in a way corresponding to their action on $M_{\mathcal{C}}$. We know that ζ_i acts on the y_j by a matrix of elements of Λ, and thus it suffices to determine this action over the generic point of $\mathrm{Spec}\,\Lambda$, i.e. the fraction field of Λ. We have that $\mathcal{C}(x, y) = 0$ and thus $y\mathcal{C}_1 x_1 + y\mathcal{C}_2 x_2 = 0$, where y is a row vector of the y_i. Thus, where x_2 is invertible, $\frac{x_1}{x_2}$ acts like $-\mathcal{C}_2\mathcal{C}_1^{-1}$ on the right on the row vector y, and where x_1 is invertible, $\frac{x_2}{x_1}$ acts like $-\mathcal{C}_1\mathcal{C}_2^{-1}$ on the right on the row vector y. Thus $\frac{x_1}{x_2}$ acts like $-\mathcal{C}_2\mathcal{C}_1^{-1}$ on the left on elements of $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1))$ written as row vectors whose entries are the coefficients of the y_i in the element. We have that θ acts in elements of $M_{\mathcal{C}}$ by $(-\mathcal{C}_2\mathcal{C}_1^{-1})^t$

on the left. Since in the correspondence between the algebraic and geometric construction on R_c we have that θ corresponds to $\frac{x_1}{x_2}$, we see that the ζ_i act on $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1))$ as they act on $\mathrm{Hom}_\Lambda(M_{\mathcal{C}}, \Lambda)$. $\qquad\square$

We can of course obtain a completely analogous geometric construction of the module $N_{\mathcal{C}}$ as $\mathrm{Hom}_\Lambda(\Gamma(\mathcal{O}_{T_{\mathcal{C}(z)}}(1)), \Lambda)$.

8. Geometric construction over an arbitrary base scheme

Notation. Given a scheme S and a locally free $\mathcal{O}_S$-module U, we let U^* denote the $\mathcal{O}_S$-module $\mathcal{H}om_{\mathcal{O}_S}(U, \mathcal{O}_S)$, even if U is also a module for another sheaf of algebras. Let $\mathbb{P}(U) = \mathrm{Proj}\, \mathrm{Sym}^* U$.

Now we replace $\mathrm{Spec}\,\Lambda$ by an arbitrary scheme S, and we consider V, U, W, locally free $\mathcal{O}_S$-modules of ranks 2, n, and n, respectively. Let $p \in V \otimes U \otimes W$ denote a global section of $V \otimes U \otimes W$. Let $f = \mathrm{Det}(p) \in \mathrm{Sym}^n V \otimes \wedge^n U \otimes \wedge^n W$. In Section 6 we constructed a balanced pair M, N of modules for f from p. In this section, we will give a geometric construction of those modules, or rather a geometric construction of M^* and N^* as we have done in the case of the universal form in Section 7. This construction of the modules M^* and N^* from $p \in V \otimes U \otimes W$ will work for all p and be functorial in S, i.e. will commute with base change in S.

The idea is to replace the subschemes of $\mathbb{P}(V)$, $\mathbb{P}(U)$, and $\mathbb{P}(W)$ cut out by the maximal minors of our tensor (called $T_p(V), T_p(U), T_p(W)$, respectively, in the Introduction) with complexes of sheaves. We will then replace π_* with the hypercohomology functors $H^0 R\pi_*$. This has already been done in the construction of R_f and the module I_f in [20, Section 3]. We face some additional challenges in this paper for $T_p(U)$ and $T_p(W)$ because the complexes involved are more complicated. One can also interpret this work as a construction of dg-schemes given by resolutions of the maximal minors instead of just a construction of schemes.

8.1. Arbitrary triple tensors. We now give a more general construction before specifying to the situation of interest in this paper. Let S be an arbitrary scheme, and let $p \in V \otimes U \otimes W$, where V, U, W are locally free $\mathcal{O}_S$-modules of ranks r_V, r_U, and r_W, respectively. Let $r = r_V$ and assume $r_W \geq r$. We can view p as a map $W^* \to V \otimes U$, and take r-minors of this map, with coefficients in U, to get $\wedge^r_U p : \wedge^r W^* \to \wedge^r V \otimes \mathrm{Sym}^r U$ or equivalently $\wedge^r_U p : \wedge^r W^* \otimes \wedge^r V^* \to \mathrm{Sym}^r U$.

Let $\pi : \mathbb{P}(U) \to S$. Let $\mathcal{O}(k)$ be the usual sheaf of $\mathbb{P}(U)$. Then since $\pi_* \mathcal{O}(r) = \mathrm{Sym}^r U$, by the adjointness of π_* and π^* we get a map

$$\wedge^r_U p : \pi^*(\wedge^r W^* \otimes \wedge^r V^*) \to \mathcal{O}(r)$$

or equivalently, for any k, we get

$$\wedge^r_U p : \pi^*(\wedge^r W^* \otimes \wedge^r V^*) \otimes_{\mathcal{O}_S} \mathcal{O}(k) \to \mathcal{O}(r + k).$$

It is an abuse of notation to call all these maps $\wedge^r_U p$, but it is better than the alternative. Locally on S, where U, V, and W are free, the map $\wedge^r_U p : \pi^*(\wedge^r W^* \otimes \wedge^r V^*) \to \mathcal{O}(r)$ has image spanned by the $\binom{r_W}{r}$ r by r minors of the matrix of the map $W^* \to V \otimes U$, an r_V by r_W

matrix with entries in U. The idea of our construction is to replace the sheaf $\mathcal{O}/\operatorname{im}(\wedge_U^r p)$ of regular functions of the subscheme of $\mathbb{P}(U)$ cut out by those r by r minors with complex that is generically a locally free resolution of the $\mathcal{O}/\operatorname{im}(\wedge_U^r p)$.

From the Eagon–Northcott complex, which resolves R modulo the ℓ by ℓ minors of a generic matrix (see [9]), we can construct a complex $\mathcal{C}(k)$ with $\mathcal{C}^{-1}(k) \to \mathcal{C}^0(k)$ given by

$$\wedge_U^r p : \pi^*(\wedge^r W^* \otimes \wedge^r V^*) \otimes_{\mathcal{O}_S} \mathcal{O}(k) \to \mathcal{O}(r+k),$$

and with $\mathcal{C}^i(k) = 0$ for $i > 0$ and $i \leq -(r_W - r) - 2$. For $-(r_W - r) - 1 \leq i \leq -2$, we have

$$\mathcal{C}^i(k) = \pi^*(\wedge^r V^* \otimes K_{-i}(r, W^*, V)) \otimes_{\mathcal{O}_S} \mathcal{O}(i + 1 + k),$$

where $K_{-i}(r, W^*, V)$ is the locally free $\mathcal{O}_S$-module built from V and W that is the ith term in the Eagon–Northcott complex for a map $\alpha : W^* \to V$, and d_i is canonically constructed from p (and explained in the next paragraph). Note that $K_{-i}(r, W^*, V)$ only depends on V and W and does not depend on α.

We now show how to construct the d_i. From the construction of the Eagon–Northcott complex, there is a map

$$\mathcal{H}om_{\mathcal{O}_Y}(W^*, V) \to \mathcal{H}om_{\mathcal{O}_Y}(\wedge^r V^* \otimes K_{-i}(r, W^*, V), \wedge^r V^* \otimes K_{-i+1}(r, W^*, V))$$

that sends $\alpha \mapsto d_i$, where d_i is the map in the Eagon–Northcott complex for α. We can extend that linear map to

$$\begin{aligned}
&\mathcal{H}om_{\mathcal{O}_Y}(W^*, V) \otimes U \\
&\quad \to \mathcal{H}om_{\mathcal{O}_Y}(\wedge^r V^* \otimes K_{-i}(r, W^*, V), \wedge^r V^* \otimes K_{-i+1}(r, W^*, V)) \otimes_{\mathcal{O}_Y} U
\end{aligned}$$

to get the maps when there are coefficients. Let H_i be the $\mathcal{O}_Y$-module $\wedge^r V^* \otimes K_{-i}(r, W^*, V)$. We obtain $d_i : H_i \to H_{i+1} \otimes U$, or equivalently $d_i : H_i \otimes H_{i+1}^* \to U$. Using adjointness of π_* and π^*, this is equivalent to $d_i : \pi^*(H_i \otimes H_{i+1}^*) \to \mathcal{O}_U(1)$, which gives us

$$d_i : \pi^*(H_i) \otimes \mathcal{O}_U(i + 1 + k) \to \pi^*(H_{i+1}) \otimes \mathcal{O}_U(i + 2 + k).$$

The complex $\mathcal{C}(-r)$ of sheaves on $\mathbb{P}(U)$ has a commutative, homotopy-associative differential graded algebra structure from the commutative, homotopy-associative differential graded algebra structure on the Eagon–Northcott complex (which every resolution of a cyclic module has [5, Proposition 1.1]), and the complex $\mathcal{C}(-r + 1)$ is a differential graded module for $\mathcal{C}(-r)$. Now we make an important calculation about the cohomology of $\mathcal{C}(-r)$ and $\mathcal{C}(-r + 1)$.

Theorem 8.1. *Let $p \in V \otimes U \otimes W$, where V, U, W are locally free $\mathcal{O}_S$-modules of ranks r, r_U, and r_W, respectively. Assume that $r_W \geq r \geq 2$ and that we are in one of the following cases:*

(i) *$r = 2$ and $r_U \geq r_W$,*

(ii) *$r_U = 2$ and $r = r_W$.*

If $k = -r$ or $k = -r + 1$, then $R\pi_*\mathcal{C}(k)$ has no cohomology in any degree except 0.

Proof. Let $j \neq 0$, and we will compute that each term of the complex $\mathcal{C}(k)$ has trivial $R^j \pi_*$. By the projection formula, we can ignore the term that is pulled back from S. We have $R^j \pi_*$ of the ith term $i \leq -1$ of $\mathcal{C}(k)$, in the ith place, is $R^{j-i} \pi_*$ of $\mathcal{C}^i(k)$ viewed as a complex in the 0th place. We have that $R^{j-i} \pi_* \mathcal{O}(i+1+k) = 0$ unless either (1) $j = i$ and $i+1+k \geq 0$ or (2) $j - i = r_U - 1$ and $i+1+k \leq -r_U$. Since $i+1 \leq 0$ and $k \leq -r+1 \leq -1$, we can never have $i+1+k \geq 0$. We consider the two assumptions of the theorem in cases.

(i) $r = 2$ and $r_U \geq r_W$. In this case, we have
$$i + 1 + k \geq -(r_W - r) + k \geq -(r_W - r) - r \geq -r_U$$
and thus we can only have $i + 1 + k \leq -r_U$ if $i = -(r_W - r) - 1$, and $k = -r$, and $r_U = r_W$. However, that implies that $i = -r_U + 1$ and thus $j = 0$.

(ii) $r_U = 2$ and $r = r_W$. In this case, we only are considering the case $i = -1$, and thus $j - i = r_U - 1$ implies $j = 0$.

We now need to consider $R^j \pi_*$ of the 0th term of $\mathcal{C}(k)$ (for $j \neq 0$). We have that $R^j \pi_* \mathcal{O}(r+k) = 0$ unless (1) $j = 0$ and $r + k \geq 0$ or (2) $j = r_U - 1$ and $r + k \leq -r_U$. However, we are assuming $j \neq 0$ and $r + k \geq 0$, and thus this can never happen. Thus we conclude that for $k = -r$ and $k = -r + 1$, under our assumptions about r_U, r, and r_W, the complex $\mathcal{C}(k)$ has no cohomology in any degree except 0. □

Corollary 8.2. *We have that $\mathcal{R}_{\wedge_U^r p} = H^0 R\pi_* \mathcal{C}(-r)$ is a sheaf of algebras on S, and $\mathcal{I}_{\wedge_U^r p} = H^0 R\pi_* \mathcal{C}(-r+1)$ is a sheaf of $\mathcal{R}$-modules on S.*

Proof. Since $R\pi_* \mathcal{C}(-r)$ is equivalent to a single sheaf in degree 0, it has no non-trivial homotopies. Thus $H^0 R\pi_* \mathcal{C}(-r)$ has an $\mathcal{O}_S$-algebra structure that is not just homotopy-associative but in fact associative. Since $\mathcal{C}(-r+1)$ is a module for $\mathcal{C}(-r)$, we have that $\mathcal{I}_{\wedge_U^r p} = H^0 R\pi_* \mathcal{C}(-r+1)$ is an $H^0 R\pi_* \mathcal{C}(-r)$-module. □

We can also view the construction of $\mathcal{R}_{\wedge_U^r p}$ as taking the pushforward of the regular functions on the dg-scheme given by our resolution of $\mathcal{O}/\operatorname{im}(\wedge_U^r p)$, instead of on the scheme cut out by $\wedge_U^r p$.

When p is the universal tensor (of any size), then the Eagon–Northcott complex, and thus $\mathcal{C}(k)$, is exact at every spot except the 0th. Thus, when p is the universal tensor, $\mathcal{C}(k)$ is quasi-isomorphic to $\mathcal{O}(k)/\operatorname{im}(\wedge_U^r p)$. The sheaf $\mathcal{O}(k)/\operatorname{im}(\wedge_U^r p)$ is supported on the scheme defined by the r by r minors in $\operatorname{im}(\wedge_U^r p)$, and is isomorphic on that scheme to the pullback of $\mathcal{O}(k)$ from $\mathbb{P}(U)$. Thus, when $\mathcal{C}$ is the universal tensor in $\Lambda^2 \otimes \Lambda^n \otimes \Lambda^n$, we have that $\mathcal{R}_{\wedge_U^r \mathcal{C}}$ is the sheaf of rings given by the global sections $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}})$ (as defined in Section 7), and $\mathcal{I}_{\wedge_U^r \mathcal{C}}$ is the $\mathcal{R}_{\wedge_U^r \mathcal{C}}$-module given by the global sections $\Gamma(\mathcal{O}_{T_{\mathcal{C}(y)}}(1))$.

Theorem 8.1 also allows us to see that the constructions of $\mathcal{R}_{\wedge_U^r p}$ and $\mathcal{I}_{\wedge_U^r p}$ commute with base change on S.

Corollary 8.3. *Let $p \in V \otimes U \otimes W$, where U, V, W are locally free $\mathcal{O}_S$-modules of ranks r_U, r, and r_W, respectively. Assume that $r_W \geq r \geq 2$ and that we are in one of the following cases:*

(i) *$r = 2$ and $r_U \geq r_W$,*

(ii) *$r_U = 2$ and $r = r_W$.*

Then the constructions of $\mathcal{R}_{\wedge_U^r p}$ and $\mathcal{I}_{\wedge_U^r p}$ commute with base change. More precisely, let $\phi : S' \to S$ be a map of schemes. Let $p' \in \phi^* U \otimes \phi^* V \otimes \phi^* W$ be the pullback of p. Then the natural map from cohomology

$$\mathcal{R}_{\wedge_U^r p} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to \mathcal{R}_{\wedge_{\phi^* U}^r p'}$$

is an isomorphism of $\mathcal{O}_{S'}$-algebras. Also, the natural map from cohomology

$$\mathcal{I}_{\wedge_U^r p} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \to \mathcal{I}_{\wedge_{\phi^* U}^r p'}$$

is, in fact, an isomorphism of $\mathcal{R}_{\wedge_{\phi^* U}^r p'}$-modules, where the $\mathcal{R}_{\wedge_{\phi^* U}^r p'}$-module structure on $\mathcal{I}_{\wedge_U^r p} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$ comes from the $(\mathcal{R}_{\wedge_U^r p} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})$-module structure.

Proof. The key to this proof is to compute all the cohomology of the pushforward of the complex $\mathcal{C}(k)$ for $k = -r$ and $k = -r + 1$. We already know from Theorem 8.1 that there is only cohomology in degree 0. Theorem 8.4 will tell us that $H^0 R\pi_* \mathcal{C}(k)$ is locally free for $k = -r$ and $k = -r + 1$. Thus since all $H^i R\pi_*(\mathcal{C}(k))$ are flat, by [12, Corollaire 6.9.9], we have that cohomology and base change commute. Note that the base change morphisms respect the algebra and module structures on $\mathcal{R}_{\wedge_U^r p}$ and $\mathcal{I}_{\wedge_U^r p}$, and thus since they are isomorphisms, they are algebra and module isomorphisms. $\square$

8.2. $\mathcal{O}_S$-module structure of $\mathcal{R}_{\wedge_U^r p}$ and $\mathcal{I}_{\wedge_U^r p}$. Now we consider a base scheme S, and V, U, W locally free $\mathcal{O}_S$-modules of ranks 2, n, and n, respectively. In this case, we construct the $\mathcal{O}_S$-algebra and module pairs $\mathcal{R}_{\wedge_V^n p}$ and $\mathcal{I}_{\wedge_V^n p}$, $\mathcal{R}_{\wedge_U^2 p}$ and $\mathcal{I}_{\wedge_U^2 p}$, and $\mathcal{R}_{\wedge_W^2 p}$ and $\mathcal{I}_{\wedge_W^2 p}$. We will now find the $\mathcal{O}_S$-module structure of all of these constructions. This has already been done for the $\wedge_V^n p$ construction in [20, Section 3.1], and so we consider here the $\wedge_U^2 p$ (as the $\wedge_W^2 p$ constructions will follow identically).

Theorem 8.4. *We have an exact sequence of $\mathcal{O}_S$-modules*

$$0 \to \mathcal{O}_S \to \mathcal{R}_{\wedge_U^2 p} \to (\mathrm{Sym}^{n-2} V)^* \otimes \wedge^2 V^* \otimes \wedge^n W^* \otimes \wedge^n U^* \to 0,$$

and an $\mathcal{O}_S$-module isomorphism $\mathcal{I}_{\wedge_U^2 p} \cong U$.

Proof. From Theorem 8.1, we know that, for $k = -r$ and $k = -r + 1$, $\mathcal{C}(k)$ has trivial $H^j R\pi_*$ for all $j \neq 0$, and all components $C(k)_i^i$ of the complex (the ith term of $C(k)$ sitting as a complex in the ith place) have $H^0 R\pi_*(C(k)_i^i) = 0$ except for possibly the two extremal terms $i = 0$ and $i = -n + 1$. Thus, by the long exact sequence of cohomology, we have the exact sequences

$$0 \to H^0 R\pi_*(\mathcal{O})$$
$$\to \mathcal{R}_{\wedge_U^2 p} \to H^{n-1} R\pi_*\big(\pi^*(\wedge^2 V^* \otimes K_{-n+1}(2, W^*, V)) \otimes_{\mathcal{O}_S} \mathcal{O}(-n)\big) \to 0$$

and

$$0 \to H^0 R\pi_*(\mathcal{O})$$
$$\to \mathcal{R}_{\wedge_U^2 p} \to H^{n-1} R\pi_*\big(\pi^*(\wedge^2 V^* \otimes K_{-n+1}(2, W^*, V)) \otimes_{\mathcal{O}_S} \mathcal{O}(-n + 1)\big) \to 0.$$

We see that

$$
\begin{aligned}
H^{n-1} R\pi_* &\big(\pi^*(\wedge^2 V^* \otimes K_{-n+1}(2, W^*, V)) \otimes_{\mathcal{O}_S} \mathcal{O}(-n)\big) \\
&= \wedge^2 V^* \otimes K_{-n+1}(2, W^*, V) \otimes_{\mathcal{O}_S} H^{n-1} R\pi_*(\mathcal{O}(-n)) \\
&= \wedge^2 V^* \otimes K_{-n+1}(2, W^*, V) \otimes_{\mathcal{O}_S} \wedge^n U^*.
\end{aligned}
$$

Since $K_{-n+1}(2, W^*, V) = (\mathrm{Sym}^{n-2} V)^* \otimes \wedge^n W^*$, we obtain the exact sequence desired. Also, note that $H^{n-1} R\pi_*(\mathcal{O}(-n+1)) = 0$. $\qquad\square$

We can see that the three $\mathcal{O}_S$-algebras constructed from a tensor $p \in V \otimes U \otimes W$ are isomorphic. In the case that V, U, and W are free, then p is a pull-back from the universal tensor, in which case we know the algebras are isomorphic from Theorem 7.1. If one checks that the algebra isomorphism given by Theorem 7.1 is canonical, that it doesn't depend on the choice of bases of V, U, and W, then that would show that the three $\mathcal{O}_S$-algebras constructed from a tensor $p \in V \otimes U \otimes W$ are all isomorphic because locally, V, U and W are free, and if the isomorphisms between algebras do not depend on the choice of bases, they will agree on overlaps. In fact, the constructions made in Lemma 7.6 to give the isomorphism of S'-schemes $T'_{\mathcal{C}(y)} \cong T'_{\mathcal{C}(x)}$ are all given by minors of matrices and in fact are canonical.

Finally, we can see that the construction of the $\mathcal{R}_{\wedge^2_U p}$-module $\mathcal{I}^*_{\wedge^2_U p}$ from this section agrees with the construction of a module structure on U^* given in the proof of Theorem 6.2. By Theorem 8.4, $\mathcal{I}^*_{\wedge^2_U p}$ is also an $\mathcal{R}_{\wedge^2_U p}$-module structure on U^*. Thus, we just need to check that the $\mathcal{R}_{\wedge^2_U p}$ actions agree, which we can check locally. By Corollary 8.3, the construction of $\mathcal{I}^*_{\wedge^2_U p}$ commutes with base change, and the construction of a module structure on U^* from the proof of Theorem 6.2 commutes with base change by construction. Thus it suffices to check that the module structures agree in the case of the universal tensor, which was done in Theorem 7.7.

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