# TALK 1

## Serre-type Conj. for tame n-dimensional mod p Gal. reps.

(Florian Herzig)

- Notations: $p$ prime

Fix $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$$\begin{array}{l} G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \\ \cup \\ D_p = \text{decomp. gp.} \\ \cup \\ I_p = \text{inertia gp.} \end{array}$$

(1) $GL_2$  $k \geq 2$

$f \in S_k(\Gamma_1(N), \mathbb{C})$ eigenform : $T_\ell f = a_\ell f$, $S_\ell f = b_\ell f$

$\circlearrowleft T_\ell$, $\circlearrowleft S_\ell$ $(\ell \nmid pN)$  $a_\ell, b_\ell \in \overline{\mathbb{Z}}$

$\rightsquigarrow \rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ unram. outside $Np$ s.t.

(i) $\det(1 - \rho_f(\mathrm{Frob}_\ell)X) = 1 - \bar{a}_\ell X + \ell \bar{b}_\ell X^2$  $\forall \ell \nmid pN$

(ii) $\rho_f(\text{cx. conj.}) \sim \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ if $p > 2$  ("odd").

Serre's Conjecture: If $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is irr. + odd, it arises from some $f$ of level $N(\rho)$ and weight $k(\rho) \geq 2$

$N(\rho) \uparrow \{\rho|_{I_\ell} : \ell \neq p\}$ (prime to $p$)

$k(\rho) \uparrow \{\rho|_{I_p}\}$.

Passing to cohomology:

Eichler-Shimura: $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\mathbb{C}^2) \cong M_k(N) \oplus \overline{S_k(N)}$
(group coho., via $\Gamma_1(N) \subset SL_2(\mathbb{Z}) \curvearrowright \mathbb{C}^2$) (Hecke equiv.)

For $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$ irr.,

$\rho$ attached to reduction "mod $p$" of Hecke evals. in wt. $k$, level prime to $p$

$\Leftrightarrow$ - - - - - - - - Hecke evals. in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\overline{\mathbb{F}}_p^2)$, some $(N,p)=1$.

$(*)\ \{ \Leftrightarrow$ - - - - - - - - - - - - - - - - $H^1(\Gamma_1(N), F)$ - - - - - - some $F \in JH(\mathrm{Sym}^{k-2}$

$\rightarrow$ Ash-Stevens 1986

Serre weights = irr. reps. of $GL_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$

$F(a,b) \cong \mathrm{Sym}^{a-b}\overline{\mathbb{F}}_p^2 \otimes \det^b \quad (0 \le a-b \le p-1)$

(So there are $p(p-1)$.)

Let $W(\rho) = \{\text{Serre wts. } F \mid (*) \text{ holds}\}$.

Translate between $k(\rho)$ and $W(\rho)$ (exercise)

"$\rightarrow$": use
- $W(\rho \otimes \omega) \cong W(\rho) \otimes \det$
- $\rho$ modular of wt. 2 $\Rightarrow$ wt. $p+1$
- $\rho$ modular of wt. $k$ $\Rightarrow$ $k$ determined mod $p-1$ by $\det \rho|_{I_p}$

"$\leftarrow$": need to work out $JH(\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2)$ for $k \le p^2-1$.

(Rk: $k(\rho)$ minimal wt. in prime-to-$p$ level.)

E.g.: $\rho|_{I_p} \sim \begin{pmatrix} \omega^i & * \\ & \omega^j \end{pmatrix}$: $k(\rho) = i + pj + 1$

$p-2 > i > j+1 > 0$

$$W(\rho) = \begin{cases} \{F(i-1,j),\, F(j+p-2,\, i)\} & * \text{ split} \\ \{F(i-1,j)\} & * \text{ non-split} \end{cases}$$

(see also BDJ, thm. 3.15)

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② $GL_n$, $n \geqslant 2$

$$\Gamma_1(N) = \left\{ \gamma \in SL_n(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} & * & \\ 0 & \cdots & 0 \; 1 \end{pmatrix} \bmod N \right\}, \quad (N,p) = 1$$

$F$ ... Serre weight: irr. rep. of $GL_n(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$.

$\rightsquigarrow$ $H^i(\Gamma_1(N), F)$ $\circlearrowleft$ $T_{\ell,1}, \ldots, T_{\ell,n}$ $(\forall\, \ell \nmid pN)$.

If $\rho : G_{\mathbb{Q}} \longrightarrow GL_n(\overline{\mathbb{F}}_p)$ irr. + odd (i.e. $\rho(\text{cx. conj.}) \sim \begin{pmatrix} \pm 1 & & \\ & \mp 1 & \\ & & \pm 1 \\ & & & \ddots \end{pmatrix}$ if $p > 2$)

let $W(\rho) = \{$ Serre wts. $F \mid \exists$ Hecke eigenvec. in $H^i(\Gamma_1(N), F)$ attached to $\rho$, some $i \geqslant 0$, $(N,p) = 1 \}$.

$\rightarrow$ Ash, Doud, Pollack, Sinnott

Rks.: – $W(\rho)$ should only depend on $\rho|_{I_p}$.

– expect $\underset{\phi}{W(\rho|_{I_p})} \neq \subset W(\rho|_{I_p}^{ss})$

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③ Modular representations

$G = GL_n$ [or split reductive $/\mathbb{F}_p$, $G'$ simply conn.]

$\cup$

$B$ Borel $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$\cup$

$T$ max. torus $\begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$X(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $Y(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ free $/\mathbb{Z}$ of rk. $n$

$\langle , \rangle : X(T) \times Y(T) \longrightarrow \mathbb{Z}$ perfect

$R^+ \subset R \subset X(T)$ (pos.) roots

$\alpha \in R \longleftrightarrow \alpha^\vee \in R^\vee \subset Y(T)$

simple coroots $\alpha_i^\vee(t) = \begin{array}{c} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & t & & \\ & & & t^{-1} & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \\ \uparrow \\ i \end{array}$, $1 \le i \le n-1$

dominant wts. $X(T)_+$ ($a_1 \geq \dots \geq a_n$ in $\mathbb{Z}$)

Weyl gp. $W = N(T)/T \cong S_n$

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restricted wts. $X_1(T) = \{\lambda \in X(T) ; 0 \le \langle \lambda, \alpha_i^\vee \rangle \le p-1 \ \forall i\} \subset X(T)$

$\cup$

regular wts. $X_{reg}(T) =$ – – – – – – $0 \le$ – – $< p-1$ – – –
[non-standard] terminology

$\cup$

$X^0(T) =$ – – – – $0 =$ – – –

**Thm.** (Chevalley)

$$\{\text{irr. reps. of } G \,/\overline{\mathbb{F}}_p\} \longleftrightarrow X(T)_+$$
$$F(\lambda) \longleftrightarrow \lambda$$

**Thm.** (Steinberg + $\varepsilon$)

(i) Every irr. rep. of $G(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$ (Serre wt.) is of form $F(\lambda)$, some $\lambda \in X_1(T)$.

(ii) If $\lambda, \lambda' \in X_1(T)$, $F(\lambda) \cong F(\lambda')$ as reps. of $G(\mathbb{F}_p)$ $\iff$ $\lambda - \lambda' \in (p-1)X^0(T)$.

E.g. Serre wts. for $n = 3$: $F(a,b,c)$ : $0 \leq a-b,\ b-c \leq p-1$.

reg. ―――――― : ―――― ―――――― $< p-1$.

$$F(\underline{a} + \underline{1}) \cong F(\underline{a}) \otimes \det.$$

Dual Weyl module: for $\lambda \in X(T)_+$, let

$$W(\lambda) = \{\text{morphisms } G(\overline{\mathbb{F}}_p) \xrightarrow[f]{} \overline{\mathbb{F}}_p : f(\bar{b}g) = \lambda(\bar{b}) f(g)\ \forall \bar{b} \in \overline{B} \text{ (opp. Borel)},\ g \in G\}$$

(action by right translation)

→ works over any comm. ring
→ satisfies the Weyl character formula
→ $F(\lambda) \cong \mathrm{soc}_G W(\lambda)$ [unique irr. submod.]

E.g. ($n = 2$) $W(m, 0) \cong \mathrm{Sym}^m \overline{\mathbb{F}}_p^2$

## Alcoves:

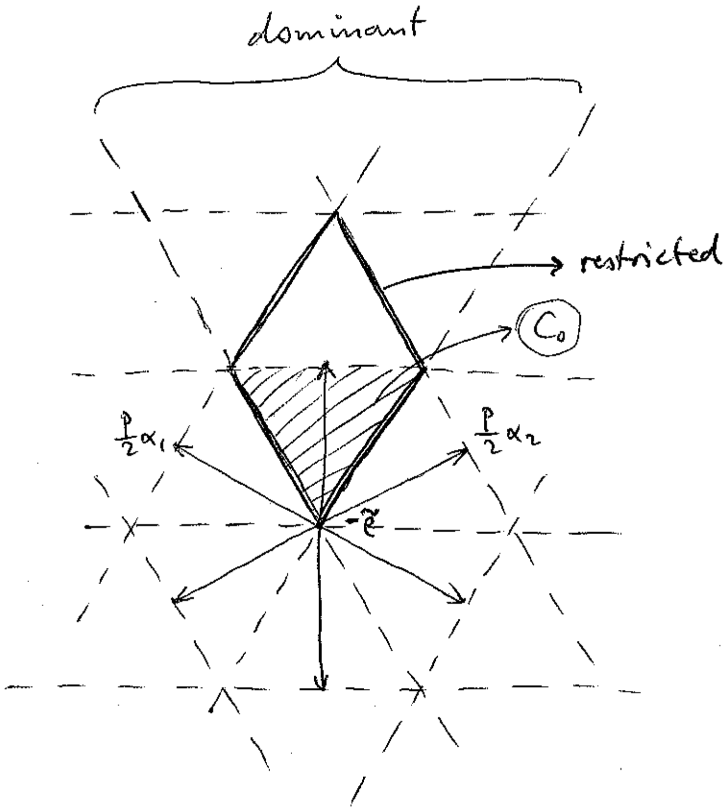
Choose $\tilde{\rho} \in X(T)$ s.t. $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1 \ \forall i$, e.g. $\tilde{\rho} = (n-1, n-2, \dots, 1, 0)$

For $\alpha \in R$, $m \in \mathbb{Z}$,

affine hyperplane $H_{\alpha,m} \subset X(T) \otimes \mathbb{R}$: $\langle \lambda + \tilde{\rho}, \alpha^\vee \rangle = mp$.

Alcoves = conn. components of $X(T) \otimes \mathbb{R} - \bigcup_{\alpha,m} H_{\alpha,m}$ (open!)

n = 3: [really the projection under $X(T) \otimes \mathbb{R} \twoheadrightarrow X(T \cap Sl_3) \otimes \mathbb{R}$]




lowest alcove:

$$C_0 = \{ \lambda : \quad 0 < \langle \lambda + \tilde{\rho}, \alpha^\vee \rangle < p \quad \forall \alpha \in R^+ \}.$$

$\boxed{\text{Assume for simplicity } C_0 \cap X(T) \neq \emptyset}$ $(p > n-1)$

Affine Weyl group: – generated by aff. reflections $s_{\alpha,m}$ in $H_{\alpha,m}$
– fund. domain $C_0$.

For $\lambda, \lambda' \in X(T)$, say $\lambda \uparrow \lambda'$ if $\exists$ aff. reflections $s_i = s_{\alpha_i, m_i}$ s.t.

$$\lambda \leq s_1\lambda \leq s_2 s_1 \lambda \leq \cdots \leq s_r s_{r-1} \cdots s_1 \lambda = \lambda'$$

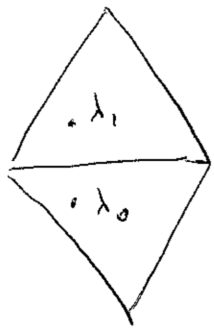
(in particular: $\lambda, \lambda'$ in same aff. Weyl gp. orbit)

Linkage Principle: $[W(\lambda) : F(\mu)] \neq 0 \Rightarrow \mu \uparrow \lambda$
(multiplicity as JH constit.)

Translation Principle:
Suppose $\lambda, \mu$ lie in same aff. Weyl gp. orbit and on no $H_{\alpha,m}$. Then $[W(\lambda) : F(\mu)]$ only depends on the alcoves $\lambda, \mu$ lie in.

Cor.: $W(\lambda) = F(\lambda)$ if $\lambda \in C_0$. [consider highest wt.]

Eg. $(n=3)$




If $a - c > p-2$,
$a-b,\ b-c < p-1$, then

$$0 \longrightarrow \underset{(\lambda_1)}{F(a,b,c)} \longrightarrow W(a,b,c) \longrightarrow \underset{(\lambda_0)}{F(c+p-2, b, a-p+2)} \longrightarrow 0$$

$(n=2)$ If $p-1 < m < 2p-1$,

$$0 \longrightarrow F(m,0) \longrightarrow W(m,0) \longrightarrow F(p-1, m-p+1) \longrightarrow 0$$

S **Rk:** It is possible that $\mu \uparrow \lambda$ with $\lambda$ restricted and $\mu$ dominant, but not restricted. (e.g. $GL_n, n \geq 4$; $G_2$).

④ <u>Serre-type Conjecture</u>

$\omega_r : I_p \longrightarrow \mathbb{F}_{p^r}^\times$ tame fundamental character

Assume for $\forall s \mid r$, $N_{\mathbb{F}_{p^r}/\mathbb{F}_{p^s}} \circ \omega_r = \omega_s$.

If $\mu = (a_1, \dots, a_n) \in X(T)$, let $\bar{\mu} = (a_1, \dots, a_n) \in Y(T)$.
[duality]

If $w \in W$, $\mu \in X(T)$: choose $r \geq 1$ s.t. $w^r = 1$.

Let $$\tau(w, \mu) := \prod_{j=0}^{r-1} \underset{\substack{\uparrow \\ \text{Frob.}}}{(\mathrm{Fr} \circ w)}^j (\bar{\mu} \circ \omega_r) : I_p \to T(\mathbb{F}_{p^r}) \subset GL_n(\overline{\mathbb{F}}_p).$$

<u>E.g.</u> $(n=3)$
$$\tau((123), (a, b, c)) = \begin{pmatrix} \omega_3^{a+pb+p^2c} & & \\ & (\cdot)^p & \\ & & (\cdot)^{p^2} \end{pmatrix}.$$
(take $r = 3$)

Given $\rho$, $\rho|_{I_p}^{ss} \sim \tau(w, \mu)$ for some $w, \mu$ (not unique)

If $\mathrm{rk}_{\mathbb{Z}}(Y(T)^w)$ is maximal, say $ccl(w) =:$ <u>niveau of $\rho$</u>.
$(= \#$ orbits of $w)$ $\quad$ among all possible expressions

In examples, niveau is $ccl((123)) \iff a + pb + p^2c \not\equiv 0 \ (p^2+p+1)$
$ccl(1)$, otherwise

## Ingredients for the conj:

(i) Given $\rho$ tame at $p$:

$V(\rho|_{I_p})$ is a certain Deligne-Lusztig rep. of $GL_n(\mathbb{F}_p)$ over $\overline{\mathbb{Q}}_p$.

(genuine rep. up to sign, parabolic ind. of cuspidal)

(ii) $\mathcal{R}$ : {Serre weights} $\longrightarrow$ {regular Serre wts.} $\cong (\mathbb{Z}/(p-1))^n$

$F(a_1, \dots, a_n) \mapsto (\bar{a}_1, \dots, \bar{a}_n)$.

$$\mathcal{R}(F(b_1, \dots, b_n)) := (\overline{b_n - (n-1)}, \dots, \overline{b_2 - 1}, \bar{b}_1).$$

**Conj. (H.)** If $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{F}}_p)$ odd, irred. and $\rho|_{I_p}$ tame, then

$$W_{reg}(\rho) = \underbrace{\mathcal{R}(JH(\overline{V(\rho|_{I_p})}))}_{=: W^?(\rho).}$$

regular Serre wts. in $W(\rho)$

→ prev. conjecture by Ash, Doud, Pollack, Sinnott for $GL_3$ (fewer wts. predicted, computations)

→ evidence:
- computations (Doud-Pollack)
- examples of $\rho$ for $\underbrace{GL_3}_{\text{Ash et al}}$, $GL_4$ modular of certain predicted weight
- companion forms for $GSp_4$. (Tilouine)

Generic case: (use Jantzen)

Assume $\rho|_{I_p}$ tame and $(\rho|_{I_p}$ generic or $\nu$ generic$)$.

Then $F(\nu) \in W^?(\rho) \Leftrightarrow \left(\begin{array}{l} \exists\, \nu' \uparrow \nu \\ \text{dom. restricted} \\ \text{s.t. } \rho|_{I_p} \sim \tau(w, \nu' + \tilde{\rho}) \text{ for some } w \in W \end{array}\right.$

Terminology: $\nu' = \nu$: $F(\nu)$ is "obvious" weight (there are $\#W = n!$)

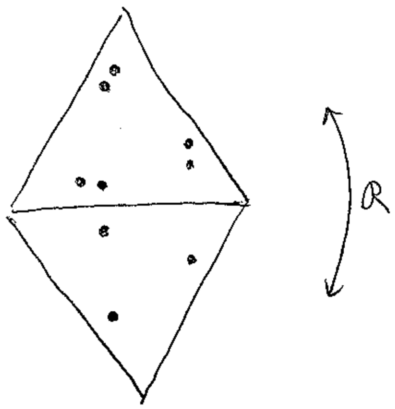
$\nu' \neq \nu$: $F(\nu)$ is "shadow" weight

Examples:

$GL_2$:


2 obvious

$GL_3$:




6 obvious
3 shadow