

**ON THE EXISTENCE OF ADMISSIBLE SUPERSINGULAR  
REPRESENTATIONS OF  $p$ -ADIC REDUCTIVE GROUPS  
(PRELIMINARY VERSION)**

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ABSTRACT. Suppose that  $\mathbf{G}$  is a connected reductive group over a finite extension  $F/\mathbb{Q}_p$ , and that  $C$  is a field of characteristic  $p$ . We prove that the group  $\mathbf{G}(F)$  admits an irreducible admissible supercuspidal, or equivalently supersingular, representation over  $C$ .

CONTENTS

1. Introduction	2
1.1. Acknowledgements	4
1.2. Notation	4
2. Iwahori–Hecke algebras	5
2.1. Definitions	5
2.2. Dominant monoids	6
2.3. Bernstein elements	8
2.4. Supersingular modules	8
3. On supercuspidal representations	9
3.1. Supercuspidality criterion	9
3.2. Change of coefficient field	11
3.3. Reduction to an absolutely simple adjoint group	13
4. Proof of the main theorem for most simple groups	18
4.1. Discrete Iwahori–Hecke modules	18
4.2. Characters	20
4.3. Discrete simple modules with supersingular reduction	22
4.4. Admissible integral structure via discrete cocompact subgroups	27
4.5. Reduction to rank 1 and $\mathrm{PGL}_n(D)$	29
5. Supercuspidal representations of rank 1 groups	30
5.1. Preliminaries	30
5.2. Parahoric subgroups	30
5.3. Pro- $p$ Iwahori–Hecke algebras	31
5.4. Diagrams	32
5.5. Supercuspidal representations via homology	35
6. Supersingular representations of $\mathrm{PGL}_n(D)$	36
6.1. Notation and conventions	36

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6.2. On the Jacquet–Langlands correspondence	37
6.3. On lifting non-supersingular Hecke modules	37
6.4. A reducibility lemma	42
6.5. On base change and descent for compact unitary groups	43
6.6. Supersingular representations of $\mathrm{GL}_n(D)$	43
Appendix A. Base change (by Sug Woo Shin)	49
References	52

## 1. INTRODUCTION

Suppose that  $F$  is a non-archimedean field of residue characteristic  $p$  and that  $\mathbf{G}$  is any connected reductive algebraic group over  $F$ . There has been a growing interest in understanding the smooth representation theory of the  $p$ -adic group  $G := \mathbf{G}(F)$  over a field  $C$  of characteristic  $p$ , going back to the work of Barthel–Livné [BL94] and Breuil [Bre03] in the case of  $\mathbf{G} = \mathbf{GL}_2$ . The latter work in particular demonstrated the relevance of the mod  $p$  representation theory of  $p$ -adic reductive groups to the  $p$ -adic Langlands programme.

The results of [AHHV17] when  $C$  is algebraically closed and [HV] in general give a classification of irreducible admissible representations in terms of supercuspidal  $C$ -representations of Levi subgroups of  $G$ . Here, an irreducible admissible smooth representation  $\pi$  is said to be *supercuspidal* if it does not occur as subquotient of any parabolic induction  $\mathrm{Ind}_P^G \sigma$ , where  $P$  is a proper parabolic subgroup of  $G$  and  $\sigma$  an irreducible admissible representation of the Levi quotient of  $P$ . Unfortunately, so far, the supercuspidal representations themselves remain mostly mysterious, outside anisotropic groups,  $\mathrm{GL}_2(\mathbb{Q}_p)$  [BL94], [Bre03], and some related cases ([Abd14], [Che13], [Koz16], [KX15]). If  $F/\mathbb{Q}_p$  is a non-trivial unramified extension, then irreducible supercuspidal representations of  $\mathrm{GL}_2(F)$  were first constructed by Paškūnas [Paš04]; however, it seems hopelessly complicated to classify them [BP12], [Hu10]. One additional challenge in constructing supercuspidal representations is that irreducible smooth representations need not be admissible in general (unlike what happens over  $\mathbb{C}$ ), as was shown recently by Daniel Le [Le].

There are two ways to characterise supercuspidality in terms of Hecke actions. The first description assumes  $C$  is algebraically closed and uses weights and Hecke eigenvalues for any fixed choice  $K$  of special parahoric subgroup (a weight is then an irreducible representation of  $K$ ). It was shown to be equivalent to supercuspidality in [AHHV17]. The second description uses the centre of the pro- $p$  Iwahori–Hecke algebra. The equivalence between the second Hecke description and supercuspidality was shown in [OV] when  $C$  is algebraically closed and by [HV] in general. In either description, supercuspidality is characterised by the vanishing of certain Hecke operators, which is why supercuspidal representations are also called *supersingular*.

Suppose from now on that  $F$  is of characteristic zero, i.e. that  $F/\mathbb{Q}_p$  is a finite extension.

**Theorem A.** *If  $\mathbf{G}$  is any connected reductive algebraic group over  $F$  and  $C$  any field of characteristic  $p$ , then  $G$  admits an irreducible admissible supersingular, or equivalently supercuspidal, representation over  $C$ .*

This theorem is new outside the low rank cases mentioned above.

We now briefly explain our argument, which uses several completely different ideas. First, in Section 3 we reduce to the cases where  $C$  is finite and  $\mathbf{G}$  is absolutely simple adjoint. If  $\mathbf{G}$  is moreover anisotropic, then  $G$  is compact and any irreducible smooth representation of  $G$  is finite-dimensional (hence admissible) and supercuspidal. If  $\mathbf{G}$  is isotropic, we distinguish three cases.

For most groups  $\mathbf{G}$  we show in Section 4 that there exists a discrete series representation  $\pi$  of  $G$  over  $\mathbb{C}$  that admits invariants under an Iwahori subgroup  $\mathfrak{B}$ , and that has moreover the following property: the module  $\pi^{\mathfrak{B}}$  of the Iwahori–Hecke algebra  $H(G, \mathfrak{B})$  admits a  $\mathbb{Z}[q^{1/2}]$ -integral structure whose reduction modulo the maximal ideal of  $\mathbb{Z}[q^{1/2}]$  with residue field  $\mathbb{F}_p$  is supersingular. The Hecke modules  $\pi^{\mathfrak{B}}$  are constructed either from characters (using [Bor76]) or reflection modules (using [Lus83] and [GS05]; the latter is needed to handle unramified non-split forms of  $\mathbf{PSO}_8$ ).

The  $p$ -adic version of the de George–Wallach limit multiplicity formula ([DKV84, App. 3] plus [Kaz86, Thm. K]) implies that the representation  $\pi$  above embeds in  $C^\infty(\Gamma \backslash G, \mathbb{C})$  for some discrete cocompact subgroup  $\Gamma$  of  $G$ . By construction we deduce that the Hecke module  $C^\infty(\Gamma \backslash G / \mathfrak{B}, \mathbb{F}_p) = C^\infty(\Gamma \backslash G, \mathbb{F}_p)^{\mathfrak{B}}$  of  $\mathfrak{B}$ -invariants admits a supersingular submodule. Crucially, by cocompactness of  $\Gamma$  we know that  $C^\infty(\Gamma \backslash G, \mathbb{F}_p)$  is an admissible representation of  $G$ . Picking any non-zero supersingular vector  $v \in C^\infty(\Gamma \backslash G / \mathfrak{B}, \mathbb{F}_p)$ , the  $G$ -subrepresentation of  $C^\infty(\Gamma \backslash G, \mathbb{F}_p)$  generated by  $v$  admits an irreducible quotient, which is admissible (as  $\text{char } F = 0$ ) and supersingular.

Unfortunately, this argument does not work for all groups  $\mathbf{G}$ . We have the following exceptional cases:

- (i)  $\text{PGL}_n(D)$ , where  $n \geq 2$  and  $D$  a central division algebra over  $F$ ;
- (ii)  $\text{PU}(h)$ , where  $h$  is a split hermitian form in 3 variables over a ramified quadratic extension of  $F$  or a non-split hermitian form in 4 variables over the unramified quadratic extension of  $F$ .

Note that for the group  $\text{PGL}_n(D)$  with  $n \geq 2$  the only discrete series representations  $\pi$  having  $\mathfrak{B}$ -invariant vectors are the unramified twists of the Steinberg representation (by Proposition 4.1.4(i) and the classification of Bernstein–Zelevinsky and Tadić), but then  $\pi^{\mathfrak{B}}$  is one-dimensional with non-supersingular reduction.

In the second exceptional case, where  $G \cong \text{PU}(h)$  for certain hermitian forms  $h$ , we use the theory of coefficient systems and diagrams, building on ideas of Paškūnas [Paš04] (in turn building on earlier work of Schneider–Stuhler and Ronan–Smith). See Section 5. Note that  $\mathbf{G}$  is of relative rank 1, so the reduced Bruhat–Tits building of  $G$  is a tree, and our method works for all such groups. In order to carry it out, we may apply the reductions in Section 3 and assume that  $\mathbf{G}$  is absolutely simple and simply connected. Given a supersingular module  $\Xi$  for the *pro- $p$*  Iwahori–Hecke algebra of  $G$ , we naturally construct a  $G$ -equivariant coefficient system  $\mathcal{D}_\Xi$  on the Bruhat–Tits tree of  $G$ . The homology of  $\mathcal{D}_\Xi$  admits a smooth  $G$ -action, and any irreducible admissible quotient will be supersingular (by Proposition 3.1.2). To construct such a quotient, we define an auxiliary coefficient system  $\mathcal{D}'$ , which is built out of injective envelopes of representations of certain parahoric subgroups, along with a morphism  $\mathcal{D}_\Xi \rightarrow \mathcal{D}'$ . The image of the induced map on homology is admissible, and admits an irreducible quotient  $\pi'$  which is itself admissible (since  $\text{char } F = 0$ ) and supersingular.

In the first exceptional case, where  $G \cong \mathrm{PGL}_n(D)$ , we use a global method (see Section 6). We find a totally real number field  $\tilde{F}^+$  and a compact unitary group  $\underline{G}$  over  $\tilde{F}^+$  such that  $\underline{G}(\tilde{F}_v^+)$  is isomorphic to  $\mathrm{GL}_n(D)$  for a suitable place  $v|p$  of  $\tilde{F}^+$ . Then, fixing a level away from  $v$  and taking the limit over all levels at  $v$ , the space  $S$  of algebraic automorphic forms of  $\underline{G}(\mathbb{A}_{\tilde{F}^+}^\infty)$  over  $\overline{\mathbb{F}}_p$  affords an admissible smooth action of  $\underline{G}(\tilde{F}_v^+)$ . Using automorphic induction and descent we construct an automorphic representation  $\pi$  of  $\underline{G}(\mathbb{A}_{\tilde{F}^+})$  whose associated Galois representation  $r_\pi$  has the property that its reduction modulo  $p$  is irreducible locally at  $v$ . From  $\pi$  we get a maximal ideal  $\mathfrak{m}$  in the Hecke algebra (at good places outside  $p$ ), and we claim that any irreducible subrepresentation of the localisation  $S_{\mathfrak{m}}$  is supercuspidal.

To prove the claim, we use the pro- $p$  Iwahori–Hecke criterion for supercuspidality and argue by contradiction. If one of the relevant Hecke operators has a non-zero eigenvalue, we lift to characteristic zero by a Deligne–Serre argument and construct an automorphic representation  $\pi'$  with Galois representation  $r_{\pi'}$  having the same reduction  $\bar{r}_\pi$  modulo  $p$ . Using local-global compatibility at  $p$  for  $r_{\pi'}$  and some basic  $p$ -adic Hodge theory we show that the non-zero Hecke eigenvalue in characteristic  $p$  implies that  $r_{\pi'}$  is reducible locally at  $v$ , obtaining the desired contradiction.

For our automorphic base change and descent argument we require results going slightly beyond [Lab11], since our group  $\underline{G}$  is typically not quasi-split at all finite places. In the appendix, Sug Woo Shin explains the necessary modifications.

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**1.2. Notation.** Fix a prime number  $p$ , and let  $F$  be a non-archimedean local field of residue characteristic  $p$  (we will later assume that  $\mathrm{char} F = 0$ , i.e. that  $F$  is a finite extension of  $\mathbb{Q}_p$ ). The field  $F$  comes equipped with ring of integers  $\mathcal{O}_F$  and residue field  $k_F$  of cardinality  $q$ , a power of  $p$ . We fix a uniformizer  $\varpi$ , and let  $|\cdot|_F$  denote the normalised absolute value of  $F$ .

If  $\mathbf{H}$  is an algebraic  $F$ -group, we denote by  $H$  its group of  $F$ -points  $\mathbf{H}(F)$ .

Let  $\mathbf{G}$  be a connected reductive  $F$ -group,  $\mathbf{T}$  a maximal  $F$ -split subtorus of  $\mathbf{G}$ ,  $\mathbf{B}$  a minimal  $F$ -parabolic subgroup of  $\mathbf{G}$  containing  $\mathbf{T}$ , and  $x_0$  a special point of the apartment of the adjoint Bruhat–Tits building defined by  $\mathbf{T}$ . We associate to  $x_0$  and the triple  $(\mathbf{G}, \mathbf{T}, \mathbf{B})$  the following data:

- the center  $\mathbf{Z}(\mathbf{G})$  of  $\mathbf{G}$ ,
- the root system  $\Phi \subset X^*(T)$ ,
- the set of simple roots  $\Delta \subset \Phi$ ,
- the centralizer  $\mathbf{Z}$  of  $\mathbf{T}$ ,
- the normalizer  $\mathcal{N}$  of  $\mathbf{T}$ ,
- the unipotent radical  $\mathbf{U}$  of  $\mathbf{B}$  (hence  $\mathbf{B} = \mathbf{Z}\mathbf{U}$ ), and the opposite unipotent radical  $\mathbf{U}^{\mathrm{op}}$ ,
- the triples  $(\mathbf{G}^{\mathrm{sc}}, \mathbf{T}^{\mathrm{sc}}, \mathbf{B}^{\mathrm{sc}})$  and  $(\mathbf{G}^{\mathrm{ad}}, \mathbf{T}^{\mathrm{ad}}, \mathbf{B}^{\mathrm{ad}})$ , corresponding to the simply-connected covering of the derived subgroup and the adjoint group of  $\mathbf{G}$ ,
- the apartment  $\mathcal{A} := X_*(T)/X_*(Z(G)^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$  associated to  $\mathbf{T}$  in the semisimple Bruhat–Tits building,
- the alcove  $\mathcal{C}$  of  $\mathcal{A}$  with vertex  $x_0$  lying in the dominant Weyl chamber with vertex  $x_0$ ,

- the Iwahori subgroups  $\mathfrak{B}$  and  $\mathfrak{B}^{\text{sc}}$  of  $G$  and  $G^{\text{sc}}$ , respectively, fixing  $\mathcal{C}$  pointwise,
- the pro- $p$ -Sylow subgroup  $\mathfrak{U}$  of  $\mathfrak{B}$ .

We fix a field  $C$  of characteristic  $c \in \{0, 2, 3, 5, 7, \dots\}$ , which will serve as the field of coefficients for the modules and representations appearing below. In our main result we will assume  $c = p$ .

Suppose  $K$  is a compact open subgroup of  $G$ , and  $R$  is a commutative ring. We define the Hecke algebra associated to this data to be the  $R$ -algebra

$$H_R(G, K) := \text{End}_G R[K \backslash G].$$

If  $R = \mathbb{Z}$ , we simply write  $H(G, K)$ . In our applications below, we will often assume that  $K = \mathfrak{B}$  or  $K = \mathfrak{U}$ .

Given a module or algebra  $X$  over some ring  $R$  and a ring extension  $R'/R$ , we let  $X_{R'} := X \otimes_R R'$  denote the extension of scalars.

Other notation will be introduced as necessary in subsequent sections.

## 2. IWAHORI–HECKE ALGEBRAS

In this section we review some basic facts concerning Iwahori–Hecke algebras and their (supersingular) modules. We will use these algebras extensively in our construction of supercuspidal  $G$ -representations. See [Vig16], [Vig14], and [Vig17] for references.

**2.1. Definitions.** Recall that we have defined the Iwahori–Hecke ring as

$$H(G, \mathfrak{B}) = \text{End}_G \mathbb{Z}[\mathfrak{B} \backslash G].$$

We have an analogous ring  $H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}})$  for the simply-connected group. The natural ring homomorphism  $H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}}) \rightarrow H(G, \mathfrak{B})$  (induced by the covering  $G^{\text{sc}} \rightarrow G$  of the derived subgroup) is injective, so we identify  $H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}})$  with a subring of  $H(G, \mathfrak{B})$ . We first discuss presentations for these rings.

There is a canonical isomorphism

$$j^{\text{sc}} : H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}}) \xrightarrow{\sim} H(W, S, q_s),$$

where  $H := H(W, S, q_s)$  is the Hecke ring of an affine Coxeter system  $(W, S)$  with parameters  $\{q_s := q^{d_s}\}_{s \in S}$ . The  $d_s$  are positive integers, which we will abusively also refer to as the parameters of  $G$ . Thus,  $H(W, S, q_s)$  is a free  $\mathbb{Z}$ -module with basis  $\{T_w\}_{w \in W}$ , satisfying the braid and quadratic relations:

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{for } w, w' \in W, \ell(w) + \ell(w') = \ell(ww'), \\ (T_s - q_s)(T_s + 1) &= 0 && \text{for } s \in S. \end{aligned}$$

Here  $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$  denotes the length function with respect to  $S$ . We identify  $H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}})$  with  $H$  via  $j^{\text{sc}}$ .

In order to describe  $H(G, \mathfrak{B})$ , we require a larger affine Weyl group. The unique parahoric subgroup  $Z_0$  of  $Z$  is contained in the maximal compact subgroup  $\tilde{Z}_0$  of  $Z$ . (When the group  $\mathbf{G}$  is  $F$ -split or semisimple and simply connected, we have  $Z_0 = \tilde{Z}_0$ .) We define the *extended affine Weyl group* to be

$$\widetilde{W} := \mathcal{N}/Z_0.$$

The group  $\widetilde{W}$  acts on the apartment  $\mathcal{A}$ , and permutes the alcoves of  $\mathcal{A}$  transitively. We let  $\Omega$  denote the subgroup of  $\widetilde{W}$  stabilizing  $\mathcal{C}$ . The affine Weyl group  $W$  is isomorphic

to a normal subgroup of  $\widetilde{W}$ , and permutes the alcoves *simply* transitively. We therefore have a semidirect product decomposition

$$\widetilde{W} = W \rtimes \Omega.$$

The function  $\ell$  extends to  $\widetilde{W}$  by setting  $\ell(uw) = \ell(wu) = \ell(w)$  for  $u \in \Omega, w \in W$ .

Let  $\Sigma$  denote the reduced root system whose extended Dynkin diagram  $\text{Dyn}$  is equal to the Dynkin diagram of  $(W, S)$ , and let  $\text{Dyn}'$  denote the Dynkin diagram  $\text{Dyn}$  decorated with the parameters  $\{d_s\}_{s \in S}$ . The quotient of  $\Omega$  by the pointwise stabilizer of  $\mathcal{C}$  in  $\Omega$  is isomorphic to a subgroup  $\Psi$  of the group of automorphisms  $\text{Aut}(W, S, d_s)$  of  $\text{Dyn}'$ . Thus,  $\Omega$  acts on  $\text{Dyn}'$  and consequently on  $H(W, S, q_s)$ , and the isomorphism  $j^{\text{sc}}$  extends to an isomorphism

$$(2.1.1) \quad j : H(G, \mathfrak{B}) \xrightarrow{\sim} \mathbb{Z}[\Omega] \widetilde{\otimes} H(W, S, q_s),$$

where  $\widetilde{\otimes}$  denotes the twisted tensor product. The generalized affine Hecke ring  $\widetilde{H} := \mathbb{Z}[\Omega] \widetilde{\otimes} H(W, S, q_s)$  as above is the free  $\mathbb{Z}$ -module with basis  $\{T_w\}_{w \in \widetilde{W}}$ , satisfying the braid and quadratic relations:

$$(2.1.2) \quad T_w T_{w'} = T_{ww'} \quad \text{for } w, w' \in \widetilde{W}, \ell(w) + \ell(w') = \ell(ww'),$$

$$(2.1.3) \quad (T_s - q_s)(T_s + 1) = 0 \quad \text{for } s \in S.$$

The group  $\widetilde{W}$  forms a system of representatives for the space of double cosets  $\mathfrak{B} \backslash G / \mathfrak{B}$ . Under the isomorphism  $j$ , the element  $T_w \in \widetilde{H}$  for  $w \in \widetilde{W}$  corresponds to the endomorphism sending the characteristic function of  $\mathfrak{B}$  to the characteristic function of  $\mathfrak{B}n\mathfrak{B}$ , where  $n \in \mathcal{N}$  lifts  $w$ .

Finally, let  $\mathbf{P} = \mathbf{M}\mathbf{N}$  denote a standard parabolic  $F$ -subgroup of  $\mathbf{G}$ , and suppose that  $\mathbf{M}$  contains  $\mathbf{T}$ . Then the group  $M \cap \mathfrak{B}$  is an Iwahori subgroup of  $M$ . We can therefore form the algebra

$$H(M, M \cap \mathfrak{B}) = \text{End}_M \mathbb{Z}[(M \cap \mathfrak{B}) \backslash M].$$

It is *not* a subalgebra of  $H(G, \mathfrak{B})$  in general. The basis of  $H(M, M \cap \mathfrak{B})$  will be denoted  $T_w^M$ , where  $w$  is an element of the extended affine Weyl group associated to  $M$ .

**2.2. Dominant monoids.** The subgroup

$$\Lambda := Z/Z_0$$

of  $\widetilde{W} = \mathcal{N}/Z_0$  is commutative and finitely generated, and its torsion subgroup is equal to  $\widetilde{Z}_0/Z_0$ . It acts by translation on  $\mathcal{A}$ , and the quotient map  $\mathcal{N}/Z_0 \rightarrow \mathcal{N}/Z$  splits, identifying the (finite) Weyl group  $W_0$  of  $\Sigma$  with  $\text{Stab}_W(x_0)$ . We obtain semidirect product decompositions

$$\Lambda \rtimes W_0 = \widetilde{W}$$

and

$$\Lambda^{\text{sc}} \rtimes W_0 = W,$$

where  $\Lambda^{\text{sc}} := \Lambda \cap W$ . An element  $\lambda \in \Lambda$  is called *dominant* (and  $\lambda^{-1}$  is called *anti-dominant*), if

$$z(U \cap \mathfrak{B})z^{-1} \subset U \cap \mathfrak{B}$$

for any  $z \in Z$  which lifts  $\lambda$ . We let  $\Lambda^+$  denote the monoid consisting of dominant elements of  $\Lambda$ , and similarly for any subgroup  $\Lambda' \leq \Lambda$  define  $\Lambda'^+ := \Lambda' \cap \Lambda^+$ . The set of

invertible elements in the dominant monoid  $\Lambda^+$  is exactly the subgroup  $Z(G)\tilde{Z}_0/Z_0 \subset \Lambda^+$ , and the invertible elements of  $\Lambda^{\text{sc},+}$  are trivial.

Given a subgroup  $J$  of  $Z$ , we define

$$\Lambda_J := JZ_0/Z_0 \subset \Lambda.$$

We let  $T_0$  denote the maximal compact subgroup of  $T$ , and note that  $T_0 = Z_0 \cap T$ . The map

$$\begin{aligned} X_*(T) &\xrightarrow{\sim} \Lambda_T \cong T/T_0 \\ \mu &\longmapsto \lambda_\mu := \mu(\varpi)Z_0/Z_0 \end{aligned}$$

is a  $W_0$ -equivariant isomorphism. We say a cocharacter  $\mu$  is *dominant* if  $\lambda_\mu$  is, and let  $X_*(T)^+$  denote the monoid consisting of dominant elements of  $X_*(T)$ . For  $\lambda \in \Lambda$  and a sufficiently large  $n \in \mathbb{N}$ ,  $\lambda^n$  lies in  $\Lambda_T$ , and we let  $\mu_{\lambda^n} \in X_*(T)$  denote the corresponding cocharacter. The pointwise stabilizer of the alcove  $\mathcal{C}$  in  $\Lambda$  is equal to  $\Lambda_{Z(G)\tilde{Z}_0}$ , and this group may be described as

$$\Lambda_{Z(G)\tilde{Z}_0} = \left\{ \lambda \in \Lambda \mid \langle \alpha, \mu_{\lambda^n} \rangle = 0 \text{ for all } n \text{ such that } \lambda^n \in \Lambda_T \text{ and all } \alpha \in \Delta \right\}.$$

**Lemma 2.2.1.** *The pointwise stabilizer of the alcove  $\mathcal{C}$  in  $\Omega$  is  $\Lambda_{Z(G)\tilde{Z}_0}$ .*

*Proof.* Suppose  $u \in \Omega$  fixes  $\mathcal{C}$  pointwise, and write  $u = \lambda w_0$  with  $\lambda \in \Lambda, w_0 \in W_0$ . Since  $w_0$  and  $u$  both fix  $x_0$ ,  $\lambda$  does also. However,  $\lambda$  acts by translation on  $\mathcal{A}$ , so we see that  $\lambda$  must fix  $\mathcal{C}$ , and therefore  $\lambda \in \Lambda_{Z(G)\tilde{Z}_0}$ . Finally, since  $\lambda$  and  $u$  both fix  $\mathcal{C}$ ,  $w_0$  must also fix  $\mathcal{C}$ , which implies  $w_0 = 1$ . The converse is immediate.  $\square$

The action of  $\Omega$  on the alcove  $\mathcal{C}$  induces an isomorphism

$$(2.2.2) \quad \Omega/\Lambda_{Z(G)\tilde{Z}_0} \xrightarrow{\sim} \Psi,$$

and the embeddings of  $\Lambda$  and  $\Omega$  into  $\widetilde{W}$  induce

$$(2.2.3) \quad \Lambda/(\Lambda_{Z(G)\tilde{Z}_0} \times \Lambda^{\text{sc}}) \xrightarrow{\sim} \widetilde{W}/(\Lambda_{Z(G)\tilde{Z}_0} \times W) \xleftarrow{\sim} \Omega/\Lambda_{Z(G)\tilde{Z}_0}.$$

**Lemma 2.2.4.** *The subgroup  $\Lambda_{Z(G)} \times \Lambda^{\text{sc}}$  (resp.  $\Lambda_T$ ) of  $\Lambda$  is finitely generated of finite index. The submonoid  $\Lambda_{Z(G)} \times \Lambda^{\text{sc},+}$  (resp.  $\Lambda_T^+$ ) of the dominant monoid  $\Lambda^+$  is finitely generated of finite index.*

Here, we say that a submonoid  $N$  of a commutative monoid  $M$  has finite index if  $M = \bigcup_{i=1}^n (N + x_i)$  for some  $x_i \in M$ . If  $M$  is finitely generated, then  $dM$  is of finite index in  $M$  for all  $d \geq 1$ .

*Proof.* The group  $\tilde{Z}_0/Z_0$  is finite, and equations (2.2.2) and (2.2.3) imply that  $\Lambda/(\Lambda_{Z(G)\tilde{Z}_0} \times \Lambda^{\text{sc}})$  is isomorphic to the finite group  $\Psi$ . Thus, we see that the commutative group  $\Lambda_{Z(G)} \times \Lambda^{\text{sc}}$  is a finitely generated, finite index subgroup of  $\Lambda$ . On the other hand,  $\Lambda_T$  is finite free and it is well known that it is of finite index in  $\Lambda$ . Gordan's lemma implies the second assertion (as in the proof of [HV15, 7.2 Lem.]).  $\square$

**2.3. Bernstein elements.** Let  $w \in \widetilde{W}$ , and let  $w = us_1 \dots s_n$  be a reduced expression, with  $u \in \Omega$ ,  $s_i \in S$ . We set

$$q_w := q_{s_1} \dots q_{s_n},$$

and define  $T_s^* := T_s - q_s + 1$  and  $T_w^* := T_u T_{s_1}^* \dots T_{s_n}^*$ . Then

$$T_w T_{w^{-1}}^* = q_w,$$

and the linear map defined by  $T_w \mapsto (-1)^{\ell(w)} T_w^*$  is an automorphism of  $\widetilde{H}$ .

Let  $\lambda \in \Lambda \subset \widetilde{W}$ . The  $\widetilde{W}$ -conjugacy class of  $\lambda$  is finite, and consists of the  $W_0$ -orbit of  $\lambda$ . By [Vig14, Thm. 1.2], a basis of the center of  $\widetilde{H}$  is given by the elements

$$\sum_{\lambda \in \mathcal{O}} E_\lambda \quad \text{for all } W_0\text{-orbits } \mathcal{O} \subset \Lambda.$$

Here, the elements  $E_\lambda$  are the integral Bernstein elements of  $\widetilde{H}$  corresponding to the spherical orientation induced by  $\Delta$  ([Vig16, Cor. 5.28, Ex. 5.30]).

Precisely, they are characterized by the relations

$$(2.3.1) \quad E_\lambda = \begin{cases} T_\lambda & \text{if } \lambda \text{ is anti-dominant,} \\ T_\lambda^* & \text{if } \lambda \text{ is dominant,} \end{cases}$$

$$(2.3.2) \quad E_{\lambda_1} E_{\lambda_2} = (q_{\lambda_1} q_{\lambda_2} q_{\lambda_1 \lambda_2}^{-1})^{1/2} E_{\lambda_1 \lambda_2} \quad \text{if } \lambda_1, \lambda_2 \in \Lambda.$$

If  $\lambda_1, \lambda_2$  are both dominant (or anti-dominant), then  $E_{\lambda_1} E_{\lambda_2} = E_{\lambda_1 \lambda_2}$ . Given  $\mu \in X_*(T)$ , we let  $\mathcal{O}_\mu \subset \Lambda$  denote the  $W_0$ -orbit of  $\lambda_\mu$ , and define the central element

$$z_\mu := \sum_{\lambda \in \mathcal{O}_\mu} E_\lambda.$$

When  $\mu \in X_*(T^{\text{sc}})$ , the element  $z_\mu$  lies in  $H$ .

The generalized affine Hecke ring  $\widetilde{H}$  contains a commutative subring  $\mathcal{A}$  with  $\mathbb{Z}$ -basis  $\{E_\lambda\}_{\lambda \in \Lambda}$ . When  $G = Z$ , the Bernstein elements  $E_\lambda^Z$  are simply the classical elements  $T_\lambda^Z$ , and the Iwahori–Hecke ring  $H(Z, Z_0)$  and the subring  $\mathcal{A}$  are both isomorphic to  $\mathbb{Z}[\Lambda]$ . In general,  $\mathcal{A}$  is not isomorphic to  $\mathbb{Z}[\Lambda]$ , but the subring  $\mathcal{A}^+$  which has basis  $\{E_\lambda\}_{\lambda \in \Lambda^+}$  is isomorphic to  $\mathbb{Z}[\Lambda^+]$ . Denote by  $\mathcal{A}_T$  the subring whose basis is  $\{E_{\lambda_\mu}\}_{\mu \in X_*(T)}$ .

**2.4. Supersingular modules.** We now discuss supersingular Hecke modules.

Recall that  $C$  is our coefficient field. We define  $H_C := H \otimes_{\mathbb{Z}} C$  and  $\widetilde{H}_C := \widetilde{H} \otimes_{\mathbb{Z}} C$ , which are isomorphic to the Iwahori–Hecke algebras  $H_C(G^{\text{sc}}, \mathfrak{B}^{\text{sc}})$  and  $H_C(G, \mathfrak{B})$ , respectively. For simplicity, we suppose in this subsection that  $(W, S)$  is irreducible.

**Definition 2.4.1.** Let  $M$  be a non-zero right  $\widetilde{H}_C$ -module. An element  $v \in M$  is called *supersingular* if  $v \cdot z_\mu^n = 0$  for all  $\mu \in X_*(T)^+$  such that  $-\mu \notin X_*(T)^+$ , and all sufficiently large  $n$ . The  $\widetilde{H}_C$ -module  $M$  is called *supersingular* if all its elements are supersingular<sup>1</sup>. We make a similar definition for modules over  $H_C$ , using the monoid  $X_*(T^{\text{sc}})^+$ .

**Lemma 2.4.2.**

<sup>1</sup>This is slightly different from [Vig17, Def. 6.10]; the definition given there requires that  $M \cdot z_\mu^n = 0$  for all  $\mu \in X_*(T)^+$  such that  $-\mu \notin X_*(T)^+$  and  $n$  sufficiently large.



- (i) Any simple  $\tilde{H}_C$ -module is finite dimensional, and is semisimple as an  $H_C$ -module.
- (ii) When  $c \nmid p|W_0|$ ,  $\tilde{H}_C$  does not admit any simple supersingular modules.
- (iii) When  $c = p$ , a simple  $\tilde{H}_C$ -module is supersingular if and only if its restriction to  $H_C$  is supersingular.
- (iv) When  $c = p$ , the simple supersingular  $H_C$ -modules are those characters which are not special or trivial (see Section 4.2).

*Proof.* (i) The first statement follows from [Vig07, §5.3]. For the second part, note that there exists a finite index subgroup  $\Omega'$  of  $\Omega$  which acts trivially on  $H$  (for example, we may take  $\Omega' = \Lambda_{Z(G)\tilde{z}_0}$ ). Set  $H'_C := C[\Omega'] \otimes_C H_C$ . Any simple  $H_C$ -module  $N$  extends trivially to an  $H'_C$ -module  $N'$ , and the restriction of  $N' \otimes_{H'_C} \tilde{H}_C$  to  $H_C$  is a finite direct sum  $\bigoplus_{u \in \Omega/\Omega'} N^u$  of (simple) conjugates  $N^u$  of  $N$  by elements  $u \in \Omega$ . If  $M$  is a simple  $\tilde{H}_C$ -module and  $N$  is contained in  $M|_{H_C}$ , then  $M$  is a quotient of  $N' \otimes_{H'_C} \tilde{H}_C$  (and thus the restriction of  $M$  is semisimple).

(ii) Let  $M$  denote a simple supersingular module. Since  $\mathcal{A}_{T,C}$  is commutative and  $M$  is finite dimensional, there exists a one-dimensional submodule, given by a character  $\chi : \mathcal{A}_{T,C} \rightarrow C$ . Let  $v'$  denote a basis for the underlying vector space. Supersingularity implies that if  $\mu \in X_*(T)^+$  with  $-\mu \notin X_*(T)^+$ , then

$$(2.4.3) \quad 0 = v' \cdot z_\mu = \chi(z_\mu)v'.$$

On the other hand, the product formula 2.3.2 gives

$$z_\mu z_{-\mu} = q_{\lambda_\mu} |\mathcal{O}_\mu| z_0 + \sum_{\substack{\mu' \in X_*(T)^+ \\ q_{\lambda_{\mu'}} > 1}} a_{\mu'} z_{\mu'}$$

for some  $a_{\mu'} \in C$ ; applying  $\chi$  to both sides and using (2.4.3) (for varying  $\mu'$ ) gives  $q_{\lambda_\mu} |\mathcal{O}_\mu| = 0$ , a contradiction.

(iii) This follows from [Vig17, Cor. 6.13] and part (i).

(iv) This follows from [Vig17, Thm. 6.15].

□

### 3. ON SUPERCUSPIDAL REPRESENTATIONS

**3.1. Supercuspidality criterion.** In this section we collect various results concerning supercuspidal  $C$ -representations of  $G$ . We also state the key Proposition 3.1.2, which will be used to check that the representations we construct below are supercuspidal.

We first describe the scalar extension of an irreducible admissible  $C$ -representation  $\pi$  of  $G$  [HV]. Given such a  $\pi$ , the commutant  $D := \text{End}_C(\pi)$  is a division algebra of finite dimension over  $C$ . Let  $E$  denote the center of  $D$ ,  $E_s/C$  the maximal separable extension contained in  $E/C$  and  $\delta$  the reduced degree of  $D/E$ . Let  $L^{\text{alg}}$  be an algebraically closed field containing  $E$  and  $\pi_{L^{\text{alg}}}$  the scalar extension of  $\pi$  from  $C$  to  $L^{\text{alg}}$ .

**Proposition 3.1.1** ([HV, Thms. I.1, III.4]). *The length of  $\pi_{L^{\text{alg}}}$  is  $\delta[E : C]$  and*

$$\pi_{L^{\text{alg}}} \cong \bigoplus_{i \in \text{Hom}_C(E_s, L^{\text{alg}})} \pi_i^{\oplus \delta}$$

where each  $\pi_i$  is indecomposable with commutant  $L^{\text{alg}} \otimes_{i, E_s} E$ , descends to a finite extension  $C'$  of  $C$ , has length  $[E : E_s]$ , and its irreducible subquotients are pairwise isomorphic, say to  $\rho_i$ . The  $\rho_i$  are admissible, with commutant  $L^{\text{alg}}$ ,  $\text{Aut}_C(L^{\text{alg}})$ -conjugate, pairwise non-isomorphic, and descend to a finite extension of  $C$ . Any descent of  $\rho_i$  to a finite extension  $C'/C$ , viewed as  $C$ -representation of  $G$ , is  $\pi$ -isotypic of finite length.

*Proof.* By [HV, Thms. I.1, III.4], it suffices to prove that if  $\rho_i$  descends to a  $C'$ -representation  $\rho'_i$  with  $C'/C$  finite, then  $\rho'_i$  is  $\pi$ -isotypic of finite length. Then  $(\rho'_i)_{L^{\text{alg}}}$  injects into  $\pi_{L^{\text{alg}}}$ , and so  $\rho'_i$  injects into  $\pi_{C'}$  by [HV, Rk. II.2], which implies the claim.  $\square$

In particular, any irreducible admissible  $C$ -representation  $\pi$  with commutant  $C$  is absolutely irreducible in the sense that its base change  $\pi_L$  is irreducible for any field extension  $L/C$ . For example, this holds when  $C$  is algebraically closed.

Given an irreducible admissible  $C$ -representation  $\pi$ , the space  $\pi^{\mathfrak{U}}$  of  $\mathfrak{U}$ -invariants comes equipped with an action of the pro- $p$  Iwahori–Hecke algebra  $H_C(G, \mathfrak{U})$ . This algebra has a similar structure to that of  $H_C(G, \mathfrak{B})$ . In particular, we have analogous definitions of the Bernstein elements  $E_\lambda$  and the central elements  $z_\mu$ , as well as an analogous notion of supersingularity for (right)  $H_C(G, \mathfrak{U})$ -modules (cf. Definition 2.4.1). We say an irreducible admissible  $C$ -representation  $\pi$  is *supersingular* if the right  $H_C(G, \mathfrak{U})$ -module  $\pi^{\mathfrak{U}}$  is supersingular.

Finally, recall that an irreducible admissible  $C$ -representation  $\pi$  of  $G$  is said to be *supercuspidal* if it is not a subquotient of  $\text{Ind}_P^G \tau$  for any parabolic subgroup  $P = MN \subsetneq G$  and any irreducible admissible representation  $\tau$  of the Levi quotient  $M$ .

**Proposition 3.1.2** (Supercuspidality criterion). *Assume  $c = p$ . Suppose that  $\pi$  is an irreducible admissible  $C$ -representation of  $G$ . The following are equivalent:*

- (i)  $\pi$  is supercuspidal;
- (ii)  $\pi$  is supersingular;
- (iii)  $\pi^{\mathfrak{U}}$  contains a non-zero supersingular element;
- (iv) every subquotient of  $\pi^{\mathfrak{U}}$  is supersingular;
- (v) some subquotient of  $\pi^{\mathfrak{U}}$  is supersingular.

*Proof.* We have (i)  $\Leftrightarrow$  (ii)  $\Leftrightarrow$  (iii) by [HV, Thms. I.13, III.17]. Since (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v), it suffices to show that (v)  $\Rightarrow$  (ii). Let  $C^{\text{alg}}$  denote an algebraic closure of  $C$ . Say  $\pi^{\mathfrak{U}}$  has supersingular subquotient  $M$ . Then  $(\pi^{\mathfrak{U}})_{C^{\text{alg}}} \cong (\pi_{C^{\text{alg}}})^{\mathfrak{U}}$  has subquotient  $M_{C^{\text{alg}}}$ , and  $M_{C^{\text{alg}}}$  is clearly supersingular. By Proposition 3.1.1 there exists an irreducible admissible constituent  $\rho$  of  $\pi_{C^{\text{alg}}}$  such that the  $H_C(G, \mathfrak{U})$ -module  $\rho^{\mathfrak{U}}$  shares an irreducible constituent with  $M_{C^{\text{alg}}}$ . In particular,  $\rho^{\mathfrak{U}}$  has a supersingular subquotient, and [OV, Thm. 3] implies  $\rho$  is supersingular. Then [HV, Lem. III.16 2)] implies that  $\pi$  is supersingular.  $\square$

We now discuss how supercuspidality behaves under extension of scalars. We require a preliminary lemma.

**Lemma 3.1.3.** *Suppose that  $C'/C$  is a finite extension and that  $\pi'$  is an irreducible admissible  $C'$ -representation of  $G$ . Then  $\pi'|_{C[G]} \cong \pi^{\oplus n}$  for some irreducible admissible  $C$ -representation of  $G$  and some  $n \geq 1$ .*

*Proof.* Let  $C^{\text{alg}}$  be an algebraic closure of  $C$ . Then the finite-dimensional  $C^{\text{alg}}$ -algebra  $A := C' \otimes_C C^{\text{alg}}$  is of finite length over itself. The simple  $A$ -modules are given by

$C^{\text{alg}}$  with  $C'$  acting via the various  $C$ -embeddings  $C' \rightarrow C^{\text{alg}}$ . It follows that  $\pi'|_{C[G]} \otimes_C C^{\text{alg}} \cong \pi' \otimes_{C'} A$  is of finite length as  $C^{\text{alg}}$ -representation by Proposition 3.1.1. So  $\pi'|_{C[G]}$  is of finite length. If  $\pi$  denotes an irreducible submodule, then  $\sum_i \lambda_i \pi = \pi'|_{C[G]}$ , where  $(\lambda_i)_{i=1}^m$  is a basis of  $C'/C$ . It follows that  $\pi'|_{C[G]} \cong \pi^{\oplus n}$  for some  $n \leq m$ . Moreover  $\pi$  is admissible, as  $\pi'|_{C[G]}$  is.  $\square$

**Proposition 3.1.4.** *Let  $L^{\text{alg}}$  denote an algebraically closed field containing  $C$ . If  $c \neq p$ , we assume that  $L^{\text{alg}} = C^{\text{alg}}$  is an algebraic closure of  $C$ .*

*A  $C$ -representation  $\pi$  is supercuspidal if and only if some irreducible subquotient  $\rho$  of  $\pi_{L^{\text{alg}}}$  is supercuspidal, if and only if every irreducible subquotient  $\rho$  of  $\pi_{L^{\text{alg}}}$  is supercuspidal.*

*Proof.* If  $c = p$ , we note that  $\pi$  is supercuspidal if and only if  $\pi$  is supersingular by Proposition 3.1.2. This is equivalent to some/every subquotient of  $\pi_{L^{\text{alg}}}$  being supersingular [HV, Lem. III.16 2)], or equivalently supercuspidal (again by [HV, Thm. I.13]).

Now suppose that  $c \neq p$  and  $L^{\text{alg}} = C^{\text{alg}}$ . Recall that parabolic induction  $\text{Ind}_P^G$  is exact, and commutes with scalar extensions and restrictions [HV, Prop. III.12(i)]. If  $\pi$  is not supercuspidal, then  $\pi$  is a subquotient of  $\text{Ind}_P^G \tau$  for some proper parabolic  $P = MN$  and irreducible admissible  $C$ -representation  $\tau$  of  $M$ . Then  $\pi_{C^{\text{alg}}}$  is a subquotient of  $(\text{Ind}_P^G \tau)_{C^{\text{alg}}} \cong \text{Ind}_P^G(\tau_{C^{\text{alg}}})$ . In particular, each irreducible (admissible) subquotient  $\pi'$  of  $\pi_{C^{\text{alg}}}$  is a subquotient of  $\text{Ind}_P^G \tau'$  for some irreducible (admissible) subquotient  $\tau'$  of  $\tau_{C^{\text{alg}}}$ . Hence none of the  $\pi'$  are supercuspidal.

For the converse, suppose by contradiction that  $\pi_{C^{\text{alg}}}$  has an irreducible subquotient  $\rho$  that is not supercuspidal, i.e.  $\rho$  is a subquotient of  $\text{Ind}_P^G \tau$  for some proper parabolic  $P = MN$  and irreducible admissible  $C^{\text{alg}}$ -representation  $\tau$  of  $M$ . By [Vig96, II.4.7] (as  $c \neq p$ ), respectively by Proposition 3.1.1, we can choose a finite extension  $C'/C$  such that  $\tau$ , respectively all irreducible constituents of  $\text{Ind}_P^G \tau$  and  $\pi_{C^{\text{alg}}}$ , can be defined over  $C'$ . Write  $\tau \cong (\tau')_{C^{\text{alg}}}$  for some  $C'$ -representation  $\tau'$ . Say the irreducible subquotients of  $\text{Ind}_P^G \tau'$  are  $\sigma_1, \dots, \sigma_n$ . So by our choice of  $C'$ , we know that  $\rho \cong (\sigma_i)_{C^{\text{alg}}}$  for some  $i$ . As  $\sigma_i$  is a subquotient of  $\text{Ind}_P^G \tau'$ , we see that  $\sigma_i|_{C[G]}$  is a subquotient of  $\text{Ind}_P^G(\tau'|_{C[M]})$ . But  $\sigma_i|_{C[G]}$  is  $\pi$ -isotypic by Proposition 3.1.1 and  $\tau'|_{C[M]}$  has finite length by Lemma 3.1.3, so  $\pi$  is a subquotient of  $\text{Ind}_P^G \tau''$  for some irreducible (admissible) subquotient  $\tau''$  of  $\tau'|_{C[M]}$ .  $\square$

**3.2. Change of coefficient field.** This section contains the proof of the following result.

**Proposition 3.2.1** (Change of coefficient field).

- (i) *If  $G$  admits an irreducible admissible supercuspidal representation over some finite field of characteristic  $p$ , then  $G$  admits an irreducible admissible supercuspidal representation over any field of characteristic  $p$ .*
- (ii) *If  $G$  admits an irreducible admissible supercuspidal representation over some field of characteristic  $c \neq p$ , then  $G$  admits an irreducible admissible supercuspidal representation over any algebraic extension of the prime field of characteristic  $c$ .*

*Proof.* Let  $F_c$  be the prime field of characteristic  $c$  (so that  $F_0 = \mathbb{Q}$  and  $F_c = \mathbb{F}_c$  if  $c \neq 0$ ).

*Step 1:* We show that, if  $c \neq p$  and  $G$  admits an irreducible admissible supercuspidal  $C$ -representation  $\pi$ , then  $G$  admits one over a finite extension of  $F_c$ .

Indeed, by Proposition 3.1.4 we can suppose  $C = C^{\text{alg}}$ . Twisting by a  $C^{\text{alg}}$ -character of  $G$ , we may suppose that the central character of  $\pi$  takes values in  $F_c^{\text{alg}}$ . As  $c \neq p$ , by [Vig96, II.4.9] the representation  $\pi$  descends to a finite extension  $F'_c/F_c$ . Since descent preserves irreducibility, admissibility and supercuspidality, we obtain an irreducible admissible supercuspidal  $F'_c$ -representation of  $G$ .

*Step 2:* We show that if  $G$  admits an irreducible admissible supercuspidal representation over a finite extension of  $F_c$  then  $G$  admits such a representation over  $F_c$ .

Suppose  $C/F_c$  is a finite field extension and  $\pi$  an irreducible admissible  $C$ -representation of  $G$ . Then  $\pi$  is admissible and finitely generated as a  $F_c$ -representation of  $G$ . This implies that  $\pi$  contains an irreducible admissible  $F_c$ -representation  $\pi'$  (this follows from [HV12, Lemma 7.10] if  $c = p$ , and from the next lemma if  $c \neq p$ ). By adjunction,  $\pi$  is a quotient of the scalar extension  $\pi'_C$  of  $\pi'$  from  $F_c$  to  $C$ .

We now show that if  $\pi$  is supercuspidal, then  $\pi'$  is also supercuspidal. Assume that  $\pi'$  is not supercuspidal, so that it is a subquotient of  $\text{Ind}_P^G \tau'$ , where  $P$  is a proper parabolic subgroup of  $G$  and  $\tau'$  is an irreducible admissible  $F_c$ -representation of the Levi subgroup  $M$  of  $P$ . Since parabolic induction is compatible with scalar extension from  $F_c$  to  $C$ , the representation  $\pi'_C$  is a subquotient of  $\text{Ind}_P^G \tau'_C$ , and therefore the same is true of  $\pi$ . The  $C$ -representation  $\tau'_C$  of  $M$  has finite length and its irreducible subquotients are admissible by [HV, Thm. III.4]. Hence,  $\pi$  is a subquotient of  $\text{Ind}_P^G \rho$  for some irreducible admissible subquotient  $\rho$  of  $\tau'_C$ , and we conclude that  $\pi$  is not supercuspidal.

*Step 3:* We show that if  $G$  admits an irreducible admissible supercuspidal  $\mathbb{F}_p$ -representation (resp.,  $F_c$ -representation, where  $c \neq p$ ), then  $G$  does so over any field of characteristic  $p$  (resp., any algebraic extension of  $F_c$ ). More generally we show that if  $L/C$  is any field extension, assumed to be algebraic if  $c \neq p$ , and  $G$  admits an irreducible admissible supercuspidal  $C$ -representation then the same is true over  $L$ .

Let  $L/C$  be a field extension as above, and choose compatible algebraic closures  $L^{\text{alg}}/C^{\text{alg}}$ . Suppose  $\pi$  is an irreducible admissible supercuspidal  $C$ -representation of  $G$ , and let  $\tau$  be an irreducible subquotient of the scalar extension  $\pi_L$  of  $\pi$  from  $C$  to  $L$ . By [HV, Lem. III.1(ii)],  $\tau$  is admissible. The scalar extension  $\tau_{L^{\text{alg}}}$  of  $\tau$  from  $L$  to  $L^{\text{alg}}$  is a subquotient of the scalar extension  $\pi_{L^{\text{alg}}}$  of  $\pi_L$  from  $L$  to  $L^{\text{alg}}$  (the latter being equal to the scalar extension of  $\pi$  from  $C$  to  $L^{\text{alg}}$ ). By Propositions 3.1.1 and 3.1.4,  $\pi_{L^{\text{alg}}}$  has finite length and its irreducible subquotients are admissible and supercuspidal. Therefore, the same is true of  $\tau_{L^{\text{alg}}}$ . By Proposition 3.1.4 this implies that  $\tau$  is supercuspidal.  $\square$

It remains to prove the lemma announced in Step 2.

**Lemma 3.2.2.** *Assume  $c \neq p$ . An admissible, finitely generated  $C$ -representation  $\pi$  of  $G$  has finite length, and its irreducible subquotients are admissible.*

*Proof.* The extension  $\pi_{C^{\text{alg}}}$  of  $\pi$  from  $C$  to  $C^{\text{alg}}$  is also admissible and finitely generated. By [Vig96, II.2.8, II.5.10], the representation  $\pi_{C^{\text{alg}}}$  has finite length and its irreducible subquotients are admissible. Proposition 3.1.1 implies the same for  $\pi$ .  $\square$

The tensor product  $\pi \otimes_C \sigma$  of an irreducible admissible  $C$ -representation  $\pi$  of  $G$  and of a finite dimensional  $C$ -representation  $\sigma$  of  $G$  is admissible and finitely generated. When  $\sigma$  is of dimension 1,  $\pi \otimes_C \sigma$  is irreducible. The next lemma (which is valid for all  $C$ ) will be used in the proof of Prop. 3.3.9.

**Lemma 3.2.3.** *Let  $\pi$  be an irreducible admissible  $C$ -representation of  $G$  and  $H$  a finite commutative quotient of  $G$ . Then the representation  $\pi \otimes_C C[H]$  of  $G$ , with the natural action of  $G$  on  $C[H]$ , has finite length and its irreducible subquotients are admissible.*

*Proof.* The scalar extension of the  $C$ -representation  $\pi$  (resp.  $C[H]$ ) to  $C^{\text{alg}}$  has finite length with irreducible admissible quotients  $\pi_i$  (resp.  $\chi_j$ , of dimension 1). Therefore  $(\pi \otimes_C C[H])_{C^{\text{alg}}} \cong \pi_{C^{\text{alg}}} \otimes_{C^{\text{alg}}} C^{\text{alg}}[H]$  has finite length with irreducible admissible subquotients  $\pi_i \otimes_{C^{\text{alg}}} \chi_j$ , implying the same for  $\pi \otimes_C C[H]$ .  $\square$

**3.3. Reduction to an absolutely simple adjoint group.** We now show that in order to prove the existence of irreducible, admissible, supercuspidal  $C$ -representations of  $G = \mathbf{G}(F)$ , it suffices to assume  $\mathbf{G}$  is absolutely simple, adjoint, and isotropic.

As is well known, the adjoint group  $\mathbf{G}^{\text{ad}}$  of  $\mathbf{G}$  is  $F$ -isomorphic to a finite direct product of connected reductive  $F$ -groups

$$(3.3.1) \quad \mathbf{G}^{\text{ad}} \cong \mathbf{H} \times \prod_i \text{Res}_{F'_i/F}(\mathbf{G}'_i),$$

where  $\mathbf{H}$  is anisotropic, the  $F'_i/F$  are finite separable extensions, and  $\text{Res}_{F'_i/F}(\mathbf{G}'_i)$  are scalar restrictions from  $F'_i$  to  $F$  of isotropic, absolutely simple, connected adjoint  $F'_i$ -groups  $\mathbf{G}'_i$ .

**Proposition 3.3.2.** *Assume that the field  $C$  is algebraically closed or finite. If, for each  $i$ , the group  $\mathbf{G}'_i(F'_i)$  admits an irreducible admissible supercuspidal  $C$ -representation, then  $G$  admits an irreducible admissible supercuspidal  $C$ -representation.*

The proposition is the combination of Propositions 3.3.3, 3.3.6, 3.3.8, and 3.3.11 below, corresponding to the operations of finite product, central extension, and scalar restriction (all when  $C$  algebraically closed or finite). We also note that if  $\mathbf{G}$  is anisotropic, then  $G$  is compact and any irreducible smooth representation of  $G$  is finite-dimensional (hence admissible) and supercuspidal.

We remark that when  $C = \mathbb{C}$  and  $\mathbf{G}$  is a reductive group over a *finite* field of characteristic  $p$ , the analogous statement is proved in [Kre, Prop. 2.1].

**3.3.1. Finite product.** Let  $\mathbf{G}_1$  and  $\mathbf{G}_2$  be two connected reductive  $F$ -groups, and  $\sigma$  and  $\tau$  irreducible admissible  $C$ -representations of  $G_1$  and  $G_2$ , respectively.

**Proposition 3.3.3.** *Assume that  $C$  is algebraically closed.*

- (i) *The tensor product  $\sigma \otimes_C \tau$  is an irreducible admissible  $C$ -representation of  $G_1 \times G_2$ .*
- (ii) *Every irreducible admissible  $C$ -representation of  $G_1 \times G_2$  is of this form.*
- (iii) *The  $C$ -representation  $\sigma \otimes_C \tau$  determines  $\sigma$  and  $\tau$  (up to isomorphism).*
- (iv) *The  $C$ -representation  $\sigma \otimes_C \tau$  is supercuspidal if and only if  $\sigma$  and  $\tau$  are supercuspidal.*

*Proof.* Note first that  $\sigma \otimes_C \tau$  is admissible: for compact open subgroups  $K_1$  of  $G_1$  and  $K_2$  of  $G_2$ , we have a natural isomorphism ([Bou12, §12.2 Lem. 1])

$$\mathrm{Hom}_{K_1}(\mathbf{1}_{K_1}, \sigma) \otimes_C \mathrm{Hom}_{K_2}(\mathbf{1}_{K_2}, \tau) \xrightarrow{\sim} \mathrm{Hom}_{K_1 \times K_2}(\mathbf{1}_{K_1} \otimes_C \mathbf{1}_{K_2}, \sigma \otimes_C \tau),$$

where  $\mathbf{1}_{K_i}$  denotes the trivial representation of  $K_i$ . Thus, the admissibility of  $\sigma$  and  $\tau$  implies the admissibility of  $\sigma \otimes_C \tau$ .

Suppose now  $C$  algebraically closed.

(i) Proposition 3.1.1 implies that the commutant of  $\sigma$  is  $C$ . Irreducibility of  $\sigma \otimes_C \tau$  then follows from [Bou12, §12.2 Cor. 1].

(ii) Let  $\pi$  be an irreducible admissible  $C$ -representation of  $G_1 \times G_2$ , and let  $K_1, K_2$  be any compact open subgroups of  $G_1, G_2$ , respectively, such that  $\pi^{K_1 \times K_2} \neq 0$ .

If  $c = p$ , the  $C$ -representation of  $G_1$  generated by  $\pi^{1 \times K_2}$  is admissible (since  $\pi^{K_1 \times K_2}$  is finite dimensional for any  $K_1$ ). By [HV12, Lemma 7.10], it contains an irreducible admissible  $C$ -subrepresentation  $\sigma$ . Set  $\tau := \mathrm{Hom}_{G_1}(\sigma, \pi) \neq 0$ , with the natural action of  $G_2$ . The representation  $\sigma \otimes_C \tau$  embeds naturally in  $\pi$ . As  $\pi$  is irreducible, it is isomorphic to  $\sigma \otimes_C \tau$ , and  $\tau$  is irreducible. As  $\pi$  is admissible,  $\tau$  is admissible as well. (This proof is due to Henniart.)

If  $c \neq p$ , the space  $\pi^{K_1 \times K_2}$  is a simple right  $H_C(G_1 \times G_2, K_1 \times K_2)$ -module ([Vig96, I.4.4, I.6.3]), and we have

$$H_C(G_1 \times G_2, K_1 \times K_2) \cong H_C(G_1, K_1) \otimes_C H_C(G_2, K_2).$$

By [Bou12, §12.1 Prop. 2], the finite-dimensional simple  $H_C(G_1, K_1) \otimes_C H_C(G_2, K_2)$ -modules are factorizable, meaning  $\pi^{K_1 \times K_2} \cong \sigma^{K_1} \otimes_C \tau^{K_2}$  for irreducible admissible  $C$ -representations  $\sigma, \tau$  of  $G_1, G_2$ , respectively (this uses [Vig96, I.4.4, I.6.3] again). Thus, we obtain  $\pi \cong \sigma \otimes_C \tau$ . (We can also imitate the argument above for  $c = p$  with Hecke algebras to obtain the result.)

(iii) As a  $C$ -representation of  $G_1$ ,  $\sigma \otimes_C \tau$  is  $\sigma$ -isotypic. Similarly, as a  $C$ -representation of  $G_2$ ,  $\sigma \otimes_C \tau$  is  $\tau$ -isotypic. The result follows.

(iv) The parabolic subgroups of  $G_1 \times G_2$  are products of parabolic subgroups of  $G_1$  and of  $G_2$ . Let  $P, Q$  be parabolic subgroups of  $G_1, G_2$ , respectively, with Levi subgroups  $M, L$ , respectively, and let  $\pi'$  be an irreducible admissible  $C$ -representation of the product  $M \times L$ . By part (ii), the  $C$ -representation  $\pi'$  is factorizable, say  $\pi' = \sigma' \otimes_C \tau'$  for irreducible admissible  $C$ -representations  $\sigma'$  of  $M$  and  $\tau'$  of  $L$ . We then obtain a natural isomorphism

$$\mathrm{Ind}_P^{G_1} \sigma' \otimes_C \mathrm{Ind}_Q^{G_2} \tau' \xrightarrow{\sim} \mathrm{Ind}_{P \times Q}^{G_1 \times G_2} \pi'.$$

We conclude that the irreducible subquotients of  $\mathrm{Ind}_{P \times Q}^{G_1 \times G_2} \pi'$  are tensor products of the irreducible subquotients of  $\mathrm{Ind}_P^{G_1} \sigma'$  and of  $\mathrm{Ind}_Q^{G_2} \tau'$ , which gives the result.  $\square$

We assume from now until the end of §3.3.1 that  $C$  is a finite field.

**Proposition 3.3.4.** *Assume that  $C$  is finite. Let  $\pi$  be an irreducible admissible  $C$ -representation of  $G$ . The commutant of  $\pi$  is a finite field extension  $D$  of  $C$  and the scalar extension  $\pi_D$  of  $\pi$  from  $C$  to  $D$  is isomorphic to*

$$\pi_D \cong \bigoplus_{i \in \mathrm{Gal}(D/C)} \pi_i,$$

where the  $\pi_i$  are irreducible admissible  $D$ -representations of  $G$ . Moreover, the  $\pi_i$  each have commutant  $D$ , are pairwise non-isomorphic, form a single  $\text{Gal}(D/C)$ -orbit, and, viewed as  $C$ -representations, are isomorphic to  $\pi$ .

*Proof.* The commutant  $D$  of  $\pi$  is a division algebra of finite dimension over  $C$ . Since the Brauer group of a finite field is trivial,  $D$  is a finite Galois extension of  $C$ . The result now follows from [HV, Thms. I.1, III.4] by taking  $R' = D$ . (Note also that as a  $C$ -representation,  $\pi_D$  is  $\pi$ -isotypic of length  $[D : C]$ .)  $\square$

Recall that we have fixed irreducible admissible  $C$ -representations  $\sigma$  and  $\tau$  of  $G_1$  and  $G_2$ , respectively. Their respective commutants  $D_\sigma$  and  $D_\tau$  are finite extensions of  $C$  of dimensions  $d_\sigma$  and  $d_\tau$ , respectively. We embed them into  $C^{\text{alg}}$ , and consider:

- the field  $D$  generated by  $D_\sigma$  and  $D_\tau$ , which has  $C$ -dimension  $\text{lcm}(d_\sigma, d_\tau)$ ,
- the field  $D' := D_\sigma \cap D_\tau$ , which has  $C$ -dimension  $\text{gcd}(d_\sigma, d_\tau)$ .

The fields  $D_\sigma, D_\tau$  are linearly disjoint over  $D'$ , we have  $D_\sigma \otimes_{D'} D_\tau \cong D$  and

$$(3.3.5) \quad D_\sigma \otimes_C D_\tau \cong \prod_{k=1}^{[D':C]} D.$$

**Proposition 3.3.6.** *Assume that  $C$  is finite. The  $C$ -representation  $\sigma \otimes_C \tau$  of  $G_1 \times G_2$  is isomorphic to*

$$\sigma \otimes_C \tau \cong \bigoplus_{k=1}^{\text{gcd}(d_\sigma, d_\tau)} \pi_k,$$

where the  $\pi_k$  are irreducible admissible  $C$ -representations with commutant  $D$ , which are pairwise non-isomorphic. The  $C$ -representations  $\sigma$  and  $\tau$  are supercuspidal if and only if all the  $\pi_k$  are supercuspidal, if and only if some  $\pi_k$  is supercuspidal.

*Proof.* By Proposition 3.3.4, we have

$$\sigma_D \cong \bigoplus_{i \in \text{Gal}(D_\sigma/C)} \sigma_i, \quad \tau_D \cong \bigoplus_{j \in \text{Gal}(D_\tau/C)} \tau_j,$$

where the  $\sigma_i$  (resp.  $\tau_j$ ) are irreducible admissible  $D$ -representations of  $G_1$  (resp.  $G_2$ ) with commutant  $D$ , which are pairwise non-isomorphic, form a single  $\text{Gal}(D/C)$ -orbit, descend to  $D_\sigma$  (resp.  $D_\tau$ ) and their descents, viewed as  $C$ -representations, are isomorphic to  $\sigma$  (resp.  $\tau$ ). The  $C$ -representation  $\sigma \otimes_C \tau$  of  $G_1 \times G_2$  is admissible, and its scalar extension from  $C$  to  $D$  is equal to

$$(3.3.7) \quad (\sigma \otimes_C \tau)_D \cong \sigma_D \otimes_D \tau_D \cong \bigoplus_{(i,j) \in \text{Gal}(D_\sigma/C) \times \text{Gal}(D_\tau/C)} \sigma_i \otimes_D \tau_j.$$

The  $D$ -representation  $\sigma_i \otimes_D \tau_j$  of  $G_1 \times G_2$  is admissible and has commutant  $D \otimes_D D = D$  ([Bou12, §12.2 Lem. 1]). Hence,  $\sigma_i \otimes_D \tau_j$  is absolutely irreducible and equation (3.3.7) implies  $(\sigma \otimes_C \tau)_D$  is semisimple. By [Bou12, §12.7 Prop. 8], this implies that  $\sigma \otimes_C \tau$  is semisimple; its commutant is isomorphic to  $D_\sigma \otimes_C D_\tau$  by [Bou12, §12.2 Lem. 1]. From equation (3.3.5) we see that  $\sigma \otimes_C \tau$  has length  $[D' : C] = \text{gcd}(d_\sigma, d_\tau)$ , its irreducible constituents  $\pi_k$  are admissible and pairwise non-isomorphic with commutant  $D$ .

Applying Proposition 3.3.3 over  $C^{\text{alg}}$  and Proposition 3.1.4 (several times), we see that  $\sigma$  and  $\tau$  are supercuspidal if and only if some/every  $\sigma_i$  and some/every  $\tau_j$  are

supercuspidal, if and only if some/every  $\sigma_i \otimes_D \tau_j$  is supercuspidal. From Proposition 3.1.4 again, this is also equivalent to  $\pi_k$  being supercuspidal for some/every  $k$ .  $\square$

3.3.2. *Central extension.* Recall that we have a short exact sequence of  $F$ -groups

$$1 \rightarrow \mathbf{Z}(\mathbf{G}) \rightarrow \mathbf{G} \xrightarrow{i} \mathbf{G}^{\text{ad}} \rightarrow 1,$$

which induces an exact sequence between  $F$ -points

$$1 \rightarrow Z(G) \rightarrow G \xrightarrow{i} G^{\text{ad}} \rightarrow H^1(F, \mathbf{Z}(\mathbf{G})).$$

The image  $i(G)$  of  $G$  is a closed cocompact normal subgroup of  $G^{\text{ad}}$  and  $H^1(F, \mathbf{Z}(\mathbf{G}))$  is commutative.

Until the end of §3.3.2, we assume that  $\text{char } F = 0$ . The group  $H^1(F, \mathbf{Z}(\mathbf{G}))$  is then finite ([PR94, Thm. 6.14]), implying that  $i(G)$  is an *open* normal subgroup of  $G^{\text{ad}}$  and the quotient  $G^{\text{ad}}/i(G)$  is finite and commutative. Our next task will be to prove the following:

**Proposition 3.3.8.**  *$G^{\text{ad}}$  admits an irreducible admissible supercuspidal  $C$ -representation if and only if  $G$  admits such a representation with a trivial action of  $Z(G)$ .*

Inflation from  $i(G)$  to  $G$  identifies representations of  $i(G)$  with representations of  $G$  having trivial  $Z(G)$ -action; this inflation functor respects irreducibility and admissibility. The composite functor

$$(\text{inflation from } i(G) \text{ to } G) \circ (\text{restriction from } G^{\text{ad}} \text{ to } i(G))$$

from  $C$ -representations of  $G^{\text{ad}}$  to representations of  $G$  trivial on  $Z(G)$  will be denoted by  $- \circ i$ .

Suppose  $\tilde{\rho}$  is an irreducible admissible  $C$ -representation of  $G$  with trivial action of  $Z(G)$ . Then  $\tilde{\rho}$  is the inflation of a representation  $\rho$  of the open, normal, finite-index subgroup  $i(G)$  of  $G^{\text{ad}}$ . The  $C$ -representation  $\rho$  of  $i(G)$  is irreducible and admissible, and therefore the induced representation  $\text{Ind}_{i(G)}^{G^{\text{ad}}} \rho$  of  $G^{\text{ad}}$  is admissible of finite length.

Any irreducible quotient  $\pi$  of  $\text{Ind}_{i(G)}^{G^{\text{ad}}} \rho$  is admissible (if  $c = p$ , this uses the assumption  $\text{char } F = 0$ ; see [Hen09, §4, Thm. 1]). By adjunction,  $\pi|_{i(G)}$  contains a subrepresentation isomorphic to  $\rho$  and, by inflation from  $i(G)$  to  $G$ ,  $\tilde{\rho}$  is isomorphic to a subquotient of  $\pi \circ i$ .

Conversely, suppose  $\pi$  is an irreducible admissible  $C$ -representation of  $G^{\text{ad}}$ . The restriction  $\pi|_{i(G)}$  of  $\pi$  to  $i(G)$  is semisimple of finite length, and its irreducible constituents  $\rho$  are  $G^{\text{ad}}$ -conjugate and admissible (see [Vig96, I.6.12]<sup>2</sup>). Hence, the  $C$ -representation  $\pi \circ i$  of  $G$  is semisimple of finite length, and its irreducible constituents are the inflations  $\tilde{\rho}$  of the irreducible constituents  $\rho$  of  $\pi|_{i(G)}$ .

Proposition 3.3.8 now follows from:

**Proposition 3.3.9.** *Let  $\pi, \rho$  and  $\tilde{\rho}$  be as above. Then  $\pi$  is supercuspidal if and only if some  $\tilde{\rho}$  is supercuspidal, if and only if all  $\tilde{\rho}$  are supercuspidal.*

*Proof.* We first check first the compatibility of parabolic induction with  $- \circ i$ . The parabolic  $F$ -subgroups of  $\mathbf{G}$  and of  $\mathbf{G}^{\text{ad}}$  are in bijection via the map  $i$  ([Bor91, 22.6 Thm.]). If the parabolic  $F$ -subgroup  $\mathbf{P}$  of  $\mathbf{G}$  corresponds to the parabolic  $F$ -subgroup

<sup>2</sup> In [Vig96, I.6.12], the condition that the index is invertible in  $C$  is not necessary and not used in the proof.



$\mathbf{Q}$  of  $\mathbf{G}^{\text{ad}}$ , then  $i$  restricts to an isomorphism between their unipotent radicals, and sends a Levi subgroup  $\mathbf{M}$  of  $\mathbf{P}$  onto a Levi subgroup  $\mathbf{L}$  of  $\mathbf{Q}$ . Further, we have an exact sequence between  $F$ -points:

$$1 \rightarrow Z(G) \rightarrow M \xrightarrow{i} L \rightarrow H^1(F, \mathbf{Z}(\mathbf{G})).$$

We have  $G^{\text{ad}} = Qi(G)$  and  $Q \cap i(G) = i(P) = i(M)U$ , where  $i(M)$  is an open normal subgroup of  $L$  having finite commutative quotient, and  $U$  is the unipotent radical of  $Q$ . Thus, if  $\sigma$  is a smooth  $C$ -representation of  $L$ , the Mackey decomposition implies  $(\text{Ind}_Q^{G^{\text{ad}}} \sigma)|_{i(G)} \cong \text{Ind}_{i(P)}^{i(G)}(\sigma|_{i(M)})$  and, by inflation from  $i(G)$  to  $G$ , we obtain:

$$(3.3.10) \quad (\text{Ind}_Q^{G^{\text{ad}}} \sigma) \circ i \cong \text{Ind}_P^G(\sigma \circ i).$$

We may now proceed with the proof. It suffices to prove:

- (i) if  $\pi$  is non-supercuspidal, then all  $\tilde{\rho}$  are non-supercuspidal,
- (ii) if some  $\tilde{\rho}$  is non-supercuspidal, then  $\pi$  is non-supercuspidal.

To prove (i), let  $\pi$  be an irreducible admissible non-supercuspidal  $C$ -representation of  $G^{\text{ad}}$ , which is isomorphic to a subquotient of  $\text{Ind}_Q^{G^{\text{ad}}} \sigma$  for  $Q \subsetneq G^{\text{ad}}$  and  $\sigma$  an irreducible admissible  $C$ -representation of  $L$ . Therefore,  $\pi \circ i$  is isomorphic to a subquotient of  $(\text{Ind}_Q^{G^{\text{ad}}} \sigma) \circ i$ , and by equation (3.3.10), each  $\tilde{\rho}$  is isomorphic to a subquotient of  $\text{Ind}_P^G \tilde{\tau}$  for some irreducible subquotient  $\tilde{\tau}$  of  $\sigma \circ i$  (depending on  $\rho$ ). Since  $\tilde{\tau}$  is admissible and  $P \subsetneq G$ , all the  $\tilde{\rho}$  are non-supercuspidal.

To prove (ii), let  $\pi$  be an irreducible admissible  $C$ -representation of  $G^{\text{ad}}$  such that some irreducible constituent  $\tilde{\rho}$  of  $\pi \circ i$  is non-supercuspidal. Suppose  $\tilde{\rho}$  is isomorphic to a subquotient of  $\text{Ind}_P^G \tau'$  for  $P \subsetneq G$  and  $\tau'$  an irreducible admissible  $C$ -representation of  $M$ . The central subgroup  $Z(G)$  acts trivially on  $\tilde{\rho}$ , and hence also on  $\tau'$ . Therefore  $\tau' = \tilde{\tau}$  for some irreducible subquotient  $\tau$  of  $\sigma|_{i(G)}$ , where  $\sigma$  is an irreducible admissible  $C$ -representation of  $L$ . The representation  $\tilde{\rho}$  is isomorphic to a subquotient of  $\text{Ind}_P^G(\sigma \circ i)$ . By equation (3.3.10) and exactness of parabolic induction,  $\text{Ind}_{i(G)}^{G^{\text{ad}}}(\rho)$ , and hence its quotient  $\pi$ , is isomorphic to a subquotient of  $\text{Ind}_{i(G)}^{G^{\text{ad}}}((\text{Ind}_Q^{G^{\text{ad}}} \sigma)|_{i(G)})$ . This representation is isomorphic to

$$\text{Ind}_{i(M)U}^{G^{\text{ad}}}(\sigma|_{i(M)}) \cong \text{Ind}_Q^{G^{\text{ad}}}(\text{Ind}_{i(M)}^L(\sigma|_{i(M)})) \cong \text{Ind}_Q^{G^{\text{ad}}}(\sigma \otimes_C C[i(M)\backslash L]).$$

By Lemma 3.2.3, the  $C$ -representation  $\sigma \otimes_C C[i(M)\backslash L]$  of  $L$  has finite length and its irreducible subquotients  $\nu$  are admissible. Therefore  $\pi$  is isomorphic to a subquotient of  $\text{Ind}_Q^{G^{\text{ad}}} \nu$  for some  $\nu$  and some  $Q \subsetneq G^{\text{ad}}$ , and therefore  $\pi$  is non-supercuspidal.  $\square$

**3.3.3. Scalar restriction.** Now let  $F'/F$  be a finite separable extension,  $\mathbf{G}'$  a connected reductive  $F'$ -group and  $\mathbf{G} := \text{Res}_{F'/F}(\mathbf{G}')$  the scalar restriction of  $\mathbf{G}'$  from  $F'$  to  $F$ . As topological groups,  $G' := \mathbf{G}'(F')$  is equal to  $G := \mathbf{G}(F)$ . By [BT65, 6.19. Cor.],  $G'$  and  $G$  have the same parabolic subgroups. Hence:

**Proposition 3.3.11.**  *$G'$  admits an irreducible admissible supercuspidal  $C$ -representation if and only if  $G$  does.*

## 4. PROOF OF THE MAIN THEOREM FOR MOST SIMPLE GROUPS

**4.1. Discrete Iwahori–Hecke modules.** Let  $\text{Rep}_C(G, \mathfrak{B})$  denote the category of  $C$ -representations of  $G$  generated by their  $\mathfrak{B}$ -invariant vectors, and let  $\text{Mod}(H_C(G, \mathfrak{B}))$  denote the category of right  $H_C(G, \mathfrak{B})$ -modules. The functor of  $\mathfrak{B}$ -invariants

$$\begin{aligned} \text{Rep}_C(G, \mathfrak{B}) &\rightarrow \text{Mod}(H_C(G, \mathfrak{B})) \\ \pi &\mapsto \pi^{\mathfrak{B}} \end{aligned}$$

admits a left adjoint

$$\begin{aligned} \mathfrak{I} : \text{Mod}(H_C(G, \mathfrak{B})) &\rightarrow \text{Rep}_C(G, \mathfrak{B}) \\ M &\mapsto M \otimes_{H_C(G, \mathfrak{B})} C[\mathfrak{B} \backslash G]. \end{aligned}$$

**Fact 4.1.1.** *When  $c \neq p$ , the functor  $\pi \mapsto \pi^{\mathfrak{B}}$  induces a bijection between the isomorphism classes of irreducible  $C$ -representations  $\pi$  of  $G$  with  $\pi^{\mathfrak{B}} \neq 0$  and isomorphism classes of simple right  $H_C(G, \mathfrak{B})$ -modules ([Vig96, I.4.4, I.6.3]). When  $C = \mathbb{C}$ , the functors are inverse equivalences of categories (see [Ber84, Cor. 3.9(ii)]).*

When  $C = \mathbb{C}$ , the Bernstein ring embedding  $H_{\mathbb{C}}(Z, Z_0) \xrightarrow{\theta} \tilde{H}_{\mathbb{C}}$  is the linear map defined by sending  $T_{\lambda}^Z$  to  $\theta_{\lambda} := q_{\lambda}^{-1/2} E_{\lambda}$  for  $\lambda \in \Lambda$ . Its image is equal to  $\mathcal{A}_{\mathbb{C}}$ . Note that if  $\lambda \in \Lambda$  is anti-dominant and  $z \in Z$  lifts  $\lambda$ , we have  $q_{\lambda} = \delta_B(z)$ , where  $\delta_B$  denotes the modulus character of  $B$ .

We now recall some properties of the category  $\text{Rep}_{\mathbb{C}}(G, \mathfrak{B})$ , including Casselman’s criterion of square integrability modulo center, before giving the definition of a discrete simple right  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module. Recall that  $\pi_U$  denotes the space of  $U$ -coinvariants (i.e., the unnormalized Jacquet module) of a representation  $\pi$ .

**Lemma 4.1.2.** *Suppose that  $\pi$  is an admissible  $\mathbb{C}$ -representation of  $G$ . Then the natural map  $\pi \rightarrow \pi_U$  induces an isomorphism  $\varphi : \pi^{\mathfrak{B}} \xrightarrow{\sim} \pi_U^{Z_0}$ . Moreover, we have  $\varphi(v \cdot \theta_{\lambda^{-1}}) = \delta_B^{-1/2}(t)(t \cdot \varphi(v))$  for  $\lambda \in \Lambda_T$ ,  $t \in T$  lifting  $\lambda$ , and  $v \in \pi^{\mathfrak{B}}$ .*

*Proof.* Recall that  $\mathfrak{B}$  has an Iwahori decomposition with respect to  $Z, U, U^{\text{op}}$ . Then [Cas, Prop. 4.1.4] implies that the map  $\pi \rightarrow \pi_U$  induces an isomorphism  $\pi^{\mathfrak{B}} \cdot T_{\lambda^{-1}} \xrightarrow{\sim} \pi_U^{Z_0}$  for  $\lambda \in \Lambda_T$  with  $\max_{\alpha \in \Delta} |\alpha(\lambda)|_F$  sufficiently small. By [Vig16, Prop. 4.13(1)] the operator  $T_{\lambda^{-1}}$  is invertible in  $H_{\mathbb{C}}(G, \mathfrak{B})$ , so  $\pi^{\mathfrak{B}} \cdot T_{\lambda^{-1}} = \pi^{\mathfrak{B}}$ .

To show the last statement, we may assume that  $\lambda \in \Lambda_T^+$ . Then, in our terminology, [Cas, Lemma 4.1.1] says that  $\varphi(|\mathfrak{B}t\mathfrak{B}/\mathfrak{B}|^{-1}[\mathfrak{B}t\mathfrak{B}] \cdot v) = t \cdot \varphi(v)$ , where  $[\mathfrak{B}t\mathfrak{B}]$  denotes the usual double coset operator on  $\pi^{\mathfrak{B}}$ . Now  $[\mathfrak{B}t\mathfrak{B}] \cdot v = v \cdot T_{t^{-1}}$  and  $T_{t^{-1}} = E_{t^{-1}} = q_{t^{-1}}^{1/2} \theta_{t^{-1}}$ . Moreover,  $|\mathfrak{B}t\mathfrak{B}/\mathfrak{B}| = q_t = q_{t^{-1}} = \delta_B(t^{-1})$ . Putting this all together, we obtain the claim.  $\square$

*Remark 4.1.3.* The lemma and its proof hold when  $\mathfrak{B}$  is replaced by  $\mathfrak{U}$  and  $Z_0$  is replaced by  $Z_0 \cap \mathfrak{U}$ .

**Proposition 4.1.4.** *Let  $\pi$  be an irreducible  $\mathbb{C}$ -representation of  $G$  with  $\pi^{\mathfrak{B}} \neq 0$ .*

- (i)  *$\pi$  is isomorphic to a subrepresentation of  $\text{Ind}_B^G \sigma$ , where  $\sigma$  is a  $\mathbb{C}$ -character of  $Z$  trivial on  $Z_0$ .*

- (ii) **Casselman's criterion:**  $\pi$  is square integrable modulo center (as defined in [Cas, §2.5]) if and only if its central character is unitary and

$$|\chi(\mu(\varpi))|_{\mathbb{C}} < 1$$

for all  $\mu \in X_*(T)^+$  such that  $-\mu \notin X_*(T)^+$ , and all characters  $\chi$  of  $T$  contained in  $\delta_B^{-1/2} \pi_U$ .

*Proof.* (i) Since  $\pi$  is irreducible and smooth, it is admissible by [Vig96, II.2.8], and [Cas, 3.3.1] implies  $\pi_U$  is admissible as well. By Lemma 4.1.2 and the assumption  $\pi^{\mathfrak{B}} \neq 0$ , we see that  $\pi_U \neq 0$ . The claim now follows by choosing an irreducible quotient  $\pi_U \rightarrow \sigma$  for which  $\sigma^{Z_0} \neq 0$  and applying Frobenius reciprocity.

(ii) This follows from [Cas, Thm. 6.5.1].  $\square$

**Definition 4.1.5.** We say a simple right  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module is *discrete* if it is isomorphic to  $\pi^{\mathfrak{B}}$  for an irreducible admissible square-integrable modulo center  $\mathbb{C}$ -representation  $\pi$  of  $G$ .

**Proposition 4.1.6.** *A simple right  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module  $M$  is discrete if and only if any  $\mathbb{C}$ -character  $\chi$  of  $\mathcal{A}$  contained in  $M$  satisfies the following condition: the restriction of  $\chi$  to  $\Lambda_{Z(G)}$  is a unitary character, and*

$$(4.1.7) \quad |\chi(\theta_{\lambda_{\mu}^{-1}})|_{\mathbb{C}} < 1$$

for any  $\mu \in X_*(T)^+$  such that  $-\mu \notin X_*(T)^+$ .

*Proof.* Note that  $M = \pi^{\mathfrak{B}}$  for an irreducible (admissible)  $\mathbb{C}$ -representation  $\pi$  of  $G$ . Then  $\pi$  has unitary central character if and only if  $\Lambda_{Z(G)}$  acts by a unitary character on  $M$ . As any irreducible  $\mathcal{A}_{\mathbb{C}}$ -module is a character, by Casselman's criterion (Proposition 4.1.4) and Lemma 4.1.2,  $M$  is discrete if and only condition (4.1.7) holds.  $\square$

*Remark 4.1.8.* Some authors view  $\pi^{\mathfrak{B}}$  as a left  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module. One may pass between left and right modules by twisting by the anti-automorphism  $T_w \mapsto T_{w^{-1}}$ , that is, we may define

$$T_w \cdot v = v \cdot T_{w^{-1}}$$

for  $w \in \widetilde{W}, v \in \pi^{\mathfrak{B}}$ . The space  $\pi^{\mathfrak{B}}$ , viewed as either a left or right  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module, is then called discrete if  $\pi$  is square integrable modulo center. For left modules, the proposition above holds with “anti-dominant” replaced by “dominant.”

**Lemma 4.1.9.** *For a character  $\chi : \mathcal{A} \rightarrow \mathbb{C}$  such that  $\chi|_{\Lambda_{Z(G)}}$  is unitary, the following conditions are equivalent:*

- (i)  $|\chi(\theta_{\lambda_{\mu}^{-1}})|_{\mathbb{C}} < 1$  for any  $\mu \in X_*(T)^+$  such that  $-\mu \notin X_*(T)^+$ ,
- (ii)  $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} < 1$  for any  $\lambda \in \Lambda^{\text{sc},+}$  such that  $\lambda^{-1} \notin \Lambda^{\text{sc},+}$ ,
- (iii)  $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} < 1$  for any  $\lambda \in \Lambda^+$  such that  $\lambda^{-1} \notin \Lambda^+$ .

*Proof.* We first recall that the invertible elements in  $\Lambda^+$  consist of  $Z(G)\widetilde{Z}_0/Z_0$ , so  $|\chi(\theta_{\lambda})|_{\mathbb{C}} = 1$  for all invertible elements of  $\Lambda^+$ .

As  $\Lambda_T \cong X_*(T)$ , we see that (iii) implies (i) and (ii). To prove that (ii) implies (iii), we need to show that  $|\chi(\theta_{\lambda^{-1}})|_{\mathbb{C}} = 1$  for  $\lambda \in \Lambda^+$  implies  $\lambda^{-1} \in \Lambda^+$ . By Lemma 2.2.4 pick  $n \geq 1$  such that  $\lambda^n \in \Lambda_{Z(G)} \times \Lambda^{\text{sc},+}$ . Then  $\lambda^n \lambda_0 \in \Lambda^{\text{sc},+}$  for some  $\lambda_0 \in \Lambda_{Z(G)}$ . As  $|\chi(\theta_{\lambda^{-n} \lambda_0^{-1}})|_{\mathbb{C}} = 1$  we deduce from (ii) that  $\lambda^n \lambda_0 \in \Lambda^{\text{sc},+} \cap (\Lambda^{\text{sc},+})^{-1}$ , which is contained

in  $\Lambda^+ \cap (\Lambda^+)^{-1}$ . Therefore  $\lambda^n \in \Lambda^+ \cap (\Lambda^+)^{-1}$ . From the definition of dominance it follows that  $\lambda \in \Lambda^+ \cap (\Lambda^+)^{-1}$ .

The proof that (i) implies (iii) is similar but easier.  $\square$

**Proposition 4.1.10.** *A simple right  $H_{\mathbb{C}}(G, \mathfrak{B})$ -module  $M$  is discrete if and only if  $\Lambda_{Z(G)}$  acts on  $M$  by a unitary character and if its restriction to  $H_{\mathbb{C}}(G^{\text{sc}}, \mathfrak{B}^{\text{sc}})$  is discrete.*

*Proof.* This follows from Proposition 4.1.6 and Lemma 4.1.9.  $\square$

**4.2. Characters.** In this section we continue to assume  $C$  is a field of characteristic  $c$ , and suppose further that  $\mathbf{G}$  is absolutely simple and isotropic. We determine the characters  $H = H(G^{\text{sc}}, \mathfrak{B}^{\text{sc}}) \rightarrow C$  which extend to  $\tilde{H} = H(G, \mathfrak{B})$ . This is an exercise, which is already in the literature when  $C = \mathbb{C}$  (cf. [Bor76]).

For distinct reflections  $s, t \in S$ , the order  $n_{s,t}$  of  $st$  is finite, except if the type of  $\Sigma$  is  $A_1$ . In the finite case, the braid relations (2.1.2) imply

$$(4.2.1) \quad (T_s T_t)^r = (T_t T_s)^r \quad \text{if } n_{s,t} = 2r,$$

$$(4.2.2) \quad (T_t T_s)^r T_t = T_t (T_s T_t)^r \quad \text{if } n_{s,t} = 2r + 1.$$

The  $T_s$  for  $s \in S$  and the relations (2.1.3), (4.2.1) and (4.2.2) give a presentation of  $H$ . A presentation of  $\tilde{H}$  is given by the  $T_u, T_s$  for  $u \in \Omega, s \in S$  and the relations (2.1.3), (4.2.1), (4.2.2) and

$$(4.2.3) \quad T_u T_{u'} = T_{uu'} \quad \text{if } u, u' \in \Omega,$$

$$(4.2.4) \quad T_u T_s = T_{u(s)} T_u \quad \text{if } u \in \Omega, s \in S,$$

where  $u(s)$  denotes the action of  $\Omega$  on  $S$ .

We have a disjoint decomposition

$$S = \bigsqcup_{i=1}^m S_i,$$

where  $S_i$  is the intersection of  $S$  with a conjugacy class of  $W$ . The  $S_i$  are precisely the connected components of Dyn when all multiple edges are removed (see [Bou02, VI.4.3 Th. 4] and [Bor76, 3.3]). Thus, we have

$$m = \begin{cases} 1 \\ 2 \\ 3 \end{cases} \quad \text{when the type of } \Sigma = \begin{cases} A_\ell (\ell \geq 2), D_\ell (\ell \geq 4), E_6, E_7, \text{ or } E_8; \\ A_1, B_\ell (\ell \geq 3), F_4, \text{ or } G_2; \\ C_\ell (\ell \geq 2). \end{cases}$$

When  $m > 1$ , we fix a labeling of the  $S_i$  such that  $|S_1| \geq |S_2|$ , and when the type of  $\Sigma$  is  $C_\ell$  ( $\ell \geq 2$ ), we let  $S_2 = \{s_2\}$  and  $S_3 = \{s_3\}$  denote the endpoints of Dyn. The parameters  $d_s$  are equal on each component  $S_i$ ; we denote this common value by  $d_i$ .

**Lemma 4.2.5.** *Suppose  $\{T_s\}_{s \in S} \rightarrow C$  is an arbitrary map.*

- *When  $c \neq p$ , the above map extends to a character of  $H$  if and only if it is constant on each  $S_i$ , and takes the value  $-1$  or  $q^{d_i}$  on each  $T_s, s \in S_i$ . There are  $2^m$  characters if  $q^{d_i} + 1 \neq 0$  in  $C$  for each  $i$ .*
- *When  $c = p$ , the above map extends to a character of  $H$  if and only if its values are  $-1$  or  $0$  on each  $T_s, s \in S$ . There are  $2^{|S|}$  characters.*

*Proof.* When  $c \neq p$ , this follows from the presentation of  $H$  and the fact that the  $T_w$  are invertible (so that the map must be constant on conjugacy classes). When  $c = p$ , this follows from [Vig17, Prop. 2.2].  $\square$

**Definition 4.2.6.** The unique character  $\chi : H \rightarrow C$  with  $\chi(T_s) = q_s$  (resp.,  $\chi(T_s) = -1$ ) for all  $s \in S$  is called the *trivial* (resp., *special*)  $C$ -character. They are distinct if  $c \neq p, 0$ .

We say a character  $\chi : H \rightarrow C$  extends to  $\tilde{H}$  if there exists a character  $\tilde{\chi} : \tilde{H} \rightarrow C$  such that  $\tilde{\chi}|_H = \chi$ . We wish to determine which characters of  $H$  extend to  $\tilde{H}$ . Since the elements  $T_u$  for  $u \in \Omega$  are invertible in  $\tilde{H}$ , the relations (4.2.4) imply that a character  $\chi : H \rightarrow C$  extends to a character of  $\tilde{H}$  if and only if  $\chi(T_s) = \chi(T_{u(s)})$  for all  $s \in S$  and  $u \in \Omega$ . For example, if the image  $\Psi$  of  $\Omega$  in  $\text{Aut}(W, S, d_i)$  is trivial, then any character of  $H$  extends to  $\tilde{H}$ . The extensions are not unique in general. By their very definition, the trivial and special characters always extend, and we also refer to their extensions as trivial and special characters.

Let  $\chi : H \rightarrow C$  denote a character, and suppose  $c \neq p$ . By Lemma 4.2.5, the value of  $\chi$  on  $T_s$  for  $s \in S_i$  is constant for each  $1 \leq i \leq m$ . We define  $\chi_i := \chi(T_s) \in C$  for  $s \in S_i$ , and identify the character  $\chi$  with the  $m$ -tuple  $(\chi_i)_{1 \leq i \leq m}$ .

**Lemma 4.2.7.** *Assume  $c \neq p$ . Let  $\chi : H \rightarrow C$  denote a character of  $H$ , associated to the  $m$ -tuple  $(\chi_i)_{1 \leq i \leq m}$ . Then  $\chi$  extends to a character of  $\tilde{H}$  except in the following cases:*

- type  $A_1$ , equal parameters  $d_1 = d_2$ ,  $\Psi \neq 1$ , and  $\chi_1 \neq \chi_2$ ;
- type  $C_\ell$  ( $\ell \geq 2$ ), equal parameters  $d_2 = d_3$ ,  $\Psi \neq 1$ , and  $\chi_2 \neq \chi_3$ .

*Proof.* When  $m = 1$ , then  $\chi(T_s) = \chi(T_{u(s)})$  for all  $u \in \Omega$  and  $s \in S$ , so that  $\chi$  extends to  $\tilde{H}$ . We may therefore assume  $m > 1$ . We proceed type-by-type:

- *Type  $A_1$  with equal parameters  $d_1 = d_2$ .* The group  $\text{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$  permutes  $s_1$  and  $s_2$ . If  $\Psi = 1$  or  $\chi_1 = \chi_2$ , then  $\chi$  extends to  $\tilde{H}$ , while if  $\Psi \neq 1$  and  $\chi_1 \neq \chi_2$ , the character  $\chi$  cannot extend.
- *Type  $B_\ell$  ( $\ell \geq 3$ ).* In this case,  $\text{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$  stabilizes the sets  $S_1$  and  $S_2$ , so that  $\chi(T_s) = \chi(T_{u(s)})$  for all  $u \in \Omega$  and  $s \in S$ . Thus  $\chi$  extends to  $\tilde{H}$ .
- *Type  $C_\ell$  ( $\ell \geq 2$ ) with equal parameters  $d_2 = d_3$ .* The group  $\text{Aut}(W, S, d_s) \cong \mathbb{Z}/2\mathbb{Z}$  permutes  $s_2$  and  $s_3$ . If  $\Psi = 1$  or if  $\chi_2 = \chi_3$ , then  $\chi$  extends to  $\tilde{H}$ , while if  $\Psi \neq 1$  and  $\chi_2 \neq \chi_3$ , the character  $\chi$  does not extend.
- *Type  $A_1$  with unequal parameters  $d_1 \neq d_2$ ; Type  $F_4$ ; Type  $G_2$ ; Type  $C_\ell$  ( $\ell \geq 2$ ) with unequal parameters  $d_2 \neq d_3$ .* In these cases,  $\text{Aut}(W, S, d_s)$  (and consequently  $\Psi$ ) is trivial, and thus  $\chi$  extends to  $\tilde{H}$ .

$\square$

Before stating the next result, we require a definition.

**Definition 4.2.8.** Let  $R \subset \mathbb{C}$  be a subring of  $\mathbb{C}$ . We say a right  $\tilde{H}_{\mathbb{C}}$ -module  $M$  is  $R$ -integral if there exists an  $\tilde{H}_R$ -submodule  $M^\circ \subset M$  such that the natural map

$$M^\circ \otimes_R \mathbb{C} \rightarrow M$$

is an isomorphism of  $\tilde{H}_{\mathbb{C}}$ -modules. We call  $M^{\circ}$  an  $R$ -integral structure of  $M$ . If  $\mathfrak{p}$  is a maximal ideal of  $R$ , the  $\tilde{H}_{R/\mathfrak{p}}$ -module  $M^{\circ} \otimes_R R/\mathfrak{p}$  is called *reduction of  $M^{\circ}$  modulo  $\mathfrak{p}$* . We make similar definitions for the algebra  $H_{\mathbb{C}}$ .

We combine the above results in the following proposition.

**Proposition 4.2.9.**

- (i)  $H_{\mathbb{C}}$  admits  $2^m$   $\mathbb{C}$ -characters. They are all  $\mathbb{Z}$ -integral, and their reductions modulo  $p$  are supersingular except for the special and trivial characters.
- (ii) Suppose  $\chi : H_{\mathbb{C}} \rightarrow \mathbb{C}$  is a character, associated to the tuple  $(\chi_i)_{1 \leq i \leq m}$ , and suppose we are in one of the following two cases:
  - type  $A_1$ , equal parameters  $d_1 = d_2$ ,  $\Psi \neq 1$ , and  $\chi_1 \neq \chi_2$ ;
  - type  $C_{\ell}$  ( $\ell \geq 2$ ), equal parameters  $d_2 = d_3$ ,  $\Psi \neq 1$ , and  $\chi_2 \neq \chi_3$ .
Then the  $H_{\mathbb{C}}$ -module  $\chi \oplus \bar{\chi}$  extends to a two-dimensional,  $\mathbb{Z}$ -integral simple (left or right) module with supersingular reduction modulo  $p$ , where  $\bar{\chi} = (\chi_2, \chi_1)$  in the  $A_1$  case, and  $\bar{\chi} = (\chi_1, \chi_3, \chi_2)$  in the  $C_{\ell}$  case.
- (iii) Suppose  $\chi : H_{\mathbb{C}} \rightarrow \mathbb{C}$  is a character which does not fall into either of the two cases of the previous point. Then  $\chi$  extends to a  $\mathbb{Z}$ -integral complex character of  $\tilde{H}_{\mathbb{C}}$ , and its reduction modulo  $p$  is supersingular if  $\chi$  is not special or trivial.

*Proof.* The claims regarding integrality in all three points are immediate.

(i) This follows from Lemmas 4.2.5 and 2.4.2.

(ii) and (iii) If we are not in one of the two exceptional cases, the result follows from Lemmas 4.2.7 and 2.4.2. Otherwise, the character  $\chi$  of  $H_{\mathbb{C}}$  extends to a character  $\chi'$  of  $H' := \mathbb{C}[\Lambda_{Z(G)\tilde{Z}_0}] \otimes_{\mathbb{C}} H_{\mathbb{C}}$  trivial on  $\Lambda_{Z(G)\tilde{Z}_0}$ . The tensor product  $\chi' \otimes_{H'} \tilde{H}_{\mathbb{C}}$  is a simple right  $\tilde{H}_{\mathbb{C}}$ -module of dimension 2 (since the subgroup  $\Lambda_{Z(G)\tilde{Z}_0}$  of  $\Omega$  has index  $|\Psi| = 2$ ), whose restriction to  $H_{\mathbb{C}}$  is equal to  $\chi \oplus \bar{\chi}$ . Note that the characters  $\chi$  and  $\bar{\chi}$  in (ii) are neither special nor trivial, since the  $\chi_i$  are unequal by assumption, and therefore have supersingular reduction modulo  $p$ .  $\square$

**4.3. Discrete simple modules with supersingular reduction.** We continue to assume  $\mathbf{G}$  is absolutely simple and isotropic. Let  $\mathfrak{p}$  denote the maximal ideal of  $\mathbb{Z}[q^{1/2}] \subset \mathbb{C}$  with residue field  $\mathbb{F}_p$ . We now discuss discrete,  $\mathbb{Z}[q^{1/2}]$ -integral  $\tilde{H}_{\mathbb{C}}$ -modules with supersingular reduction modulo  $\mathfrak{p}$ .

The following is the key proposition of this section.

**Proposition 4.3.1.** *Suppose the type of  $\Sigma$  is not equal to  $A_{\ell}$  with equal parameters. Then there exists a simple discrete right  $\tilde{H}_{\mathbb{C}}$ -module  $M_{\mathbb{C}}$  with a  $\mathbb{Z}[q^{1/2}]$ -integral structure  $M$  having supersingular reduction modulo  $\mathfrak{p}$ .*

The proposition will follow from Propositions 4.3.2, 4.3.3, and 4.3.4 below. We sketch the main ideas of the proof.

Consider first the special character  $\chi : H_{\mathbb{C}} \rightarrow \mathbb{C}$ . It is  $\mathbb{Z}[q^{1/2}]$ -integral, its reduction modulo  $\mathfrak{p}$  is non-supersingular, and  $\mathfrak{T}(\chi)$  is equal to the Steinberg representation of  $G^{\text{sc}}$  over  $\mathbb{C}$ , so that  $\chi$  is discrete. Any discrete, non-special character of  $H_{\mathbb{C}}$  is  $\mathbb{Z}[q^{1/2}]$ -integral (in fact,  $\mathbb{Z}$ -integral) and Lemma 2.4.2 implies that its reduction modulo  $\mathfrak{p}$  is supersingular (since the trivial character of  $H_{\mathbb{C}}$  is not discrete). Thus, we first attempt to find a discrete non-special character of  $H_{\mathbb{C}}$ ; these have been classified by Borel in [Bor76, §5.8]. (Note that in [Bor76], the Iwahori subgroup is the pointwise stabiliser

$\tilde{Z}_0\mathfrak{B}$  of an alcove; recall that if  $\mathbf{G}$  is split or semisimple and simply connected we have  $\tilde{Z}_0 = Z_0$ .) We describe these characters in Proposition 4.3.2, and use Proposition 4.2.9 to determine which of these characters extend to  $\tilde{H}_{\mathbb{C}}$ .

When  $m = 1$ , there do not exist any discrete non-special characters of  $H_{\mathbb{C}}$ , and we use instead a reflection module of  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$  (see Proposition 4.3.3). It is free of rank  $|S|$  over  $\mathbb{Z}[q^{1/2}]$  and has supersingular reduction modulo  $\mathfrak{p}$ . When the type is  $A_{\ell}$ , this module is non-discrete, which is why we must omit this type. (We also use reflection modules in Proposition 4.3.4 to handle certain groups of type  $B_3$  for which Proposition 4.3.2 does not apply.)

We now proceed with the required propositions.

**Proposition 4.3.2.** *The algebra  $\tilde{H}_{\mathbb{C}}$  admits a discrete, non-special right module  $M_{\mathbb{C}}$ , induced from or extending a character of  $H_{\mathbb{C}}$ , when the type of  $\Sigma$  is  $B_{\ell}$  ( $\ell \geq 4$ ),  $C_{\ell}$  ( $\ell \geq 2$ ),  $F_4$ ,  $G_2$ ,  $A_1$  with parameters  $d_1 \neq d_2$ , or  $B_3$  with parameters  $(d_1, d_2) \neq (1, 2)$ .*

*The dimension of  $M_{\mathbb{C}}$  is 1, unless  $\Psi \neq 1$  and the type is*

- $C_2$  with parameters  $(1, 1, 1)$ ,  $(2, 1, 1)$ , or  $(3, 2, 2)$ ;
- $C_3$  with parameters  $(1, 1, 1)$ ,  $(1, 2, 2)$ , or  $(2, 3, 3)$ ;
- $C_4$  with parameters  $(1, 2, 2)$ , or  $(2, 3, 3)$ ;
- $C_5$  with parameters  $(1, 2, 2)$ .

*In these cases,  $M_{\mathbb{C}}$  extends the  $H_{\mathbb{C}}$ -module  $(-1, -1, q^d) \oplus (-1, q^d, -1)$  where  $d := d_2 = d_3$ , and thus the dimension of  $M_{\mathbb{C}}$  is 2.*

*Proof.* When  $m = 1$ , the only discrete character of  $H_{\mathbb{C}}$  is the special one ([Bor76, §5.7]).

Suppose  $m > 1$ . For each choice of irreducible root system  $\Sigma$ , we list the possible parameters  $(d_1, d_2)$  or  $(d_1, d_2, d_3)$  for  $G$  (from the tables in [Tit79, §4]), and describe if  $H_{\mathbb{C}}$  has a discrete non-special character (using [Bor76, §5.8]).

We start with  $m = 2$ :

$\Sigma$	Parameters	$\exists$ discrete non-special character of $H_{\mathbb{C}}$ ?
$A_1$	$(d, d)$ ( $d \geq 1$ )	N
	$(1, 3)$	Y
	$(2, 3)$	Y
	$(1, 2)$	Y
	$(1, 4)$	Y
	$(3, 4)$	Y
$B_{\ell}$ ( $\ell \geq 3$ )	$(1, 1)$	Y
	$(1, 2)$	Y (if $\ell \geq 4$ ), N (if $\ell = 3$ )
	$(2, 1)$	Y
	$(2, 3)$	Y
$F_4$	$(1, 1)$	Y
	$(1, 2)$	Y
	$(2, 1)$	Y
$G_2$	$(1, 1)$	Y
	$(1, 3)$	Y
	$(3, 1)$	Y

For every entry marked “Y,” the given discrete non-special character extends to a character of  $\tilde{H}_{\mathbb{C}}$  using the condition of Lemma 4.2.7. For the entries marked “N,” we obtain the following representatives of the strict isogeny classes of the corresponding group:  $\mathbf{PGL}_2(D)$  for a central division algebra  $D$  of dimension  $d^2$  over  $F$  (name  $A_1$  or  ${}^dA_{2d-1}$ );  $\mathbf{PU}(h)$  for a split hermitian form in 3 variables over a ramified field extension (name C-BC<sub>1</sub>);  $\mathbf{PU}(h)$  for a non-split hermitian form in 4 variables over an unramified field extension (name  ${}^2A_3''$ ); or the unramified non-split form of  $\mathbf{PSO}_8$  (name  ${}^2D_4$ ).

We now consider  $m = 3$ , that is, type  $C_{\ell}$ . In this case, the tables in [Bor76, §5.8] show that  $H_{\mathbb{C}}$  always admits a discrete, non-special character. Note also that Borel omitted the parameters (3, 2, 2) for type  $C_2$ . In order to obtain this missing case, we use the criterion of [Bor76, Eqn. 5.6(2)] to see that the only discrete non-special characters of  $H_{\mathbb{C}}$  are  $(-1, -1, 1)$  and  $(-1, 1, -1)$  (in the notation of [Bor76]). Note that the characters corresponding to parameters with  $d_2 \neq d_3$  automatically extend to  $\tilde{H}_{\mathbb{C}}$ , by Lemma 4.2.7.

$\Sigma$	Parameters	Condition that some discrete non-special character of $H_{\mathbb{C}}$ extends to $\tilde{H}_{\mathbb{C}}$
$C_{\ell}$ ( $\ell \geq 2$ )	(1, 1, 1)	$\ell \geq 4$ , or $\Psi = 1$
	(2, 1, 1)	$\ell \geq 3$ , or $\Psi = 1$
	(2, 3, 3)	$\ell = 2, \ell \geq 5$ , or $\Psi = 1$
	(2, 1, 3)	$\emptyset$
	(1, 1, 2)	$\emptyset$
	(2, 2, 3)	$\emptyset$
	(2, 1, 2)	$\emptyset$
	(1, 2, 2)	$\ell = 2, \ell \geq 6$ , or $\Psi = 1$
	(2, 1, 4)	$\emptyset$
	(2, 3, 4)	$\emptyset$
$C_2$	(3, 2, 2)	$\Psi = 1$

Finally, we remark that in all cases, Propositions 4.1.10 and 4.2.9 imply that the  $\tilde{H}_{\mathbb{C}}$ -module  $M_{\mathbb{C}}$  constructed above (either as the extension of a character of  $H_{\mathbb{C}}$ , or as the induction of a character from  $H_{\mathbb{C}}$  to  $\tilde{H}_{\mathbb{C}}$ ) is discrete.  $\square$

We consider now the types  $D_{\ell}$  ( $\ell \geq 4$ ),  $E_6$ ,  $E_7$ , and  $E_8$ . The tables in [Tit79, §4] imply that  $\mathbf{G}$  is  $F$ -split, so that  $d_s = 1$  for all  $s \in S$ , and for distinct  $s, t \in S$ , the order  $n_{s,t}$  of  $st$  is 2 or 3.

**Proposition 4.3.3.** *Assume that the type of  $\Sigma$  is  $D_{\ell}$  ( $\ell \geq 4$ ),  $E_6$ ,  $E_7$ , or  $E_8$ . Let  $M$  denote the right  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module obtained as the twist of the (left) reflection  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module by the anti-automorphism  $T_w \mapsto (-1)^{\ell(w)} T_w^*$ . Then  $M$  is free of rank  $|S|$  over  $\mathbb{Z}[q^{1/2}]$ , has supersingular reduction modulo  $\mathfrak{p}$ , and  $M_{\mathbb{C}}$  is a discrete, simple right  $\tilde{H}_{\mathbb{C}}$ -module.*



*Proof.* The left reflection  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module is the free  $\mathbb{Z}[q^{1/2}]$ -module with basis  $\{e_t\}_{t \in S}$ , with  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module structure given by

$$T_s \cdot e_t = \begin{cases} -e_t & \text{for } s = t, \\ qe_t & \text{for } s \neq t, n_{s,t} = 2, \\ qe_t + q^{1/2}e_s & \text{for } s \neq t, n_{s,t} = 3, \end{cases}$$

$$T_u \cdot e_t = e_{u(t)},$$

where  $s, t \in S, u \in \Omega$ . Twisting this module by the automorphism  $T_w \mapsto (-1)^{\ell(w)} T_w^*$  gives a left  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module  $M'$ , satisfying

$$T_s \cdot e_t = \begin{cases} qe_t & \text{for } s = t, \\ -e_t & \text{for } s \neq t, n_{s,t} = 2, \\ -e_t - q^{1/2}e_s & \text{for } s \neq t, n_{s,t} = 3, \end{cases}$$

$$T_u \cdot e_t = e_{u(t)}.$$

Thus, the reduction modulo  $\mathfrak{p}$  of  $M'$  is the  $\mathbb{F}_p$ -vector space with basis  $\{e_t\}_{t \in S}$ , with the structure of a left  $\tilde{H}_{\mathbb{F}_p}$ -module given by

$$T_s \cdot e_t = \begin{cases} 0 & \text{for } s = t, \\ -e_t & \text{for } s \neq t, \end{cases}$$

$$T_u \cdot e_t = e_{u(t)}.$$

The restriction to  $H_{\mathbb{F}_p}$  of this  $\tilde{H}_{\mathbb{F}_p}$ -module is the direct sum of the supersingular characters  $\{\chi_s\}_{s \in S}$ , where

$$\chi_s(T_t) = \begin{cases} 0 & \text{for } s = t, \\ -1 & \text{for } s \neq t. \end{cases}$$

Thus, Lemma 2.4.2 implies  $M'_{\mathbb{F}_p}$  is supersingular. The scalar extension of  $M'$  from  $\mathbb{Z}[q^{1/2}]$  to  $\mathbb{C}$  is a discrete simple  $\tilde{H}_{\mathbb{C}}$ -module by Proposition 4.1.6 (see also [Lus83, 4.23]). These properties (discrete scalar extension to  $\mathbb{C}$  and supersingular reduction modulo  $\mathfrak{p}$ ) remain true if we twist  $M'$  into a right  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module  $M$  by the anti-automorphism  $T_w \mapsto T_{w^{-1}}$ .  $\square$

Finally, we consider one of the omitted cases from Proposition 4.3.2, namely type  $B_3$  with parameters  $(1, 2)$ . We also make a specific choice of  $\mathbf{G}$  in the corresponding strict isogeny class; by the reductions in §3.3, this does not affect our main result.

**Proposition 4.3.4.** *Suppose that  $\mathbf{G}$  is an unramified (in particular, quasi-split), non-split form of  $\mathbf{PSO}_8$ . Then  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$  admits a right module  $M$ , such that  $M$  is free of rank 3 over  $\mathbb{Z}[q^{1/2}]$ , has supersingular reduction modulo  $\mathfrak{p}$ , and  $M_{\mathbb{C}}$  is a discrete, simple right  $\tilde{H}_{\mathbb{C}}$ -module.*

*Proof.* We will use the reflection module as defined in [GS05, §7].

Denote by  $\tilde{\Delta}_{\text{long}}$  the subset of simple affine roots  $\tilde{\Delta}$  which are long. We define an action of  $H_{\mathbb{Z}[q^{1/2}]}$  on the rank 3  $\mathbb{Z}[q^{1/2}]$ -module with basis  $\{e_\beta\}_{\beta \in \tilde{\Delta}_{\text{long}}}$  as follows. If

$\alpha \in \tilde{\Delta}_{\text{long}}$ , we set

$$T_{s_\alpha} \cdot e_\beta = \begin{cases} -e_\beta & \text{if } \alpha = \beta, \\ qe_\beta & \text{if } \alpha \neq \beta, n_{s_\alpha, s_\beta} = 2, \\ qe_\beta + q^{1/2}e_\alpha & \text{if } \alpha \neq \beta, n_{s_\alpha, s_\beta} = 3, \end{cases}$$

and if  $\alpha$  is the unique short root in  $\tilde{\Delta}$ , we set

$$T_{s_\alpha} \cdot e_\beta = q^2 e_\beta.$$

Twisting this reflection module by the automorphism  $T_w \mapsto (-1)^{\ell(w)} T_w^*$  gives a new left  $H_{\mathbb{Z}[q^{1/2}]}$ -module  $M'$ , with action given by

$$T_{s_\alpha} \cdot e_\beta = \begin{cases} qe_\beta & \text{if } \alpha = \beta, \\ -e_\beta & \text{if } \alpha \neq \beta, n_{s_\alpha, s_\beta} = 2, \\ -e_\beta - q^{1/2}e_\alpha & \text{if } \alpha \neq \beta, n_{s_\alpha, s_\beta} = 3, \end{cases}$$

if  $\alpha \in \tilde{\Delta}_{\text{long}}$ , and

$$T_{s_\alpha} \cdot e_\beta = -e_\beta$$

if  $\alpha \in \tilde{\Delta}$  is short. The reduction modulo  $\mathfrak{p}$  of  $M'$  is the  $\mathbb{F}_p$ -vector space with basis  $\{e_\beta\}_{\beta \in \tilde{\Delta}_{\text{long}}}$ , with the structure of a left  $H_{\mathbb{F}_p}$ -module given by

$$T_{s_\alpha} \cdot e_\beta = \begin{cases} 0 & \text{if } \alpha = \beta, \\ -e_\beta & \text{if } \alpha \neq \beta, \end{cases}$$

for  $\alpha \in \tilde{\Delta}$ . Therefore  $M'_{\mathbb{F}_p}$  is equal to the direct sum of the supersingular characters  $\{\chi_\beta\}_{\beta \in \tilde{\Delta}_{\text{long}}}$ , where

$$\chi_\beta(T_{s_\alpha}) = \begin{cases} 0 & \text{if } \alpha = \beta, \\ -1 & \text{if } \alpha \neq \beta. \end{cases}$$

for  $\alpha \in \tilde{\Delta}$ .

The algebra  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$  is generated by  $H_{\mathbb{Z}[q^{1/2}]}$  and an extra element  $T_u$ , where  $u$  is a generator of  $\Omega \cong \mathbb{Z}/2\mathbb{Z}$ . The element  $u$  exchanges the two elements of  $\tilde{\Delta}_{\text{long}}$  which are endpoints of Dyn, and fixes the other two elements of  $\tilde{\Delta}$ . Thus, the element  $T_u$  satisfies

$$T_u^2 = 1, \quad T_u T_{s_\alpha} T_u = T_{s_{u(\alpha)}}.$$

We extend the action of  $H_{\mathbb{Z}[q^{1/2}]}$  on  $M'$  to  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$  by declaring that

$$T_u \cdot e_\alpha = e_{u(\alpha)}.$$

Section 8.5 of [GS05] gives an explicit description of Hecke operators associated to the fundamental dominant coweights in terms of  $T_u$  and the  $T_{s_\alpha}$ . Using this description along with [GS05, Prop. 6.6] and Proposition 4.1.6, we see that the  $\tilde{H}_{\mathbb{C}}$ -module  $M'_{\mathbb{C}}$  is discrete. Once again, the properties of  $M'$  (supersingular reduction modulo  $\mathfrak{p}$  and discrete extension to  $\mathbb{C}$ ) remain true if we twist  $M'$  into a right  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module  $M$  via the anti-automorphism  $T_w \mapsto T_{w^{-1}}$ .  $\square$

**4.4. Admissible integral structure via discrete cocompact subgroups.** Let  $E$  be a number field with ring of integers  $\mathcal{O}_E$ ,  $\mathfrak{p}_E$  a maximal ideal of  $\mathcal{O}_E$  with residue field  $k := \mathcal{O}_E/\mathfrak{p}_E$ , and  $C/E$  a field extension.

**Definition 4.4.1.** We say that a  $C$ -representation  $\pi$  of  $G$  *descends to  $E$*  if there exists an  $E$ -representation  $\tau$  of  $G$  and a  $G$ -equivariant  $C$ -linear isomorphism

$$\varphi : C \otimes_E \tau \xrightarrow{\sim} \pi.$$

We call  $\varphi$  (and more often  $\tau$ ) an  $E$ -*structure* of  $\pi$ .

**Definition 4.4.2.** We say that a  $C$ -representation  $\pi$  of  $G$  is  $\mathcal{O}_E$ -*integral* if  $\pi$  contains a  $G$ -stable  $\mathcal{O}_E$ -submodule  $\tau^\circ$  such that, for any compact open subgroup  $K$  of  $G$ , the  $\mathcal{O}_E$ -module  $(\tau^\circ)^K$  is finitely generated, and the natural map

$$\varphi : C \otimes_{\mathcal{O}_E} \tau^\circ \rightarrow \pi$$

is an isomorphism. We call  $\varphi$  (and more often  $\tau^\circ$ ) an  $\mathcal{O}_E$ -*integral structure* of  $\pi$ . The  $G$ -equivariant map  $\tau^\circ \rightarrow k \otimes_{\mathcal{O}_E} \tau^\circ$  (and more often the  $k$ -representation  $k \otimes_{\mathcal{O}_E} \tau^\circ$  of  $G$ ) is called the *reduction of  $\tau^\circ$  modulo  $\mathfrak{p}_E$* . We say that  $\tau^\circ$  is *admissible* if  $k \otimes_{\mathcal{O}_E} \tau^\circ$  is admissible for all  $\mathfrak{p}_E$ .

For any commutative ring  $R$  and any discrete cocompact subgroup  $\Gamma$  of  $G$ , we define

$$C^\infty(\Gamma \backslash G, R) := \left\{ f : G \rightarrow R \mid \begin{array}{l} f(\gamma g k) = f(g) \text{ for all } \gamma \in \Gamma, g \in G, \\ \text{and } k \in K_f \end{array} \right\},$$

where  $K_f$  is some compact open subgroup of  $G$  depending on  $f$ . Letting  $G$  act on this space by right translation, we obtain a smooth  $R$ -representation  $\rho_R^\Gamma$ . The complex representation  $\rho_{\mathbb{C}}^\Gamma$  of  $G$  has an admissible  $\mathbb{Z}$ -integral structure given by  $\rho^\Gamma := \rho_{\mathbb{Z}}^\Gamma$ . The reduction of  $\rho^\Gamma$  modulo a prime number  $c$  is the admissible representation  $\rho_{\mathbb{F}_c}^\Gamma$ .

**Proposition 4.4.3.** *Assume  $\text{char } F = 0$  and  $\mathbf{G}$  semisimple. If  $\pi$  is a square-integrable  $\mathbb{C}$ -representation of  $G$ , then there exists a discrete cocompact subgroup  $\Gamma$  of  $G$  such that*

$$\text{Hom}_G(\pi, \rho_{\mathbb{C}}^\Gamma) \neq 0.$$

*Proof.* Since  $\text{char } F = 0$ , there exists a decreasing sequence  $(\Gamma_n)_{n \in \mathbb{N}}$  of discrete cocompact subgroups of  $G$  with trivial intersection, such that each is normal and of finite index in  $\Gamma = \Gamma_0$  (see [BH78, Thm. A]). The normalized multiplicity of  $\pi$  in  $\rho_{\mathbb{C}}^\Gamma$  is

$$m_{\Gamma, dg}(\pi) := \text{vol}_\Gamma \cdot \dim_{\mathbb{C}} (\text{Hom}_G(\pi, \rho_{\mathbb{C}}^\Gamma)),$$

where  $\text{vol}_\Gamma$  is the volume of  $\Gamma \backslash G$  for a  $G$ -invariant measure induced by a Haar measure on  $G$ . By the square-integrability assumption on  $\pi$  and the limit multiplicity formula, the sequence  $(m_{\Gamma_n, dg}(\pi))_{n \in \mathbb{N}}$  converges to a nonzero real number (see [DKV84, App. 3, Prop.] and [Kaz86, Thm. K]).  $\square$

**Proposition 4.4.4.** *Assume  $\text{char } F = 0$ . Let  $\pi$  be an irreducible  $\mathbb{C}$ -representation of  $G$  such that  $\pi^{\mathfrak{B}} \neq 0$  and  $\Gamma$  a discrete cocompact subgroup of  $G$ .*

(i) *If*

$$\varphi : \mathbb{C} \otimes_E \tau \xrightarrow{\sim} \pi$$

*is an  $E$ -structure of  $\pi$ , then*

$$\varphi^{\mathfrak{B}} : \mathbb{C} \otimes_E \tau^{\mathfrak{B}} \rightarrow \pi^{\mathfrak{B}}$$

is an  $E$ -structure of  $\pi^{\mathfrak{B}}$ , and the natural map

$$\mathbb{C} \otimes_E \mathrm{Hom}_{E[G]}(\tau, \rho_E^\Gamma) \rightarrow \mathrm{Hom}_{\mathbb{C}[G]}(\pi, \rho_{\mathbb{C}}^\Gamma)$$

is an isomorphism.

(ii) If

$$\psi : \mathbb{C} \otimes_E M \xrightarrow{\sim} \pi^{\mathfrak{B}}$$

is an  $E$ -structure of  $\pi^{\mathfrak{B}}$ , then

$$\mathfrak{T}(\psi) : \mathbb{C} \otimes_E \mathfrak{T}(M) \rightarrow \pi$$

is an  $E$ -structure of  $\pi$ .

(iii) Any irreducible subrepresentation  $\tau$  of  $\rho_E^\Gamma$  admits an admissible  $\mathcal{O}_E$ -integral structure  $\tau \cap \rho_{\mathcal{O}_E}^\Gamma$ , whose reduction modulo  $\mathfrak{p}_E$  contained in  $\rho_k^\Gamma$ .

*Proof.* We recall a general result in algebra from [Bou12, §12.2 Lem. 1]: let  $C'/C$  be a field extension and  $A$  a  $C$ -algebra. For  $A$ -modules  $M, N$ , the natural map

$$(4.4.5) \quad C' \otimes_C \mathrm{Hom}_A(M, N) \rightarrow \mathrm{Hom}_{C' \otimes_C A}(C' \otimes_C M, C' \otimes_C N)$$

is injective, and bijective if  $C'/C$  is finite or the  $A$ -module  $M$  is finitely generated.

(i) Take  $C'/C = \mathbb{C}/E$ ,  $A = E[G]$ ,  $(M, N) = (E[\mathfrak{B} \backslash G], \tau)$  or  $(\tau, \rho_E^\Gamma)$ . Then (4.4.5) is an isomorphism because  $E[\mathfrak{B} \backslash G]$  (resp.,  $\tau$ ) is an  $E$ -representation of  $G$  which is generated by the characteristic function of  $\mathfrak{B}$  (resp., irreducible).

(ii) The functor  $\pi \mapsto \pi^{\mathfrak{B}}$  and its left adjoint  $\mathfrak{T}$  commute with scalar extension from  $E$  to  $\mathbb{C}$  and are inverse equivalences of categories when  $C = \mathbb{C}$  (by Fact 4.1.1). Hence

$$\mathfrak{T}(\psi) : \mathbb{C} \otimes_E \mathfrak{T}(M) = \mathfrak{T}(\mathbb{C} \otimes_E M) \rightarrow \mathfrak{T}(\pi^{\mathfrak{B}}) = \pi$$

is an isomorphism.

(iii) For any compact open subgroup  $K$  of  $G$ , the  $\mathcal{O}_E$ -module  $(\rho_{\mathcal{O}_E}^\Gamma)^K$  is finite free and  $\rho_{\mathcal{O}_E}^\Gamma$  contains  $\tau^\circ := \tau \cap \rho_{\mathcal{O}_E}^\Gamma$  as  $\mathcal{O}_E$ -representations of  $G$ . Since the ring  $\mathcal{O}_E$  is noetherian, these facts imply the  $\mathcal{O}_E$ -submodule  $(\tau^\circ)^K$  of  $(\rho_{\mathcal{O}_E}^\Gamma)^K$  is finitely generated. The natural linear  $G$ -equivariant isomorphism

$$E \otimes_{\mathcal{O}_E} \rho_{\mathcal{O}_E}^\Gamma \xrightarrow{\sim} \rho_E^\Gamma$$

restricts to a linear  $G$ -equivariant isomorphism

$$E \otimes_{\mathcal{O}_E} \tau^\circ \xrightarrow{\sim} \tau,$$

and therefore  $\tau^\circ$  is an  $\mathcal{O}_E$ -integral structure of  $\tau$ . We have

$$\begin{aligned} \mathfrak{p}_E \rho_{\mathcal{O}_E}^\Gamma \cap \tau^\circ &= C^\infty(\Gamma \backslash G, \mathfrak{p}_E) \cap \tau^\circ \\ &= C^\infty(\Gamma \backslash G, \mathfrak{p}_E) \cap \tau \\ &= \mathfrak{p}_E(C^\infty(\Gamma \backslash G, \mathcal{O}_E) \cap \tau) \\ &= \mathfrak{p}_E \tau^\circ. \end{aligned}$$

Therefore, the reduction modulo  $\mathfrak{p}_E$  of  $\tau^\circ$  (i.e.,  $(\mathcal{O}_E/\mathfrak{p}_E) \otimes_{\mathcal{O}_E} \tau^\circ = k \otimes_{\mathcal{O}_E} \tau^\circ$ ) is contained in the reduction modulo  $\mathfrak{p}_E$  of  $\rho_{\mathcal{O}_E}^\Gamma$  (i.e.,  $\rho_k^\Gamma$ ) as  $k$ -representations of  $G$ . As  $\rho_k^\Gamma$  is admissible,  $k \otimes_{\mathcal{O}_E} \tau^\circ$  is also.  $\square$

Although the ring  $\mathbb{Z}[q^{1/2}]$  is not always the ring of integers  $\mathcal{O}_E$  of  $E = \mathbb{Q}(q^{1/2})$ , the arguments of (iii) still apply in this setting. The key Proposition 4.3.1 then implies:

**Corollary 4.4.6.** *Assume  $\text{char } F = 0$ ,  $\mathbf{G}$  absolutely simple and isotropic, and the type of  $\Sigma$  is different from  $A_\ell$  with equal parameters. Let  $M_{\mathbb{C}}$  denote a discrete right  $\tilde{H}_{\mathbb{C}}$ -module as in Proposition 4.3.1. Then the irreducible square-integrable  $\mathbb{C}$ -representation  $\mathfrak{T}(M_{\mathbb{C}})$  of  $G$  admits an admissible  $\mathbb{Z}[q^{1/2}]$ -integral structure.*

*Proof.* By Proposition 4.4.4(ii), the  $\mathbb{C}$ -representation  $\mathfrak{T}(M_{\mathbb{C}})$  admits a  $\mathbb{Q}(q^{1/2})$ -structure  $\tau$ . Propositions 4.4.3 and 4.4.4(i) imply that  $\tau$  is a subrepresentation of  $\rho_{\mathbb{Q}(q^{1/2})}^{\Gamma}$  for some discrete cocompact subgroup  $\Gamma$  of  $G$ . Finally, Proposition 4.4.4(iii) implies that  $\tau$  admits an admissible  $\mathbb{Z}[q^{1/2}]$ -integral structure.  $\square$

When  $\mathbf{G}$  is semisimple, any irreducible admissible supercuspidal  $\mathbb{C}$ -representation of  $G$  descends to a number field (see [Vig96, II.4.9]). We have:

**Corollary 4.4.7.** *Assume  $\text{char } F = 0$  and  $\mathbf{G}$  semisimple. Then any irreducible supercuspidal  $\mathbb{C}$ -representation admits an admissible  $\mathcal{O}_E$ -integral structure whose reduction modulo  $\mathfrak{p}_E$  is contained in  $\rho_k^{\Gamma}$ , for some discrete cocompact subgroup  $\Gamma$  of  $G$ .*

*Proof.* Since  $\pi$  is square-integrable, Proposition 4.4.3 implies that  $\pi$  embeds into  $\rho_{\mathbb{C}}^{\Gamma}$  for some discrete cocompact subgroup  $\Gamma$  of  $G$ . The claim follows from points (i) and (iii) of Proposition 4.4.4.  $\square$

**4.5. Reduction to rank 1 and  $\text{PGL}_n(D)$ .** We now prove that most  $p$ -adic reductive groups admit irreducible admissible supersingular representations.

**Theorem 4.5.1.** *Assume that  $\text{char } C = p$  and  $\text{char } F = 0$ . Suppose  $\mathbf{G}$  is an isotropic, absolutely simple, connected adjoint  $F$ -group, not isomorphic to any of the following groups:*

- (i)  $\text{PGL}_n(D)$ , where  $n \geq 2$  and  $D$  a central division algebra over  $F$ ;
- (ii)  $\text{PU}(h)$ , where  $h$  is a split hermitian form in 3 variables over a ramified quadratic extension of  $F$  or a non-split hermitian form in 4 variables over the unramified quadratic extension of  $F$ .

*Then  $G$  admits an irreducible admissible supercuspidal  $C$ -representation.*

*Proof.* We first note by the tables in [Tit79] that the above exceptional groups are precisely the ones where  $\Sigma$  is of type  $A_\ell$  with equal parameters. (In that reference our exceptional groups have names  $A_{m-1}$ ,  ${}^d A_{md-1}$  for  $m \geq 2$ ,  $d \geq 2$  in case (i) and  $C\text{-BC}_1$ ,  ${}^2 A_3''$  in case (ii).) By Proposition 4.3.1 there exists a right  $\tilde{H}_{\mathbb{Z}[q^{1/2}]}$ -module  $M$  whose base change  $M_{\mathbb{C}}$  is a simple discrete  $\tilde{H}_{\mathbb{C}}$ -module, and whose reduction  $M_{\mathbb{F}_p}$  is supersingular. Letting  $\pi := \mathfrak{T}(M_{\mathbb{C}})$  denote the irreducible square-integrable  $\mathbb{C}$ -representation of  $G$  corresponding to  $M_{\mathbb{C}}$ , we know by Proposition 4.4.3 that  $\pi$  injects into  $C^\infty(\Gamma \backslash G, \mathbb{C})$  for some discrete cocompact subgroup  $\Gamma \leq G$ . By passing to  $\mathfrak{B}$ -invariants it follows that the  $\tilde{H}_{\mathbb{C}}$ -module  $M_{\mathbb{C}}$  injects into  $\mathbb{C}[\Gamma \backslash G / \mathfrak{B}] = C^\infty(\Gamma \backslash G, \mathbb{C})^{\mathfrak{B}}$ .

Set

$$M' := M_{\mathbb{C}} \cap \mathbb{Z}[q^{1/2}][\Gamma \backslash G / \mathfrak{B}] = M_{\mathbb{C}} \cap C^\infty(\Gamma \backslash G, \mathbb{Z}[q^{1/2}])^{\mathfrak{B}}.$$

Then  $M$  and  $M'$  are  $\mathbb{Z}[q^{1/2}]$ -integral structures of  $M_{\mathbb{C}}$ , so their reductions  $M_{\mathbb{F}_p}$  and  $M'_{\mathbb{F}_p}$  agree up to semisimplification. In particular,  $M'_{\mathbb{F}_p}$  contains a supersingular element (as  $M_{\mathbb{F}_p}$  is supersingular) and, by construction,  $M'_{\mathbb{F}_p}$  embeds into  $\mathbb{F}_p[\Gamma \backslash G / \mathfrak{B}]$ . Therefore we can pick a non-zero supersingular element  $v$  of  $\mathbb{F}_p[\Gamma \backslash G / \mathfrak{B}] \subset C[\Gamma \backslash G / \mathfrak{B}]$ . The

$G$ -representation  $C^\infty(\Gamma \backslash G, C)$  is admissible, as  $\Gamma$  is cocompact, and hence so is its subrepresentation  $\langle G \cdot v \rangle$  generated by  $v$ . Any irreducible quotient of  $\langle G \cdot v \rangle$  (which exists by Zorn's lemma) is admissible by [Hen09, §4, Thm. 1], as  $F$  is of characteristic zero, and supersingular by Proposition 3.1.2, as it contains (the nonzero image of)  $v$ .  $\square$

The two exceptional cases will be dealt with in Sections 5, 6 below. Assuming this, we can now prove our main result.

*Proof of Theorem A.* Suppose that  $\mathbf{G}$  is a connected reductive group over  $F$ . We want to show that  $G = \mathbf{G}(F)$  admits an irreducible admissible supercuspidal representation over any field  $C$  of characteristic  $p$ . By Proposition 3.2.1 we may assume that  $C$  is finite and as large as we like. Then by Proposition 3.3.2 we may assume that  $\mathbf{G}$  is isotropic, absolutely simple, and connected adjoint. The result then follows from Theorem 4.5.1, Corollary 5.5.2, and Corollary 6.6.2.  $\square$

## 5. SUPERCUSPIDAL REPRESENTATIONS OF RANK 1 GROUPS

In this section we verify Theorem A when  $\mathbf{G}$  is a connected reductive  $F$ -group of relative semisimple rank 1. In particular, this deals with the second exceptional case in Theorem 4.5.1.

**5.1. Preliminaries.** We suppose in this section that  $C$  is algebraically closed field of characteristic  $c = p$ . We fix an embedding of the residue field  $k_F$  into  $C$ , and henceforth view  $k_F$  as a subfield of  $C$ .

We will show that  $G$  admits irreducible, admissible, supercuspidal  $C$ -representations. By Proposition 3.3.2, it suffices to assume  $\mathbf{G}$  is an absolutely simple and adjoint group of relative rank 1. We make one further reduction. Let  $\mathbf{G}^{\text{sc}}$  denote the simply-connected cover of  $\mathbf{G}$ :

$$1 \rightarrow \mathbf{Z}(\mathbf{G}^{\text{sc}}) \rightarrow \mathbf{G}^{\text{sc}} \rightarrow \mathbf{G} \rightarrow 1$$

By Proposition 3.3.8, we see that  $G^{\text{sc}}$  admits an irreducible, admissible, supercuspidal representation on which  $Z(G^{\text{sc}})$  acts trivially if and only if  $G$  does. Therefore, we may assume that our group  $\mathbf{G}$  is absolutely simple, simply connected, and has relative rank equal to 1. We will then construct irreducible, admissible, supercuspidal representations of  $G$  on which its (finite) center acts trivially.

**5.2. Parahoric subgroups.** Let  $\mathcal{B}$  denote the Bruhat–Tits building of  $G$ . By our assumptions on  $\mathbf{G}$ ,  $\mathcal{B}$  is a one-dimensional contractible simplicial complex, i.e., a tree. Recall that  $\mathcal{C}$  denotes the chamber of  $\mathcal{B}$  corresponding to the Iwahori subgroup  $\mathfrak{B}$ , and let  $x_0$  and  $x_1$  denote the two vertices in the closure of  $\mathcal{C}$ . We let  $K_0$  and  $K_1$  denote the stabilizers of  $x_0$  and  $x_1$ , respectively. We then have  $\mathfrak{B} = K_0 \cap K_1$ .

The vertices  $x_0$  and  $x_1$  are representatives of the two orbits of  $G$  on the set of vertices of  $\mathcal{B}$ , and the edge  $\mathcal{C}$  is a representative of the unique orbit of  $G$  on the edges of  $\mathcal{B}$ . By [Ser03, §4, Thm. 6], we may therefore write the group  $G$  as an amalgamated product:

$$G \cong K_0 *_\mathfrak{B} K_1.$$

Since the group  $\mathbf{G}$  is semisimple and simply connected, the stabilizers of vertices and edges in  $\mathcal{B}$  are parahoric subgroups (see, e.g., [Vig16, §3.7]). For  $i \in \{0, 1\}$ , we let  $K_i^+$  denote the pro- $p$  radical of  $K_i$ , that is, the largest open, normal, pro- $p$  subgroup

of  $K_i$ . The quotient  $\mathbb{G}_i := K_i/K_i^+$  is isomorphic to the group of  $k_F$ -points of a connected reductive group over  $k_F$  (see [HV15, §3.7]). Likewise, the pro- $p$ -Sylow  $\mathfrak{U}$  is the largest open, normal, pro- $p$  subgroup of  $\mathfrak{B}$ , and  $\mathbb{Z} := \mathfrak{B}/\mathfrak{U}$  is isomorphic to the group of  $k_F$ -points of a torus over  $k_F$ . The image of  $\mathfrak{B}$  in  $\mathbb{G}_i$  is equal to a minimal parabolic subgroup  $\mathbb{B}_i$ , with Levi decomposition  $\mathbb{B}_i = \mathbb{Z}_i\mathbb{U}_i$ . Thus, we identify the quotient  $\mathbb{Z}$  with  $\mathbb{Z}_i$ .

**5.3. Pro- $p$  Iwahori–Hecke algebras.** We work in slightly greater generality than §2. Let

$$H_C^+ := H_C(G, \mathfrak{U}) = \text{End}_G C[\mathfrak{U}\backslash G]$$

denote the pro- $p$ -Iwahori–Hecke algebra of  $G$  with respect to  $\mathfrak{U}$ . We view  $H_C^+$  as the convolution algebra of  $C$ -valued, compactly supported,  $\mathfrak{U}$ -bi-invariant functions on  $G$  (see [Vig16, §4] for more details). For  $g \in G$ , we let  $T_g$  denote the characteristic function of  $\mathfrak{U}g\mathfrak{U}$ . The algebra  $H_C^+$  is generated by two operators  $T_{\tilde{s}_0}, T_{\tilde{s}_1}$ , along with operators  $T_z$ , where  $\tilde{s}_0$  and  $\tilde{s}_1$  are lifts to the pro- $p$ -Iwahori–Weyl group of affine reflections fixing  $x_0$  and  $x_1$ , respectively, and  $z \in \mathbb{Z}$ . For  $i \in \{0, 1\}$ , we let  $H_{C,i}^+$  denote the subalgebra of  $H_C^+$  generated by  $T_{\tilde{s}_i}$  and  $T_z$  for  $z \in \mathbb{Z}$ ; this is exactly the subalgebra of functions in  $H_C^+$  with support in  $K_i$ , i.e.,

$$H_{C,i}^+ = H_C(K_i, \mathfrak{U}) = \text{End}_{K_i} C[\mathfrak{U}\backslash K_i].$$

The algebra  $H_{C,i}^+$  is canonically isomorphic to the finite Hecke algebra  $H_C(\mathbb{G}_i, \mathbb{U}_i)$  (see [CE04, §6.1]).

Since  $K_i^+$  is an open normal pro- $p$  subgroup of  $K_i$ , the irreducible smooth representations of  $K_i$  and  $\mathbb{G}_i$  are in bijection. Further, the finite group  $\mathbb{G}_i$  possesses a strongly split BN pair of characteristic  $p$  ([Vig16, Prop. 3.25]). Therefore, by [CE04, Thm. 6.12], the functor  $\rho \mapsto \rho^{\mathfrak{U}}$  induces a bijection between (isomorphism classes of) irreducible smooth representations of  $K_i$  and (isomorphism classes of) simple right  $H_{C,i}^+$ -modules, all of which are one-dimensional.

We briefly recall some facts about supersingular  $H_C^+$ -modules (compare Lemma 2.4.2). We refer to [Vig17, Def. 6.10] for the precise definition (which is analogous to Definition 2.4.1), and give instead the classification of simple supersingular  $H_C^+$ -modules. Since  $\mathbf{G}$  is simply connected, every supersingular  $H_C^+$ -module is a character. The characters  $\Xi$  of  $H_C^+$  are parametrized by pairs  $(\chi, J)$ , where  $\chi : \mathbb{Z} \rightarrow C^\times$  is a character of the finite torus and  $J$  is a subset of

$$S_\chi := \{s \in \{s_0, s_1\} : \chi(c_{\tilde{s}}) \neq 0\}$$

(here  $c_{\tilde{s}}$  is a certain element of the group algebra of the torus  $\mathbb{Z}$  which appears in the quadratic relation for  $T_{\tilde{s}}$ ; note also that the definition of  $S_\chi$  is independent of the choice of lift  $\tilde{s}$ ). The correspondence is given as follows (cf. [Vig17, Thm. 1.6]): for  $z \in \mathbb{Z}$ , we have  $\Xi(T_z) = \chi(z)$ , and for  $s \in \{s_0, s_1\}$ , we have

$$\Xi(T_{\tilde{s}}) = \begin{cases} 0 & \text{if } s \in J, \\ \chi(c_{\tilde{s}}) & \text{if } s \notin J. \end{cases}$$

Since  $\mathbf{G}$  is simple, [Vig17, Thm. 1.6] implies that  $\Xi$  is supersingular if and only if

$$(S_\chi, J) \neq (\{s_0, s_1\}, \emptyset), (\{s_0, s_1\}, \{s_0, s_1\}).$$

**5.4. Diagrams.** Since the group  $G$  is an amalgamated product of two parahoric subgroups, the formalism of diagrams used in [KX15] applies to the group  $G$ . We recall that a diagram  $D$  is a quintuple  $(\rho_0, \rho_1, \sigma, \iota_0, \iota_1)$  which consists of a smooth representation  $\rho_i$  of  $K_i$ , a smooth representation  $\sigma$  of  $\mathfrak{B}$ , and  $\mathfrak{B}$ -equivariant morphisms  $\iota_i : \sigma \rightarrow \rho_i|_{\mathfrak{B}}$ . We depict diagrams as

$$\begin{array}{ccc} & & \rho_0 \\ & \nearrow^{\iota_0} & \\ \sigma & & \\ & \searrow_{\iota_1} & \\ & & \rho_1 \end{array}$$

Morphisms of diagrams are defined in the obvious way (i.e., so that the relevant squares commute).

Let  $\Xi$  denote a supersingular character of  $H_C^+$ , associated to a pair  $(\chi, J)$ . We define a diagram  $D_\Xi$  as follows:

- set  $\sigma := \chi^{-1}$ , which we view as a character of  $\mathfrak{B}$  by inflation;
- we let  $\rho_{\Xi,i}$  denote an irreducible smooth  $K_i$ -representation such that  $\rho_{\Xi,i}^{\mathfrak{M}} \cong \Xi|_{H_{C,i}^+}$  as  $H_{C,i}^+$ -modules (by the discussion above,  $\rho_{\Xi,i}$  is unique up to isomorphism);
- let  $\iota_i$  denote the  $\mathfrak{B}$ -equivariant map given by  $\sigma = \chi^{-1} \xrightarrow{\sim} \rho_{\Xi,i}^{\mathfrak{M}} \hookrightarrow \rho_{\Xi,i}|_{\mathfrak{B}}$ .

Pictorially, we write

$$D_\Xi = \begin{array}{ccc} & & \rho_{\Xi,0} \\ & \nearrow^{\iota_0} & \\ \chi^{-1} & & \\ & \searrow_{\iota_1} & \\ & & \rho_{\Xi,1} \end{array}$$

We now wish to construct an auxiliary diagram  $D'$  into which  $D_\Xi$  injects. This will be done with the use of injective envelopes. Recall that if  $\mathcal{G}$  is a profinite group and  $\tau$  is a smooth  $C$ -representation of  $\mathcal{G}$ , an injective envelope consists of an injective  $C$ -representation  $\text{inj}_{\mathcal{G}}\tau$  of  $\mathcal{G}$  along with a  $\mathcal{G}$ -equivariant injection  $j : \tau \hookrightarrow \text{inj}_{\mathcal{G}}\tau$  which satisfies the following property: for any nonzero  $C$ -subrepresentation  $\tau' \subset \text{inj}_{\mathcal{G}}\tau$ , we have  $j(\tau) \cap \tau' \neq 0$ . This data is unique up to (non-unique) isomorphism.

**Lemma 5.4.1** ([Paš04, Lem. 6.13]). *Let  $\tau$  denote a smooth representation of  $\mathcal{G}$ , and let  $j : \tau \hookrightarrow \text{inj}_{\mathcal{G}}\tau$  denote an injective envelope. Let  $\mathfrak{I}$  denote an injective representation of  $\mathcal{G}$ , and suppose we have an injective map  $\phi : \tau \hookrightarrow \mathfrak{I}$ . Then  $\phi$  extends to an injection  $\tilde{\phi} : \text{inj}_{\mathcal{G}}\tau \hookrightarrow \mathfrak{I}$  such that  $\phi = \tilde{\phi} \circ j$ .*

**Lemma 5.4.2.** *Suppose  $\mathcal{G}$  has an open, normal, pro- $p$  subgroup  $\mathcal{G}^+$ . Let  $\tau$  denote a representation of  $\mathcal{G}$  such that  $\mathcal{G}^+$  acts trivially, and let  $j : \tau \hookrightarrow \text{inj}_{\mathcal{G}}\tau$  denote an injective envelope of  $\tau$  in the category of representations of  $\mathcal{G}$ . Then  $\tau \hookrightarrow (\text{inj}_{\mathcal{G}}\tau)^{\mathcal{G}^+}$  is an injective envelope of  $\tau$  in the category of representations of  $\mathcal{G}/\mathcal{G}^+$ .*



*Proof.* This is [Paš04, Lem. 6.14]; its proof does not require that  $\tau$  be irreducible, only that  $\mathcal{G}^+$  acts trivially.  $\square$

We now begin constructing  $D'$ .

**Lemma 5.4.3.** *Let  $i \in \{0, 1\}$ . We then have*

$$(\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}} \cong \bigoplus_{\xi} \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus |\mathbb{B}_i \setminus \mathbb{G}_i|},$$

where  $\xi$  runs over  $C$ -characters of  $\mathfrak{B}$  (or, equivalently, of  $\mathbb{Z}_i$ ), and we have fixed choices of injective envelopes.

*Proof.* Consider the  $\mathfrak{B}$ -representation  $(\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{\mathfrak{U}}$ . The action of  $\mathfrak{B}$  factors through the quotient  $\mathfrak{B}/\mathfrak{U} \cong \mathbb{Z}$ , which is commutative of order coprime to  $p$ . Therefore, we obtain a  $\mathfrak{B}$ -equivariant isomorphism

$$(5.4.4) \quad (\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{\mathfrak{U}} \cong \bigoplus_{\xi} \xi^{\oplus m_{\xi}}$$

for non-negative integers  $m_{\xi}$  satisfying

$$\begin{aligned} m_{\xi} &= \dim_C \mathrm{Hom}_{\mathfrak{B}}(\xi, \mathrm{inj}_{K_i} C[\mathbb{G}_i]) \\ &= \dim_C \mathrm{Hom}_{\mathfrak{B}}(\xi, (\mathrm{inj}_{K_i} C[\mathbb{G}_i])^{K_i^+}) \\ &= \dim_C \mathrm{Hom}_{\mathbb{B}_i}(\xi, \mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i]) \\ &= \dim_C \mathrm{Hom}_{\mathbb{Z}_i}(\xi, (\mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i])^{\mathbb{U}_i}). \end{aligned}$$

(The third equality follows from Lemma 5.4.2.) Since  $C[\mathbb{G}_i]$  is injective as a representation of  $\mathbb{G}_i$ , we have isomorphisms of  $\mathbb{Z}_i$ -representations

$$(\mathrm{inj}_{\mathbb{G}_i} C[\mathbb{G}_i])^{\mathbb{U}_i} \cong C[\mathbb{U}_i \setminus \mathbb{G}_i] \cong \bigoplus_{\xi} \xi^{\oplus |\mathbb{B}_i \setminus \mathbb{G}_i|},$$

so that  $m_{\xi} = |\mathbb{B}_i \setminus \mathbb{G}_i|$ .

The isomorphism (5.4.4) implies we have a  $\mathfrak{B}$ -equivariant injection

$$\bigoplus_{\xi} \xi^{\oplus |\mathbb{B}_i \setminus \mathbb{G}_i|} \hookrightarrow (\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}}.$$

Since the representation on the right-hand side is injective, Lemma 5.4.1 says that the above morphism extends to a split injection between injective  $\mathfrak{B}$ -representations

$$\bigoplus_{\xi} \mathrm{inj}_{\mathfrak{B}} \xi^{\oplus |\mathbb{B}_i \setminus \mathbb{G}_i|} \hookrightarrow (\mathrm{inj}_{K_i} C[\mathbb{G}_i])|_{\mathfrak{B}}.$$

Since the  $\mathfrak{U}$ -invariants of both representations agree, the above injection must be an isomorphism.  $\square$

**Lemma 5.4.5.** *Set  $a := \mathrm{lcm}(|\mathbb{B}_0 \setminus \mathbb{G}_0|, |\mathbb{B}_1 \setminus \mathbb{G}_1|)$ . There exists a diagram  $D'$  of the form*

$$D' = \begin{array}{ccc} & & \text{inj}_{K_0} C[\mathbb{G}_0]^{\oplus a \cdot |\mathbb{B}_0 \setminus \mathbb{G}_0|^{-1}} \\ & \nearrow^{\kappa_0} & \\ \bigoplus_{\xi} \text{inj}_{\mathfrak{B}} \xi^{\oplus a} & & \\ & \searrow_{\kappa_1} & \\ & & \text{inj}_{K_1} C[\mathbb{G}_1]^{\oplus a \cdot |\mathbb{B}_1 \setminus \mathbb{G}_1|^{-1}} \end{array}$$

where  $\kappa_0$  and  $\kappa_1$  are isomorphisms, and a morphism of diagrams

$$\psi : \begin{array}{ccccc} & & \rho_{\Xi,0} & \xleftarrow{\psi_{K_0}} & \text{inj}_{K_0} C[\mathbb{G}_0]^{\oplus a \cdot |\mathbb{B}_0 \setminus \mathbb{G}_0|^{-1}} \\ & \nearrow^{\iota_0} & & & \nearrow^{\kappa_0} \\ \chi^{-1} & \xrightarrow{\psi_{\mathfrak{B}}} & \bigoplus_{\xi} \text{inj}_{\mathfrak{B}} \xi^{\oplus a} & & \\ & \searrow_{\iota_1} & & & \searrow_{\kappa_1} \\ & & \rho_{\Xi,1} & \xleftarrow{\psi_{K_1}} & \text{inj}_{K_1} C[\mathbb{G}_1]^{\oplus a \cdot |\mathbb{B}_1 \setminus \mathbb{G}_1|^{-1}} \end{array}$$

in which all arrows are injective.

*Proof.* We fix the following injections, which are equivariant for the relevant groups:

- injective envelopes  $j_{\xi} : \xi \hookrightarrow \text{inj}_{\mathfrak{B}} \xi$  for each  $C$ -character  $\xi$  of  $\mathfrak{B}$ ;
- injective envelopes  $j_i : C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}} \hookrightarrow \text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}}$  for  $i \in \{0, 1\}$ ;
- an inclusion  $\text{inc} : \chi^{-1} \hookrightarrow \bigoplus_{\xi} \xi^{\oplus a}$ ;
- an inclusion  $c_i : \rho_{\Xi,i} \hookrightarrow C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}}$  for  $i \in \{0, 1\}$ .

Let  $i \in \{0, 1\}$ . We first construct the  $\kappa_i$ . We have an  $\mathfrak{B}$ -equivariant sequence of maps

$$\chi^{-1} \xrightarrow{\iota_i} \rho_{\Xi,i} \xrightarrow{c_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}} \xrightarrow{j_i} \text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}}$$

and thus we obtain

$$\chi^{-1} \xrightarrow{j_i \circ c_i \circ \iota_i} (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})^{\mathfrak{M}}.$$

By Lemmas 5.4.3 and 5.4.2, we have  $\bigoplus_{\xi} \xi^{\oplus a} \cong (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})^{\mathfrak{M}}$ . We fix an isomorphism  $\alpha_i : \bigoplus_{\xi} \xi^{\oplus a} \xrightarrow{\sim} (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})^{\mathfrak{M}}$  such that

$$(5.4.6) \quad \alpha_i \circ \text{inc} = j_i \circ c_i \circ \iota_i.$$

Now consider the maps of  $\mathfrak{B}$ -representations

$$\bigoplus_{\xi} \xi^{\oplus a} \xrightarrow{\alpha_i} (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})^{\mathfrak{M}} \hookrightarrow (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})|_{\mathfrak{B}}.$$

By Lemma 5.4.1, the above map extends to an  $\mathfrak{B}$ -equivariant split injection

$$\kappa_i : \bigoplus_{\xi} \text{inj}_{\mathfrak{B}} \xi^{\oplus a} \hookrightarrow (\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a \cdot |\mathbb{B}_i \setminus \mathbb{G}_i|^{-1}})|_{\mathfrak{B}}$$

such that

$$(5.4.7) \quad \kappa_i \circ \left( \bigoplus_{\xi} j_{\xi}^{\oplus a} \right) = \alpha_i.$$

Since both  $\bigoplus_{\xi} \text{inj}_{\mathfrak{B}} \xi^{\oplus a}$  and  $(\text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a} |_{\mathbb{B}_i \setminus \mathbb{G}_i^{-1}})|_{\mathfrak{B}}$  are injective  $\mathfrak{B}$ -representations and  $\kappa_i$  induces an isomorphism between their  $\mathfrak{U}$ -invariants (cf. Lemma 5.4.3), we see that  $\kappa_i$  must in fact be an isomorphism.

We now construct the morphism of diagrams. Set  $\psi_{K_i} := j_i \circ c_i$  and  $\psi_{\mathfrak{B}} := (\bigoplus_{\xi} j_{\xi}^{\oplus a}) \circ \text{inc}$ . We have

$$\psi_{K_i} \circ \iota_i \stackrel{(5.4.6)}{=} \alpha_i \circ \text{inc} \stackrel{(5.4.7)}{=} \kappa_i \circ \psi_{\mathfrak{B}},$$

and therefore we obtain the desired morphism of diagrams.  $\square$

**5.5. Supercuspidal representations via homology.** We let  $\mathcal{D}_{\Xi}$  and  $\mathcal{D}'$  denote the coefficient systems on  $\mathcal{B}$  associated to  $D_{\Xi}$  and  $D'$ , respectively (cf. [KX15, §6.3]). The homology of coefficient systems gives rise to smooth  $G$ -representations, and we define

$$\pi := \text{im} \left( H_0(\mathcal{B}, \mathcal{D}_{\Xi}) \xrightarrow{\psi_*} H_0(\mathcal{B}, \mathcal{D}') \right),$$

the image of the induced map  $\psi_*$  on homology.

**Theorem 5.5.1.** *The  $G$ -representation  $\pi$  admits an irreducible, admissible, supercuspidal quotient.*

*Proof.* We use language and notation from [Paš04] and [KX15].

*Step 1:* The representation  $\pi$  is nonzero.

Fix a basis  $v$  for  $\chi^{-1}$ , and let  $\omega_{0, \iota_0(v)}$  denote the 0-chain with support  $x_0$  satisfying  $\omega_{0, \iota_0(v)}(x_0) = \iota_0(v)$  (here we identify the  $K_0$ -representation  $\mathcal{D}_{\Xi, x_0}$  with  $\rho_{\Xi, 0}$ ). Since the maps  $\iota_0, \iota_1$  are injective, [Paš04, Lem. 5.7] implies that the image  $\bar{\omega}_{0, \iota_0(v)}$  is nonzero in  $H_0(\mathcal{B}, \mathcal{D}_{\Xi})$ . Now set  $\bar{\omega} := \psi_*(\bar{\omega}_{0, \iota_0(v)}) = \bar{\omega}_{0, \psi_{K_0} \circ \iota_0(v)} \in \pi \subset H_0(\mathcal{B}, \mathcal{D}')$ . This is the image in  $H_0(\mathcal{B}, \mathcal{D}')$  of a  $\mathcal{D}'_{x_0}$ -valued 0-chain supported on  $x_0$ , and since the maps  $\kappa_0, \kappa_1$  are isomorphisms and  $\psi$  is injective, we have  $\bar{\omega} \neq 0$  ([Paš04, Lem. 5.7] again).

*Step 2:* The representation  $\pi$  is admissible.

Since  $\kappa_0, \kappa_1$  are isomorphisms, [Paš04, Prop. 5.10] gives

$$\pi|_{\mathfrak{B}} \subset H_0(\mathcal{B}, \mathcal{D}')|_{\mathfrak{B}} \cong \mathcal{D}'_C \cong \bigoplus_{\xi} \text{inj}_{\mathfrak{B}} \xi^{\oplus a},$$

which implies  $\pi^{\mathfrak{U}} \subset \bigoplus_{\xi} \xi^{\oplus a}$ , so that  $\pi$  is admissible.

*Step 3:* The  $H_C^+$ -module  $\pi^{\mathfrak{U}}$  contains  $\Xi$ .

The element  $\bar{\omega}_{0, \iota_0(v)} \in H_0(\mathcal{B}, \mathcal{D}_{\Xi})$  is  $\mathfrak{U}$ -invariant and stable by the action of  $H_C^+$ , and the vector space it spans is isomorphic to  $\Xi$  as an  $H_C^+$ -module (for all of this, see the proof of [KX15, Prop. 7.3]). Since  $\psi_*$  is  $G$ -equivariant, the same is true for  $\bar{\omega} \in \pi$ .

*Step 4:* The vector  $\bar{\omega}$  generates  $\pi$ .

Since  $\bar{\omega}_{0, \iota_0(v)}$  generates  $H_0(\mathcal{B}, \mathcal{D}_{\Xi})$  as a  $G$ -representation and  $\psi_*$  is  $G$ -equivariant,  $\bar{\omega}$  generates  $\pi$  as a  $G$ -representation.

*Step 5:* We construct the quotient  $\pi'$  and list its properties.

By the previous step, the representation  $\pi$  is generated by  $\bar{\omega}$ . Proceeding as in the end of the proof of Theorem 4.5.1, we see that any irreducible quotient of  $\pi = \langle G \cdot \bar{\omega} \rangle$  is admissible (and such quotients exist by Zorn's lemma). Let  $\pi'$  be any such quotient.

*Step 6:* We prove  $\pi'$  is supercuspidal.

Since  $\bar{\omega}$  generates  $\pi$ , its image in  $\pi'$  is nonzero. Thus, we obtain an injection of  $H_C^+$ -modules  $\Xi \cong C\bar{\omega} \hookrightarrow (\pi')^{\mathfrak{U}}$ , and supercuspidality follows from Proposition 3.1.2  $\square$

The proofs above do not actually require that  $C$  be algebraically closed; it suffices to assume that  $C$  is a finite extension of  $\mathbb{F}_p$  which contains the  $|\mathbb{G}_0|_{p'}$ -th and  $|\mathbb{G}_1|_{p'}$ -th roots of unity, where  $|\cdot|_{p'}$  denotes the prime-to- $p$  part of the order (cf. [CE04, §6.1]). Therefore, we obtain:

**Corollary 5.5.2.** *Suppose  $\mathbf{G}$  is a connected reductive  $F$ -group of relative semisimple rank 1, and  $C$  is a sufficiently large finite extension of  $\mathbb{F}_p$ . Then  $G$  admits an irreducible admissible supercuspidal  $C$ -representation.*

*Proof.* By the reductions in §5.1, it suffices to assume  $\mathbf{G}$  is absolutely simple and simply connected, and to construct a supercuspidal  $C$ -representation on which  $Z(G)$  acts trivially. Since the center of  $G$  is finite, it is contained in  $\mathfrak{B} \cap Z = Z_0$ . Hence, taking  $\Xi$  to be associated to  $(\mathbf{1}_{\mathfrak{B}}, J)$ , where  $\mathbf{1}_{\mathfrak{B}}$  is the trivial character of  $\mathfrak{B}$  and  $J \neq \emptyset, \{s_0, s_1\}$ , Theorem 5.5.1 produces an irreducible admissible supercuspidal  $C$ -representation  $\pi'$  with trivial action of the center. This gives the claim.  $\square$

*Remark 5.5.3.* The construction of  $\pi'$  above shares some similarities with the construction in §4.5. Therein, supercuspidal representations are constructed as subquotients of  $C^\infty(\Gamma \backslash G, C) \cong \text{c-Ind}_\Gamma^G \mathbf{1}_\Gamma$ , where  $\Gamma$  is a discrete, cocompact subgroup of  $G$  and  $\mathbf{1}_\Gamma$  denotes the trivial character of  $\Gamma$ . Taking  $\Gamma$  to be torsion-free, we use the Mackey formula to obtain

$$(\text{c-Ind}_\Gamma^G \mathbf{1}_\Gamma)|_{K_i} \cong \bigoplus_{\Gamma \backslash G/K_i} \text{c-Ind}_{\{1\}}^{K_i} \mathbf{1} \cong \bigoplus_{\Gamma \backslash G/K_i} \text{Ind}_{\{1\}}^{K_i} \mathbf{1} \cong \text{inj}_{K_i} C[\mathbb{G}_i]^{\oplus a'_i}$$

where  $a'_i = |\Gamma \backslash G/K_i|$ . The construction above produces supercuspidal representations as subquotients of  $H_0(\mathcal{B}, \mathcal{D}')$ , for which we have

$$H_0(\mathcal{B}, \mathcal{D}')|_{K_i} \cong \text{inj}_{K_i} (C[\mathbb{G}_i])^{\oplus a_i},$$

where  $a_i = a \cdot |\mathbb{B}_i \backslash \mathbb{G}_i|^{-1}$  (cf. [Paš04, Prop. 5.10]).

## 6. SUPERSINGULAR REPRESENTATIONS OF $\text{PGL}_n(D)$

In this section we verify Theorem A when  $\mathbf{G} = \mathbf{PGL}_n(D)$ , where  $n \geq 2$  and  $D$  a central division algebra over  $F$ . In particular, this deals with the first exceptional case in Theorem 4.5.1.

**6.1. Notation and conventions.** Throughout Section 6, we let  $\overline{\mathbb{Q}}_p$  denote a fixed algebraic closure of  $\mathbb{Q}_p$ , with ring of integers  $\overline{\mathbb{Z}}_p$  and residue field  $\overline{\mathbb{F}}_p$ . We normalise the valuation  $\text{val}$  of  $\overline{\mathbb{Q}}_p$  such that  $\text{val}(p) = 1$ . Further, we let  $D$  denote a central division algebra over  $F$  of dimension  $d^2$ .

For any field  $K$  let  $\Gamma_K$  denote the absolute Galois group for a choice of separable closure. If  $K'/K$  is a finite separable extension, then  $\Gamma_{K'}$  is a subgroup of  $\Gamma_K$ , up to conjugacy, hence the restriction of a  $\Gamma_K$ -representation to  $\Gamma_{K'}$  is well defined up to isomorphism.

If  $K/\mathbb{Q}_p$  is finite we let  $I_K$  denote the inertia subgroup of  $\Gamma_K$  and  $k_K$  the residue field of  $K$ . If  $\rho : \Gamma_K \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$  is de Rham and  $\tau : K \rightarrow \overline{\mathbb{Q}}_p$  is continuous, we let  $\text{HT}_\tau(\rho)$  denote multi-set of  $\tau$ -Hodge–Tate weights. We normalise Hodge–Tate weights so that the cyclotomic character  $\varepsilon$  has  $\tau$ -Hodge–Tate weight  $-1$  for any  $\tau$ . We let  $\text{WD}(\rho)$  denote the associated Weil–Deligne representation of  $W_K$  over  $\overline{\mathbb{Q}}_p$  (defined by Fontaine, cf. Appendix B.1 of [CDT99]).

We normalise local class field theory so that uniformisers correspond to geometric Frobenius elements under the local Artin map. Let  $\text{rec}_F$  denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of  $\text{GL}_n(F)$  over  $\mathbb{C}$  to isomorphism classes of  $n$ -dimensional Frobenius semisimple Weil–Deligne representations of  $W_F$  over  $\mathbb{C}$ . (See [HT01].)

If  $L$  is a global field, we let  $|\cdot|_L$  denote the normalised absolute value of  $\mathbb{A}_L$ .

**6.2. On the Jacquet–Langlands correspondence.** We recall some basic facts about the representation theory of  $\text{GL}_n(D)$  and the local Jacquet–Langlands correspondence. All representations in this section will be smooth and over  $\mathbb{C}$ .

For a finite-dimensional central simple algebra  $A$  let  $\text{Nrd} : A^\times \rightarrow Z(A)^\times$  (or  $\text{Nrd}_A$  for clarity) denote the reduced norm. Let  $\nu$  denote the smooth character  $|\text{Nrd}|_F$  of  $\text{GL}_m(D)$  for any  $m$ . If  $\pi_i$  are smooth representations of  $\text{GL}_{n_i}(D)$ , let  $\pi_1 \times \cdots \times \pi_r$  denote the normalised parabolic induction of  $\pi_1 \otimes \cdots \otimes \pi_r$  to  $\text{GL}_{\sum n_i}(D)$ . In particular these notions also apply to general linear groups over  $F$  (by setting  $D = F$ ).

We will say that a representation is *essentially unitarisable* if some twist of it is unitarisable.

The Jacquet–Langlands correspondence [DKV84] is a canonical bijection  $\text{JL}$  between irreducible essentially square-integrable representations of  $\text{GL}_n(D)$  and irreducible essentially square-integrable representations of  $\text{GL}_{nd}(F)$  that is compatible with character twists and preserves central characters.

On the other hand, Badulescu [Bad08] defined a map  $|\text{LJ}_{\text{GL}_n(D)}|$  in the other direction, from irreducible essentially unitarisable representations of  $\text{GL}_{nd}(F)$  to irreducible essentially unitarisable representations of  $\text{GL}_n(D)$  or zero, which in general is neither injective nor surjective. (More precisely, [Bad08] only considers unitarisable representations, but we can extend it by twisting.) In the split case  $|\text{LJ}_{\text{GL}_n(F)}|$  is the identity. It follows from Thm. 2.2 and Thm. 2.7(a) in [Bad08] that  $|\text{LJ}_{\text{GL}_n(D)}|(\text{JL}(\pi)) \cong \pi$  for any essentially square-integrable representation  $\pi$  of  $\text{GL}_n(D)$ .

If  $\rho$  is a supercuspidal representation of  $\text{GL}_m(F)$  and  $\ell \geq 1$ , then  $Z^u(\rho, \ell)$  is by definition the unique irreducible quotient of  $\rho\nu^{(1-\ell)/2} \times \rho\nu^{(3-\ell)/2} \times \cdots \times \rho\nu^{(\ell-1)/2}$ . It is essentially square-integrable and all essentially square-integrable representations of  $\text{GL}_n(F)$  arise in this way.

If  $\rho'$  is a supercuspidal representation of  $\text{GL}_m(D)$ , we can write  $\text{JL}(\rho') \cong Z^u(\rho, s)$  for some supercuspidal representation  $\rho$  and integer  $s \geq 1$ . Then  $Z^u(\rho', \ell)$  is by definition the unique irreducible quotient of  $\rho'\nu^{s(1-\ell)/2} \times \rho'\nu^{s(3-\ell)/2} \times \cdots \times \rho'\nu^{s(\ell-1)/2}$ . It is essentially square-integrable and all essentially square-integrable representations of  $\text{GL}_n(D)$  arise in this way (a result of Tadić, cf. [Bad08, §2.4]). Moreover,  $\text{JL}(Z^u(\rho', \ell)) \cong Z^u(\rho, \ell s)$  [Bad08, §3.1].

**6.3. On lifting non-supersingular Hecke modules.** Let  $B = MU$  denote the upper-triangular Borel subgroup of  $\text{GL}_n(D)$  with diagonal minimal Levi subgroup  $M \cong (D^\times)^n$  and unipotent radical  $U$ . Let  $T \cong (F^\times)^n$  denote the diagonal maximal split torus and  $U^{\text{op}}$  the lower-triangular unipotent matrices.

Let  $\mathcal{O}_D$  denote the ring of integers of  $D$ ,  $\mathfrak{m}_D$  the maximal ideal of  $\mathcal{O}_D$ , and  $k_D$  the residue field, so  $[k_D : k_F] = d$ . Let  $D(1) := 1 + \mathfrak{m}_D$ , so  $D(1) \triangleleft D^\times$ . Let  $\text{val}_D : D^\times \rightarrow \mathbb{Z}$  denote the normalised valuation of  $D$ . Let  $I(1)$  denote the pro- $p$  Iwahori subgroup

$$I(1) := \{g \in \text{GL}_n(\mathcal{O}_D) : \bar{g} \in \text{GL}_n(k_D) \text{ is upper-triangular unipotent}\}.$$

Let  $\mathcal{H} := \mathcal{H}(\mathrm{GL}_n(D), I(1))$  the corresponding pro- $p$  Iwahori–Hecke algebra over  $\mathbb{Z}$  [Vig16] and for a commutative ring  $R$  let  $\mathcal{H}_R := \mathcal{H} \otimes R$ . Similarly we define  $\mathcal{H}_M := \mathcal{H}(M, M \cap I(1))$  and  $\mathcal{H}_{M,R} := \mathcal{H}_M \otimes R$ . Note that the pro- $p$  Iwahori subgroup  $M \cap I(1)$  is normal in  $M$ . All Hecke modules we will consider are right modules. A finite-dimensional  $\mathcal{H}_{\overline{\mathbb{Q}_p}}$ -module is said to be *integral* if it arises by base change from a  $\mathcal{H}_{\overline{\mathbb{Z}_p}}$ -module that is finite free over  $\overline{\mathbb{Z}_p}$ .

Let  $W(1) := N(T)/M \cap I(1)$ ,  $W_M(1) := M/M \cap I(1)$ , and define monoids

$$M^+ := \{\mathrm{diag}(m_1, \dots, m_n) \in M : \mathrm{val}_D(m_1) \geq \dots \geq \mathrm{val}_D(m_n)\},$$

and  $W_{M^+}(1) := M^+/M \cap I(1)$ .

We recall that  $\mathcal{H}$  has an Iwahori–Matsumoto basis  $T_w$  for  $w \in W(1)$  and a Bernstein basis  $E_w$  for  $w \in W(1)$ , which in fact depends on a choice of spherical orientation. We choose our spherical orientation such that  $E_w = T_w$  for  $w \in W_{M^+}(1)$ . (This is possible by [Vig16, Ex. 5.30].) Similarly,  $\mathcal{H}_M$  has basis  $T_w^M$  for  $w \in W_M(1)$ .

Our choice of (pro- $p$ ) Iwahori subgroup determines a chamber  $C$  in the Bruhat–Tits building of  $\mathrm{GL}_n(D)$ . For  $w \in W(1)$  let  $\ell(w) \in \mathbb{Z}_{\geq 0}$  denote the length of  $w$  relative to the chamber  $C$ . (In fact, for  $w \in M/M \cap I(1)$  the length is independent of the choice of chamber.)

**Lemma 6.3.1.** *Suppose  $m = \mathrm{diag}(m_1, \dots, m_n) \in M$  with  $m_i \in D^\times$ . Then  $\ell(m) = \sum_{i < j} |\mathrm{val}_D(m_i) - \mathrm{val}_D(m_j)|$ .*

*Proof.* We will use the explicit description of the reduced apartment  $A$  of  $T$  given in [Tit79, §1.14], using the valuation  $\mathrm{val}_D|_{F^\times}$  of  $F$ . Note that  $A$  is an affine space for  $V := (X_*(T) \otimes \mathbb{R})/\mathbb{R}$ , i.e.  $\mathbb{R}^n/\mathbb{R}(1, \dots, 1)$  (using the standard basis of  $X_*(T)$ , unlike [Tit79]!), and we identify  $A$  with  $V$  by using a special point as origin. Let  $\alpha_{ij}$  for  $1 \leq i \neq j \leq n$  denote the root sending  $t = \mathrm{diag}(t_1, \dots, t_n) \in T$  to  $t_i t_j^{-1}$ . Then the affine roots are given by  $\{\alpha_{ij} + r : i \neq j, r \in \mathbb{Z}\}$ . (Note that formula [Tit79, §1.14(2)] has a typo: by the preceding line in that reference,  $\Gamma$  should be replaced by  $\omega(D^\times)$ .) Therefore the affine Weyl group is associated to the reduced root system  $\{\alpha_{ij} : i \neq j\}$ . We also note that  $m$  acts on  $A$  by translation by  $-(\mathrm{val}_D(m_1), \dots, \mathrm{val}_D(m_n))$ . A chamber is given by the convex hull of the points  $x_i := -(0, \dots, 0, 1, \dots, 1)$  (with  $1 \leq i \leq n$  giving the number of zeros), and we claim that this is the chamber  $C$  corresponding to  $I(1)$ . To see this, note that the “norms” corresponding to the points  $x_i$  in [Tit79, §2.9] are the gauge “norms” of the lattices  $\mathcal{O}_D \oplus \dots \oplus \mathcal{O}_D \oplus \mathfrak{m}_D \oplus \dots \oplus \mathfrak{m}_D$  (with  $i$  copies of  $\mathcal{O}_D$ ,  $1 \leq i \leq n$ ) and that  $I(1)$  fixes each of them. The first length formula in [Vig16, Cor. 5.11] now gives the desired result.  $\square$

If  $\pi$  is a smooth representation of  $\mathrm{GL}_n(D)$  let  $\pi_U$  denote its (unnormalised) Jacquet module.

**Lemma 6.3.2.** *Suppose that  $\pi$  is an admissible representation of  $\mathrm{GL}_n(D)$  over  $\mathbb{C}$ . Then the natural map  $p_U : \pi \rightarrow \pi_U$  induces an isomorphism  $\pi^{I(1)} \rightarrow (\pi_U)^{M \cap I(1)}$ .*

This lemma generalises to any  $p$ -adic reductive group, with the same proof.

*Proof.* Recall that  $I(1)$  has an Iwahori decomposition with respect to  $M$ ,  $U$ ,  $U^{\mathrm{op}}$ . Then [Cas, Prop. 4.1.4] says that the map  $p_U$  induces an isomorphism  $\pi^{I(1)} \cdot T_{a^{-1}} \rightarrow (\pi_U)^{M \cap I(1)}$  for  $a = \mathrm{diag}(a_1, \dots, a_n) \in T$  with  $\max_{1 \leq i < n} |a_i a_{i+1}^{-1}|_F$  sufficiently small. By [Vig16, Prop. 4.13(1)] the operator  $T_{a^{-1}}$  is invertible in  $\mathcal{H}_{\mathbb{C}}$ .  $\square$

We now fix an isomorphism  $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ .

**Proposition 6.3.3.** *Suppose that  $\Pi$  is an irreducible generic smooth representation of  $\mathrm{GL}_{nd}(F)$  over  $\mathbb{C}$  that is essentially unitarisable. Suppose that the representation  $\pi := |\mathrm{LJ}_{\mathrm{GL}_n(D)}|(\Pi)$  of  $\mathrm{GL}_n(D)$  is such that  $\iota^{-1}(\pi^{I(1)})$  is a non-zero integral  $\mathcal{H}_{\overline{\mathbb{Q}}_p}$ -module with non-supersingular reduction. Let  $v_1 \leq \dots \leq v_{nd}$  denote the valuations of the eigenvalues of a geometric Frobenius on  $\iota^{-1}(\mathrm{rec}_F(\Pi))$ . Then there exists  $1 \leq j \leq n-1$  such that*

$$\sum_{i=1}^{jd} v_i = -\frac{d^2 j(n-j)}{2} \mathrm{val}(q).$$

*Proof.*

*Step 1:* We show that there exist irreducible representations  $\rho'_1, \dots, \rho'_n$  of  $D^\times/D(1)$  such that  $\pi$  is a subquotient of  $\rho'_1 \times \dots \times \rho'_n$  and  $\mathrm{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{i=1}^n \mathrm{rec}_F(\mathrm{JL}(\rho'_i))|_{W_F}$ .

After a twist we may assume that  $\Pi$  is unitarisable. As  $\Pi$  is moreover generic, we know that  $\Pi \cong \sigma_1 \nu^{\alpha_1} \times \dots \times \sigma_r \nu^{\alpha_r}$  for some square-integrable  $\sigma_i$  of  $\mathrm{GL}_{n_i}(F)$  and real numbers  $\alpha_i \in (-\frac{1}{2}, \frac{1}{2})$  satisfying  $\alpha_i + \alpha_{r+1-i} = 0$  and  $\sigma_i = \sigma_{r+1-i}$  if  $\alpha_i \neq 0$  (see e.g. [HT01, Lemma I.3.8]). Since  $|\mathrm{LJ}_{\mathrm{GL}_n(D)}|(\Pi) \neq 0$  by assumption, it follows that  $d \mid n_i$  for all  $i$  and  $\pi = |\mathrm{LJ}_{\mathrm{GL}_n(D)}|(\Pi) \cong \sigma'_1 \nu^{\alpha_1} \times \dots \times \sigma'_r \nu^{\alpha_r}$ , where  $\sigma'_i$  is the square-integrable representation of  $\mathrm{GL}_{n_i/d}(D)$  such that  $\mathrm{JL}(\sigma'_i) \cong \sigma_i$ . (See [Bad08, §3.5].) Let  $n'_i := n_i/d$ .

From  $\pi^{I(1)} \neq 0$  and Lemma 6.3.2 it follows that the supercuspidal support of  $\pi$  is a tame representation of  $M$  (up to conjugacy), so each  $\sigma'_i$  is of the form  $Z^u(\rho''_i, n'_i)$ , where  $\rho''_i$  is an irreducible representation of  $D^\times/D(1)$ . We write  $\mathrm{JL}(\rho''_i) \cong Z^u(\rho_i, e_i)$  with  $\rho_i$  irreducible supercuspidal, so  $\sigma_i \cong Z^u(\rho_i, e_i n'_i)$ . In particular,  $\pi$  is a subquotient of the normalised induction of  $\bigotimes_{1 \leq i \leq r, 0 \leq j \leq n'_i-1} \rho''_i \nu^{\alpha_i + e_i((n'_i-1)/2-j)}$ . On the other hand,  $\Pi$  is a subquotient of the normalised induction of  $\bigotimes_{1 \leq i \leq r, 0 \leq j \leq e_i n'_i-1} \rho_i \nu^{\alpha_i + (e_i n'_i-1)/2-j}$ . As  $\mathrm{rec}_F(\Pi)|_{W_F}$  only depends on the supercuspidal support of  $\Pi$  (see the paragraph before Thm. VII.2.20 in [HT01]), we obtain

$$\mathrm{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{1 \leq i \leq r, 0 \leq j \leq e_i n'_i-1} |\cdot|_F^{\alpha_i + (e_i n'_i-1)/2-j} \mathrm{rec}_F(\rho_i)|_{W_F}.$$

Similarly,  $\mathrm{rec}_F(\mathrm{JL}(\rho''_i))|_{W_F} \cong \bigoplus_{k=0}^{e_i-1} |\cdot|_F^{(e_i-1)/2-k} \mathrm{rec}_F(\rho_i)|_{W_F}$ . Denoting by  $\rho'_1, \dots, \rho'_n$  the representations  $\rho''_i \nu^{\alpha_i + e_i((n'_i-1)/2-j)}$  in any order, a straightforward computation confirms that  $\bigoplus_{i=1}^n \mathrm{rec}_F(\mathrm{JL}(\rho'_i))|_{W_F} \cong \mathrm{rec}_F(\Pi)|_{W_F}$ .

*Step 2:* We give an explicit criterion for supersingularity of  $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -modules.

Let  $W_0 \cong S_n$  denote the Weyl group of  $T$ . Recall from [Vig17, §5, §1.3] that  $\mathcal{A}_0(\Lambda_T)$  is the free module with basis  $E_{\mu(\varpi)}$  for  $\mu \in \Lambda_T := X_*(T)$  and that the central subalgebra  $\mathcal{Z}_T := \mathcal{A}_0(\Lambda_T)^{W_0}$  of  $\mathcal{H}$  has a basis consisting of the sums  $\sum_{\mu} E_{\mu(\varpi)}$  with  $\mu$  running over the  $W_0$ -orbits in  $X_*(T)$ . For  $I \subset \{1, \dots, n\}$  let  $E_I := E_{\mu_I(\varpi)}$ , where  $\mu_I \in X_*(T) \cong \mathbb{Z}^n$  is defined by  $\mu_{I,i} = 1$  if  $i \in I$  and  $\mu_{I,i} = 0$  otherwise. For  $1 \leq i \leq n$  let  $Z_i := \sum_{I, |I|=i} E_I$ . By induction and [Vig16, Cor. 5.28] we see that the algebra  $\mathcal{Z}_T$  is generated by  $Z_1, \dots, Z_{n-1}, Z_n^{\pm 1}$ .

It follows by [Vig17, Prop. 6.9] that a finite-dimensional  $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -module  $M$  is supersingular if and only if the action of  $Z_i$  on  $M$  is nilpotent for all  $1 \leq i \leq n-1$ .

*Step 3:* We recall [OV, §2.5.2, Rk. 2.20] that we have a unique injective algebra homomorphism  $\tilde{\theta} : \mathcal{H}_{M, \overline{\mathbb{Q}}_p} \rightarrow \mathcal{H}_{\overline{\mathbb{Q}}_p}$  such that  $\tilde{\theta}(T_w^M) = T_w$  for all  $w \in W_{M^+}(1)$ . We claim that

$$(6.3.4) \quad E_{\mu(\varpi)} = q^{d^2 \sum_{r < s: \mu_r < \mu_s} (\mu_s - \mu_r)} \tilde{\theta}(T_{\mu(\varpi)}^M).$$

Let  $X_*(T)_+ := \{\mu \in X_*(T) : \mu_1 \geq \dots \geq \mu_n\}$ . If  $\mu \in X_*(T)_+$ , then  $\mu(\varpi) \in M^+$  and hence  $E_{\mu(\varpi)} = T_{\mu(\varpi)}$  and formula (6.3.4) holds. In general, choose  $\mu' \in X_*(T)_+$  such that  $\mu + \mu' \in X_*(T)_+$ . Then formula (6.3.4) follows easily from the following three assertions: (1)  $T_{\mu(\varpi)}^M T_{\mu'(\varpi)}^M = T_{\mu(\varpi)\mu'(\varpi)}^M$ ; (2)  $E_{\mu(\varpi)} E_{\mu'(\varpi)} = (q_{\mu(\varpi)} q_{\mu'(\varpi)} q_{\mu(\varpi)\mu'(\varpi)}^{-1})^{1/2} E_{\mu(\varpi)\mu'(\varpi)}$  in the notation of [Vig16, §4.4], where we take the positive square root; and (3) for  $m = \text{diag}(m_1, \dots, m_n) \in M$  we have  $q_m = q^{d \sum_{r < s} |\text{val}_D(m_r) - \text{val}_D(m_s)|}$ . Assertion (1) is clear and assertion (2) is [Vig16, Cor. 5.28]. For the last one, note that  $q_s = q^d$  for any  $s \in W(1)$  acting as a reflection in a wall of our chamber  $C$  and hence that  $q_w = q^{d\ell(w)}$  for  $w \in W(1)$  [Vig16, Prop. 3.38]. Hence assertion (3) follows from Lemma 6.3.1.

From (6.3.4) we deduce that

$$(6.3.5) \quad E_I = q^{d^2 (\sum_{i \in I} i - \binom{|I|+1}{2})} \tilde{\theta}(T_{\mu_I(\varpi)}^M).$$

*Step 4:* We compute the action of  $Z_1, \dots, Z_n$  on the Hecke module  $\iota^{-1}(\pi^{I(1)})$  and show in particular that it is scalar.

Note by Step 1 that  $\pi^{I(1)}$  is a subquotient of  $(\rho'_1 \times \dots \times \rho'_n)^{I(1)}$  and that  $\rho'_1 \times \dots \times \rho'_n \cong \text{Ind}_B^{\text{GL}_n(D)}(\rho'_1 \nu^{d(n-1)/2} \otimes \dots \otimes \rho'_n \nu^{-d(n-1)/2})$  (unnormalised induction). By [OV, Prop. 4.4] we have

$$(6.3.6) \quad (\rho'_1 \times \dots \times \rho'_n)^{I(1)} \cong (\rho'_1 \nu^{d(n-1)/2} \otimes \dots \otimes \rho'_n \nu^{-d(n-1)/2})^{M \cap I(1)} \otimes_{\mathcal{H}_{M, \tilde{\theta}}} \tilde{\theta} \mathcal{H}.$$

By [BH11, §1.5], for each  $1 \leq i \leq n$  we can find an “admissible tame pair”  $(F_i/F, \zeta_i)$  of degree  $f_i$  dividing  $d$  such that  $\rho'_i \cong \pi_D(\zeta_i)$  in the notation of that paper. Concretely this means that  $F_i/F$  is unramified of degree  $f_i$  and that  $\zeta_i : F_i^\times \rightarrow \mathbb{C}^\times$  is a tamely ramified smooth character such that all  $\text{Gal}(F_i/F)$ -conjugates of  $\zeta_i$  are distinct. Moreover,  $B_i := Z_D(F_i)$  is a central simple  $F_i$ -algebra of dimension  $e_i^2$ , where  $e_i := d/f_i$ . Define a smooth character  $\Lambda_i : B_i^\times(1 + \mathfrak{m}_D) \rightarrow \mathbb{C}^\times$  by declaring it to be  $\zeta_i \circ \text{Nrd}_{B_i}$  on  $B_i^\times$  and trivial on  $1 + \mathfrak{m}_D$ . Then  $\pi_D(\zeta_i) \cong \text{Ind}_{B_i^\times(1 + \mathfrak{m}_D)}^{D^\times} \Lambda_i$  (of dimension  $f_i$ ). The main result of [BH11] shows that  $\text{rec}_F(\text{JL}(\rho'_i)) \cong \text{Sp}_{e_i}(\text{Ind}_{W_{F_i}}^{W_F}(\eta_{F_i}^{e_i(f_i-1)} \zeta_i))$ , where  $\eta_{F_i}$  is the unramified quadratic character of  $F_i^\times$ . (We recall that the special Weil–Deligne representation  $\text{Sp}_e(\sigma)$ , for  $\sigma$  an irreducible representation of  $W_F$ , satisfies  $\text{Sp}_e(\sigma)|_{W_F} \cong \bigoplus_{k=0}^{e-1} \sigma| \cdot |_F^{\frac{e-1}{2} - k}$ .)



Let  $\zeta'_i := \iota^{-1}(\zeta_i)$ . From (6.3.5) and (6.3.6) we deduce that  $Z_j$  acts on  $\iota^{-1}(\pi^{I(1)})$  as the scalar

$$(6.3.7) \quad \begin{aligned} \lambda_j &:= \sum_{|I|=j} \left( q^{d^2(\sum_{i \in I} i - \binom{j+1}{2})} q^{d^2((n+1)j/2 - \sum_{i \in I} i)} \prod_{i \in I} \zeta'_i(\varpi)^{-e_i} \right) \\ &= q^{-d^2 \binom{j}{2}} \sum_{|I|=j} \left( q^{d^2(n-1)j/2} \prod_{i \in I} \zeta'_i(\varpi)^{-e_i} \right). \end{aligned}$$

*Step 5:* We complete the proof. By assumption, the Hecke module  $\iota^{-1}(\pi^{I(1)})$  is integral, so  $\lambda_i \in \overline{\mathbb{Z}}_p$  for all  $i$  and  $\lambda_n \in \overline{\mathbb{Z}}_p^\times$ . Moreover, as the reduction of  $\iota^{-1}(\pi^{I(1)})$  is non-supersingular we deduce by Step 2 that  $\lambda_{n-j} \in \overline{\mathbb{Z}}_p^\times$  for some  $1 \leq j \leq n-1$ .

From now on assume for convenience that the  $\zeta'_i$  are ordered such that the sequence  $\text{val}(\zeta'_i(\varpi)^{-e_i})$  is non-increasing. Consider the polynomial  $\prod_{i=1}^n (1 - q^{d^2(n-1)/2} \zeta'_i(\varpi)^{-e_i} X)$ . By (6.3.7) its Newton polygon is defined by the points  $(i, \text{val}(\lambda_i) + d^2 \binom{i}{2} \text{val}(q))$  for  $0 \leq i \leq n$ . From  $\lambda_{n-j} \in \overline{\mathbb{Z}}_p^\times$ ,  $\lambda_i \in \overline{\mathbb{Z}}_p$ , and the convexity of the quadratic function  $x(x-1)/2$  we deduce that  $(n-j, d^2 \binom{n-j}{2} \text{val}(q))$  is a vertex of the Newton polygon. It follows for the sum of the largest  $j$  root valuations that

$$(6.3.8) \quad \sum_{i=1}^j \text{val}(q^{d^2(n-1)/2} \zeta'_i(\varpi)^{-e_i}) = d^2 \left( \binom{n}{2} - \binom{n-j}{2} \right) \text{val}(q).$$

Again by convexity we obtain the root valuation bounds

$$(6.3.9) \quad \text{val}(q^{d^2(n-1)/2} \zeta'_i(\varpi)^{-e_i}) \geq d^2(n-j) \text{val}(q) \quad \forall i \leq j,$$

$$(6.3.10) \quad \text{val}(q^{d^2(n-1)/2} \zeta'_i(\varpi)^{-e_i}) \leq d^2(n-j-1) \text{val}(q) \quad \forall i > j.$$

From Steps 1 and 4 we see that

$$\text{rec}_F(\Pi)|_{W_F} \cong \bigoplus_{i=1}^n \bigoplus_{k=0}^{e_i-1} \text{Ind}_{W_{F_i}}^{W_F} (\eta_{F_i}^{e_i(f_i-1)} \zeta'_i) \cdot |_{F}^{(e_i-1)/2-k}.$$

If  $\text{Frob}_F$  denotes a geometric Frobenius of  $W_F$ , then  $\text{Frob}_F^{f_i}$  is a geometric Frobenius of  $W_{F_i}$ . We see that all eigenvalues of  $\text{Frob}_F$  on  $\text{Ind}_{W_{F_i}}^{W_F} (\eta_{F_i}^{e_i(f_i-1)} \zeta'_i)$  have valuation  $\frac{1}{f_i} \text{val}(\zeta'_i(\text{Frob}_F^{f_i})) = \frac{1}{f_i} \text{val}(\zeta'_i(\varpi))$ . Hence, for  $i \leq j$  and  $0 \leq k \leq e_i - 1$  all eigenvalues of  $\text{Frob}_F$  on  $\text{Ind}_{W_{F_i}}^{W_F} (\eta_{F_i}^{e_i(f_i-1)} \zeta'_i) \cdot |_{F}^{(e_i-1)/2-k}$  have valuation

$$\begin{aligned} \frac{1}{f_i} \text{val}(\zeta'_i(\varpi)) - \left( \frac{e_i-1}{2} - k \right) \text{val}(q) &\leq \frac{1}{d} \text{val}(\zeta'_i(\varpi)^{e_i}) + \left( \frac{e_i-1}{2} \right) \text{val}(q) \\ &< d \left( \frac{n-1}{2} - (n-j) \right) \text{val}(q) + \frac{d}{2} \text{val}(q) = \frac{d(2j-n)}{2} \text{val}(q), \end{aligned}$$

where we used (6.3.9) and that  $e_i - 1 < d$ . Similarly, for  $i > j$  and  $0 \leq k \leq e_i - 1$  we find that the eigenvalues of  $\text{Frob}_F$  on  $\text{Ind}_{W_{F_i}}^{W_F} (\eta_{F_i}^{e_i(f_i-1)} \zeta'_i) \cdot |_{F}^{(e_i-1)/2-k}$  have valuation

greater than  $\frac{d(2j-n)}{2} \text{val}(q)$ . Therefore, from (6.3.8) we deduce that

$$\begin{aligned} \sum_{i=1}^{jd} v_i &= \sum_{i=1}^j \sum_{k=0}^{e_i-1} f_i \left( \frac{1}{f_i} \text{val}(\zeta'_i(\varpi)) - \left( \frac{e_i-1}{2} - k \right) \text{val}(q) \right) = \sum_{i=1}^j \text{val}(\zeta'_i(\varpi)^{e_i}) \\ &= d^2 \left( \binom{n-j}{2} - \binom{n}{2} + \frac{j(n-1)}{2} \right) \text{val}(q) = -\frac{d^2 j(n-j)}{2} \text{val}(q). \end{aligned}$$

□

**6.4. A reducibility lemma.** Let  $F_0$  denote the maximal absolutely unramified intermediate field of  $F/\mathbb{Q}_p$ . The following lemma generalises [EGH13, Prop. 4.5.2], which dealt with regular crystalline Galois representations.

**Lemma 6.4.1.** *Suppose that  $\rho : \Gamma_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_p)$  is a de Rham Galois representation. Let  $v_1 \leq \dots \leq v_n$  denote the valuations of the eigenvalues of a geometric Frobenius element acting on  $\text{WD}(\rho)$ , and for each embedding  $\tau : F \rightarrow \overline{\mathbb{Q}}_p$  let  $h_{\tau,1} \leq \dots \leq h_{\tau,n}$  denote the  $\tau$ -Hodge–Tate weights of  $\rho$ .*

*Suppose that for some  $1 \leq j \leq n-1$  we have  $\sum_{i=1}^j v_i = [F : F_0]^{-1} \sum_{i=1}^j \sum_{\tau:F \rightarrow \overline{\mathbb{Q}}_p} h_{\tau,i}$ . Then  $\rho$  is reducible.*

*Proof.* We first choose  $E/\mathbb{Q}_p$  a sufficiently large finite subextension of  $\overline{\mathbb{Q}}_p/\mathbb{Q}_p$ , so that in particular  $\rho$  can be defined over  $E$  and all embeddings  $\tau$  have image contained in  $E$ . Choose  $F'/F$  a finite Galois extension over which  $\rho$  becomes semistable. Without loss of generality, we may assume that it is totally ramified. Let  $D := D_{\text{st}}(\rho|_{\Gamma_{F'}})$  be the covariantly associated free  $F'_0 \otimes_{\mathbb{Q}_p} E$ -module, equipped with actions of  $\varphi$ ,  $N$ ,  $\text{Gal}(F'/F)$ , where  $F'_0$  denotes the maximal absolutely unramified intermediate field of  $F'/\mathbb{Q}_p$ . Note that the action of  $\text{Gal}(F'/F)$  is linear, as  $F'/F$  is totally ramified. As usual, we write  $D \cong \bigoplus_{\sigma:F'_0 \rightarrow E} D_\sigma$ . Fix any embedding  $\sigma_0 : F'_0 \rightarrow E$ , and let  $f := [F_0 : \mathbb{Q}_p] = [F'_0 : \mathbb{Q}_p]$ .

We first note that, by construction, the eigenvalues of a geometric Frobenius on  $\text{WD}(\rho)$  have the same valuations as the eigenvalues of  $\varphi^f$  on  $D_{\sigma_0}$ . As the linear actions of  $\varphi^f$  and the finite group  $\text{Gal}(F'/F)$  on  $D_{\sigma_0}$  commute, we can choose a  $\varphi^f$ -stable and  $\text{Gal}(F'/F)$ -stable  $E$ -subspace  $D'_{\sigma_0} \subset D_{\sigma_0}$  of dimension  $j$  such that the eigenvalues of  $\varphi^f$  on  $D'_{\sigma_0}$  have valuations  $v_1 \leq \dots \leq v_j$ . Then  $D'_{\sigma_0}$  is also  $N$ -stable, since  $N\varphi = p\varphi N$ . Now for each  $\sigma : F'_0 \rightarrow E$  choose the unique  $E$ -subspace  $D'_\sigma \subset D_\sigma$  that agrees with our choice of  $D'_{\sigma_0}$  when  $\sigma = \sigma_0$  and such that  $D' := \bigoplus_{\sigma:F'_0 \rightarrow E} D'_\sigma$  is  $\varphi$ -stable. Then  $D'$  is stable under the actions of  $F'_0 \otimes_{\mathbb{Q}_p} E$ ,  $\varphi$ ,  $N$ , and  $\text{Gal}(F'/F)$ .

As in the proof of [EGH13, Prop. 4.5.2] we now compute that  $t_N(D') = \frac{[E:\mathbb{Q}_p]}{[F_0:\mathbb{Q}_p]} \sum_{i=1}^j v_i$  and that  $t_H(D') \geq \frac{[E:\mathbb{Q}_p]}{[F:\mathbb{Q}_p]} \sum_{i=1}^j \sum_{\tau:F \rightarrow E} h_{\tau,i}$ . By weak admissibility, for any  $1 \leq j \leq n$  we have

$$\sum_{i=1}^j v_i \geq [F : F_0]^{-1} \sum_{i=1}^j \sum_{\tau:F \rightarrow E} h_{\tau,i},$$

with equality when  $j = n$ . If equality holds we deduce that  $t_N(D') = t_H(D')$ , so if moreover  $j < n$  it follows that  $\rho$  is reducible. □

**6.5. On base change and descent for compact unitary groups.** The purpose of this section is to discuss base change and descent results for compact unitary groups that go slightly beyond those in [Lab11], namely allowing that the unitary group is non-quasisplit at some finite places. The proofs will be provided by Sug Woo Shin in Appendix A.

Suppose that  $\tilde{F}/\tilde{F}^+$  is a CM extension of number fields with  $\tilde{F}^+ \neq \mathbb{Q}$  and  $G$  a unitary group over  $\tilde{F}^+$  such that

- (i)  $G_{/\tilde{F}}$  is an inner form of  $\mathrm{GL}_{nd}$ ;
- (ii)  $G(\tilde{F}_u^+)$  is compact for any place  $u \mid \infty$  of  $\tilde{F}^+$ ;
- (iii)  $G$  is quasi-split at all finite places that are inert in  $\tilde{F}/\tilde{F}^+$ .

Let  $c$  denote the complex conjugation of  $\tilde{F}/\tilde{F}^+$ . Let  $\Delta^+(G)$  denote the set of finite places of  $\tilde{F}^+$  where  $G$  is not quasi-split. This is a finite set of places that split or ramify in  $\tilde{F}$ . Let  $\Delta(G)$  denote the set of places of  $\tilde{F}$  lying over a place of  $\Delta^+(G)$ .

**Proposition 6.5.1.** *Suppose that  $\pi$  is a (cuspidal) automorphic representation of  $G(\mathbb{A}_{\tilde{F}^+})$ . Then there exists a partition  $n = n_1 + \cdots + n_r$  and discrete automorphic representations  $\Pi_i$  of  $\mathrm{GL}_{n_i d}(\mathbb{A}_{\tilde{F}})$  satisfying  $\Pi_i^\vee \cong \Pi_i^c$  such that  $\Pi := \Pi_1 \boxplus \cdots \boxplus \Pi_r$  is a weak base change of  $\pi$ . More precisely, at every finite split place  $v = ww^c$  of  $\tilde{F}^+$  we have  $|\mathrm{LJ}_{G(\tilde{F}_w)}(\Pi_w) \cong \pi_v$  as representation of  $G(\tilde{F}_w) \cong G(\tilde{F}_v^+)$ , and at infinity the compatibility is as in [Lab11, Cor. 5.3].*

**Proposition 6.5.2.** *Suppose that  $\tilde{F}/\tilde{F}^+$  is unramified at all finite places and that  $\Pi$  is a cuspidal automorphic representation of  $\mathrm{GL}_{nd}(\mathbb{A}_{\tilde{F}})$  such that  $\Pi^\vee \cong \Pi^c$ ,  $\Pi_\infty$  is cohomological, and  $\Pi_w$  is supercuspidal for all  $w \in \Delta(G)$  (in particular  $|\mathrm{LJ}_{G(\tilde{F}_w)}(\Pi_w) \neq 0$ ). Then there exists a (cuspidal) automorphic representation  $\pi$  of  $G(\mathbb{A}_{\tilde{F}^+})$  such that at every finite split place  $v = ww^c$  of  $\tilde{F}^+$  we have  $|\mathrm{LJ}_{G(\tilde{F}_w)}(\Pi_w) \cong \pi_v$  as representations of  $G(\tilde{F}_w) \cong G(\tilde{F}_v^+)$ .*

**6.6. Supersingular representations of  $\mathrm{GL}_n(D)$ .** We now prove the existence of supersingular representations of  $\mathrm{GL}_n(D)$  and  $\mathrm{PGL}_n(D)$ .

**Theorem 6.6.1.** *Suppose that  $C$  is algebraically closed of characteristic  $p$ . For any smooth character  $\zeta : F^\times \rightarrow C^\times$  there exists an irreducible admissible supercuspidal  $C$ -representation of  $\mathrm{GL}_n(D)$  with central character  $\zeta$ . In particular, there exists an irreducible admissible supercuspidal  $C$ -representation of  $\mathrm{PGL}_n(D)$ .*

**Corollary 6.6.2.** *If  $C$  is any field of characteristic  $p$ , then  $\mathrm{PGL}_n(D)$  admits an irreducible admissible supercuspidal representation over  $C$ .*

As in §6.3 we fix an isomorphism  $\iota : \overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ .

Recall that if  $L/L^+$  is a CM extension of number fields and  $\Pi$  is a regular algebraic cuspidal polarisable automorphic representation of  $\mathrm{GL}_n(\mathbb{A}_L)$  (in the sense of [BLGGT14b, §2.1]) we have an associated semisimple potentially semistable  $p$ -adic Galois representation  $r_{p,\iota}(\Pi) : \Gamma_L \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  that satisfies and is determined by local-global compatibility with  $\Pi$  at all finite places [BLGGT14b, Thm. 2.1.1], [BLGGT14a].

Suppose that  $L^+ \neq \mathbb{Q}$  and that  $G$  is a unitary group over  $L^+$  as in §6.5. If  $\pi$  is an automorphic representation of  $G(\mathbb{A}_{L^+})$ , then its weak base change  $\Pi = \Pi_1 \boxplus \cdots \boxplus \Pi_r$

of Proposition 6.5.1 is regular algebraic and each  $\Pi_i$  is polarisable. By the Mœglin–Waldspurger classification of the discrete spectrum and the previous paragraph it follows that  $\Pi$  has an associated semisimple potentially semistable  $p$ -adic Galois representation  $r_{p,i}(\pi) = r_{p,i}(\Pi) : \Gamma_L \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  that satisfies and is determined by local-global compatibility with  $\pi$  at all finite places of  $L$  that split over  $L^+$  and are not contained in  $\Delta(G)$ . In particular, if  $\Pi$  is not cuspidal, then  $r_{p,i}(\pi)$  is reducible.

*Proof of Theorem 6.6.1.*

*Step 0:* We show that it suffices to prove the theorem when  $C = \overline{\mathbb{F}}_p$ .

Given a smooth character  $\zeta : F^\times \rightarrow C^\times$  we can define  $\zeta' : F^\times \rightarrow \overline{\mathbb{F}}_p^\times$  by extending  $\zeta|_{\mathcal{O}_F^\times}$  (which is of finite order and hence takes values in  $\overline{\mathbb{F}}_p^\times$ ) arbitrarily. If Theorem 6.6.1 holds over  $\overline{\mathbb{F}}_p$ , there exists an irreducible admissible supercuspidal  $\overline{\mathbb{F}}_p$ -representation  $\pi$  of  $\mathrm{GL}_n(F)$  with central character  $\zeta'$ . Then by Step 3 of the proof of Proposition 3.2.1 there exists an irreducible admissible supercuspidal  $C$ -representation of  $\mathrm{GL}_n(D)$  with central character  $\zeta'$ . As  $C$  is algebraically closed, a suitable unramified twist of  $\pi'$  has central character  $\zeta$ .

We will assume from now on that  $C = \overline{\mathbb{F}}_p$ .

*Step 1:* We find a CM field  $\tilde{F}$  with maximal totally real subfield  $\tilde{F}^+ \neq \mathbb{Q}$  and a place  $v \mid p$  of  $\tilde{F}^+$  such that

- (i)  $\tilde{F}/\tilde{F}^+$  is unramified at all finite places;
- (ii) the place  $v$  splits in  $\tilde{F}$ ;
- (iii)  $\tilde{F}_v^+ \cong F$ ;

and a cyclic totally real extension  $L^+/\tilde{F}^+$  of degree  $nd$  in which  $v$  is inert.

By Krasner’s lemma we can find a totally real number field  $H$ , a place  $u$  of  $H$ , and an isomorphism  $H_u \xrightarrow{\sim} F$ . Now we apply [Hen84, Lemma 3.6] and its proof to find finite totally real extensions  $L^+/\tilde{F}^+/H$  and a place  $v$  of  $\tilde{F}^+$  above  $u$  such that  $L^+/\tilde{F}^+$  is cyclic of degree  $nd$ ,  $\tilde{F}_v^+ = H_u$ , and  $v$  is inert in  $L^+$ . (We briefly recall the proof: pick a monic polynomial  $Q$  of degree  $nd$  over  $F$  whose splitting field is the unramified extension of degree  $nd$ . Then let  $L^+$  be the splitting field of a monic polynomial  $P$  over  $H$  that is  $u$ -adically very close to  $Q$  and let  $\tilde{F}^+$  be the decomposition field of some place above  $u$ . We can use sign changes of  $P$  at real places to ensure that  $L^+$  is totally real.)

Now pick any totally imaginary quadratic extension  $\tilde{F}/\tilde{F}^+$  in which  $v$  splits. By [CHT08, Lemma 4.1.2] we can find a finite solvable Galois totally real extension  $K^+/\tilde{F}^+$  that is linearly disjoint from  $L^+/\tilde{F}^+$ , such that  $v$  splits in  $K^+$ , and such that for any prime  $v'$  of  $\tilde{F}^+$  that ramifies in  $\tilde{F}$  and any prime  $w'$  of  $K^+$  above  $v'$  the extension  $K_{w'}^+/\tilde{F}_{v'}^+$  is isomorphic to the extension  $\tilde{F}_{v'}/\tilde{F}_{v'}^+$ . Then we can replace  $\tilde{F}/\tilde{F}^+$  by  $K^+\tilde{F}/K^+$ ,  $L^+$  by  $K^+L^+$ , and  $v$  by any place of  $K^+$  lying above  $v$  to ensure that, without loss of generality,  $\tilde{F}/\tilde{F}^+$  is unramified at all finite places. (In particular, we can always achieve  $\tilde{F}^+ \neq \mathbb{Q}$  in this way.)

We let  $w$  denote a place of  $\tilde{F}$  lying over  $v$  and fix an isomorphism of topological fields  $\tilde{F}_w \xrightarrow{\sim} F$ . We let  $L := L^+ \tilde{F}$  and let  $c$  denote the unique complex conjugation of  $L$ .

*Step 2:* Letting  $v_1 \nmid p$  denote any place of  $\tilde{F}^+$  that is inert in  $L^+$  and splits in  $\tilde{F}$ , we now find a unitary group  $G$  over  $\tilde{F}^+$  such that

- (i)  $G_{/\tilde{F}}$  is an inner form of  $\mathrm{GL}_{nd}$ ;
- (ii)  $G(\tilde{F}_u^+)$  is compact for any place  $u \mid \infty$  of  $\tilde{F}^+$ ;
- (iii)  $G(\tilde{F}_w) \cong \mathrm{GL}_n(D)$ ;
- (iv)  $G$  is quasi-split at all finite places not contained in  $\{v, v_1\}$ .

Let  $G^*$  denote the unique quasi-split outer form of  $\mathrm{GL}_{nd}$  over  $\tilde{F}^+$  that splits over  $\tilde{F}$ . By [Clo91, §2] we can find an inner form  $G$  of  $G^*$  that satisfies all the above conditions. (If  $nd$  is odd we do not need the auxiliary place  $v_1$ . If  $nd$  is even we use  $v_1$  to ensure our local conditions can be globally realised.)

We let  $\Delta_L(G)$  be the set of places of  $L$  lying above  $\Delta(G)$  (defined in §6.5). Note that any place of  $\Delta(G)$  is inert in  $L$  and splits over  $\tilde{F}^+$ .

For any finite place  $w' \notin \Delta^+(G)$  of  $\tilde{F}^+$  that splits as  $w' = w'w'^c$  in  $\tilde{F}$  we obtain an isomorphism  $\iota_{w'} : G(\tilde{F}_{w'}^+) = G(\tilde{F}_{w'}) \xrightarrow{\sim} \mathrm{GL}_{nd}(\tilde{F}_{w'})$  that is unique up to conjugacy. Moreover,  $c \circ \iota_{w'}$  and  $\iota_{w'^c}$  differ by an outer automorphism of  $\mathrm{GL}_{nd}(\tilde{F}_{w'^c})$ . We also fix an isomorphism  $\iota_w : G(\tilde{F}_v^+) = G(\tilde{F}_w) \xrightarrow{\sim} \mathrm{GL}_n(D)$ . (It is canonical, up to conjugacy, by condition (i).)

*Step 3:* We find an algebraic Hecke character  $\chi : \mathbb{A}_L^\times / L^\times \rightarrow \mathbb{C}^\times$  with associated potentially crystalline  $p$ -adic Galois representation  $\psi = r_{p,\iota}(\chi) : \Gamma_L \rightarrow \overline{\mathbb{Q}}_p^\times$  such that

- (i)  $\psi\psi^c = \varepsilon^{-(nd-1)}$ ;
- (ii) for any place  $w' \in \Delta(G)$  the induced representation  $\mathrm{Ind}_{W_{L_{w'}}}^{W_{\tilde{F}_{w'}}} \chi_{w'}$  is irreducible;
- (iii) the representation  $r := \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi$  has regular Hodge–Tate weights, i.e., for each  $\kappa' : \tilde{F} \rightarrow \overline{\mathbb{Q}}_p$  the  $nd$  integers  $\mathrm{HT}_{\kappa'}(\mathrm{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi)$  are pairwise distinct;
- (iv) the restriction  $\bar{r}|_{\Gamma_{\tilde{F}_w}}$  to  $\Gamma_{\tilde{F}_w}$  of the reduction  $\bar{r} \cong \mathrm{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \bar{\psi}$  is irreducible.

Our strategy is to carefully choose characters  $\theta_{w'} : \Gamma_{L_{w'}} \rightarrow \overline{\mathbb{Q}}_p^\times$  for any  $w' \in \Delta_L(G)$  or  $w' \mid p$  that are potentially crystalline when  $w' \mid p$  and satisfy  $(\theta_{w'} \theta_{w'^c}^c)|_{I_{L_{w'}}} = \varepsilon^{-(nd-1)}|_{I_{L_{w'}}}$ , and then deduce by [BLGGT14b, Lemma A.2.5(1)] that there exists a character  $\psi : \Gamma_L \rightarrow \overline{\mathbb{Q}}_p^\times$  such that  $\psi\psi^c = \varepsilon^{-(nd-1)}$  and  $\psi|_{I_{L_{w'}}} = \theta_{w'}|_{I_{L_{w'}}}$  for all  $w' \in \Delta_L(G)$ . In particular,  $\psi$  is potentially crystalline and we can let  $\chi$  be the associated algebraic Hecke character. In particular, condition (i) holds.

For any  $w' \in \Delta_L(G)$  we can choose a smooth character  $\zeta_{w'} : \Gamma_{L_{w'}}^{\mathrm{ab}} \cong \widehat{L_{w'}^\times} \rightarrow \overline{\mathbb{Q}}_p^\times$  such that the  $\mathrm{Gal}(L_{w'}/\tilde{F}_{w'})$ -conjugates of  $\zeta_{w'}|_{\mathcal{O}_{L_{w'}}^\times}$  are pairwise distinct. (This just uses that  $\mathrm{Gal}(L_{w'}/\tilde{F}_{w'})$  acts faithfully on  $\mathcal{O}_{L_{w'}}^\times$  and that  $\mathcal{O}_{L_{w'}}^\times$  is complete. Or concretely, we can take a faithful character of  $k_{L_{w'}}^\times$  and inflate it to  $\mathcal{O}_{L_{w'}}^\times$ .) We may assume without loss of generality that  $\zeta_{w'} \zeta_{w'^c}^c = 1$ .

If  $w' \in \Delta_L(G)$  and  $w' \nmid p$ , then we let  $\theta_{w'} := \zeta_{w'}$ . This guarantees that the  $\text{Gal}(L_{w'}/\tilde{F}_{w'})$ -conjugates of  $\psi|_{\Gamma_{L_{w'}}}$ , and hence of  $\chi_{w'}$ , are pairwise distinct so that condition (ii) holds for places  $w' \nmid p$ .

If  $w' \mid p$ , we first introduce the notation  $S_K := \text{Hom}_{\text{cts}}(K, \overline{\mathbb{Q}}_p)$  for any topological field  $K$  of characteristic zero and  $S_k := \text{Hom}(k, \overline{\mathbb{F}}_p)$  for any field  $k$  of characteristic  $p$ . Suppose that we are given any integers  $\lambda_\kappa$  ( $\kappa \in S_L$ ) satisfying  $\lambda_\kappa + \lambda_{\kappa c} = nd - 1$  for all  $\kappa \in S_L$ . Let  $\theta_{w'}^{\text{cr}} : \Gamma_{L_{w'}} \rightarrow \overline{\mathbb{Q}}_p^\times$  be any crystalline character with  $\text{HT}_\kappa(\theta_{w'}^{\text{cr}}) = \lambda_\kappa$  for all  $\kappa \in S_{L_{w'}} \subset S_L$ . Without loss of generality, by our constraint on the  $\lambda_\kappa$ , we may assume that  $\theta_{w'}^{\text{cr}}(\theta_{w'}^{\text{cr}})^c = \varepsilon^{-(nd-1)}$ . We define  $\theta_{w'} := \theta_{w'}^{\text{cr}} \zeta_{w'}$  for  $w' \in \Delta_L(G)$  and  $\theta_{w'} := \theta_{w'}^{\text{cr}}$  otherwise. If  $w' \in \Delta_L(G)$ , then by construction,  $\iota \zeta_{w'}|_{I_{L_{w'}}}$  corresponds to  $\chi_{w'}|_{\mathcal{O}_{L_{w'}}^\times}$  under the local Artin map, hence the  $\text{Gal}(L_{w'}/\tilde{F}_{w'})$ -conjugates of  $\chi_{w'}$  are pairwise distinct so that condition (ii) holds for places  $w' \mid p$ .

We will now choose the integers  $\lambda_\kappa$  ( $\kappa \in S_L$ ) so that conditions (iii) and (iv) hold. Note that condition (iii) is equivalent to the condition

(iii') for any  $\kappa' \in S_{\tilde{F}}$  the  $n$  integers  $\{\lambda_\kappa : \kappa \in S_L, \kappa|_{\tilde{F}} = \kappa'\}$  are pairwise distinct.

First choose the  $\lambda_\kappa$  for those  $\kappa \in S_L$  that do not induce either of the places  $w$ ,  $w^c$  on  $L$  so that condition (iii') holds for any  $\kappa' \in S_{\tilde{F}}$  not inducing either of the places  $w$ ,  $w^c$  on  $\tilde{F}$ . It remains to choose the  $\lambda_\kappa$  for those  $\kappa$  that induce the place  $w$  on  $L$  (since the remaining  $\lambda_\kappa$  are determined by the condition  $\lambda_\kappa + \lambda_{\kappa c} = nd - 1$  for all  $\kappa$ ), i.e. for  $\kappa \in S_{L_w}$ .

To choose the  $\lambda_\kappa$  for  $\kappa \in S_{L_w}$ , we note that  $\bar{\psi}|_{\Gamma_{\tilde{F}_w}} \cong \text{Ind}_{\Gamma_{L_w}}^{\Gamma_{\tilde{F}_w}}(\bar{\psi}|_{\Gamma_{L_w}})$  is irreducible if and only if the  $\text{Gal}(L_w/\tilde{F}_w)$ -conjugates of  $\bar{\psi}|_{\Gamma_{L_w}}$  are pairwise distinct, or equivalently if the characters  $\bar{\psi}|_{I_{L_w}}^{q^i}$  ( $0 \leq i \leq nd - 1$ ) are pairwise distinct. (Recall that  $q = \#k_F$ .) We have  $\bar{\psi}|_{I_{L_w}} \cong \bar{\theta}_w^{\text{cr}} \zeta_w|_{I_{L_w}}$ . By [GHS18, Cor. 7.1.2] we have  $\bar{\theta}_w^{\text{cr}}|_{I_{L_w}} = \prod_{\sigma \in S_{k_{L_w}}} \omega_\sigma^{-b_\sigma}$ , where  $\omega_\sigma$  corresponds to the character  $\mathcal{O}_{L_w}^\times \rightarrow k_{L_w}^\times \xrightarrow{\sigma} \overline{\mathbb{F}}_p^\times$  under local class field theory and  $b_\sigma := \sum_{\kappa \in S_{L_w} : \bar{\kappa} = \sigma} \lambda_\kappa$ . Fix any character  $\xi : I_{L_w} \rightarrow \overline{\mathbb{F}}_p^\times$  that is extendable to  $\Gamma_{L_w}$ . We claim that we can choose the  $\lambda_\kappa$  for  $\kappa \in S_{L_w}$  so that  $\bar{\psi}|_{I_{L_w}} = \xi$ . This is true because any  $\xi$  is of the form  $\prod_{\sigma \in S_{k_{L_w}}} \omega_\sigma^{c_\sigma}$  for some  $c_\sigma \in \mathbb{Z}$ . (In fact,  $\xi$  is a power of  $\omega_\sigma$  for any fixed  $\sigma$ .) By taking  $\xi$  so that the  $\xi^{q^i}$  ( $i = 0, 1, \dots, nd - 1$ ) are pairwise distinct (taking, for example,  $\xi = \omega_\sigma$  for any  $\sigma$ ), condition (iv) holds. Finally, we can ensure that condition (iii') holds for all  $\kappa' \in S_{\tilde{F}_w}$  while keeping  $\bar{\psi}|_{I_{\tilde{F}_w}}$  unchanged by varying the  $\lambda_\kappa$  (for  $\kappa \in S_{L_w}$ ) modulo  $q^{nd} - 1$ . This completes Step 3.

*Step 4:* Using automorphic induction and descent we define an automorphic representation  $\pi'$  of  $G(\mathbb{A}_{\tilde{F}_+})$  with associated Galois representation  $r = \text{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi$ .

By conditions (i) and (iii) in Step 3 we deduce that  $\chi \chi^c = |\cdot|_L^{-(nd-1)}$  and that for any place  $v' \mid \infty$  of  $\tilde{F}$  the  $nd$  characters  $\{\chi_{w'} : w' \mid v'\}$  are pairwise distinct.

Let  $\Pi''$  denote the automorphic induction of  $\chi$  with respect to the cyclic extension  $L/\tilde{F}$ . It is an automorphic representation of  $\text{GL}_{nd}(\mathbb{A}_{\tilde{F}})$  that is parabolically induced from a cuspidal representation. (For the functoriality of automorphic induction in cyclic extensions we refer to [Hen12], which shows in particular that it is compatible

with local automorphic induction at all places. Note the results of [Hen12] apply to unitary representations, but by twisting they continue to hold for twists of unitary representations.)

We claim that  $\Pi''$  is cuspidal. This follows from [Hen12], Theorems 2, 3, and Proposition 2.5, provided that the Hecke characters  $\{\chi^\sigma : \sigma \in \text{Gal}(L/\tilde{F})\}$  are pairwise distinct. Equivalently, the Galois characters  $\{\psi^\sigma : \sigma \in \text{Gal}(L/\tilde{F})\}$  are pairwise distinct, which in turn is equivalent to the condition that  $\text{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi$  is irreducible. This is a consequence of condition (iv) in Step 3, so  $\Pi''$  is cuspidal.

Let  $\Pi' := \Pi'' \otimes |\det|_F^{(nd-1)/2}$ . Then  $\chi\chi^c = |\cdot|_L^{-(nd-1)}$  implies that  $(\Pi')^\vee \cong \Pi'^c$ . On the other hand,  $\Pi'_\infty$  is cohomological by [Clo90, Lemma 3.14] (this uses the distinctness of the characters  $\{\chi_{w'} : w' \mid v'\}$  for  $v' \mid \infty$ ). It follows that  $\Pi'$  is regular algebraic and polarisable in the sense of [BLGGT14b], so we have an associated Galois representation  $r_{p,i}(\Pi')$ . By local-global compatibility at unramified places we deduce that  $r_{p,i}(\Pi') \cong \text{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi$ .

For  $w' \in \Delta(G)$  the local factor  $\Pi'_{w'}$  is supercuspidal, as  $\text{rec}_F(\Pi'_{w'})$  is irreducible by item (ii) in Step 3. It follows from what we recalled in §6.2 that  $|\text{LJ}_{G(\tilde{F}_{w'})}(\Pi'_{w'}) \neq 0$ .

By Proposition 6.5.2 we deduce that  $\Pi'$  descends to a (cuspidal) automorphic representation  $\pi'$  of  $G(\mathbb{A}_{\tilde{F}^+})$ , such that  $\pi_{v'} \cong \Pi'_{w'}$  as representations of  $G(\tilde{F}_{v'}^+) \cong \text{GL}_{nd}(\tilde{F}_{v'}^+)$  for all finite places  $v' \notin \Delta^+(G)$  of  $\tilde{F}^+$  that split as  $v' = w'w'^c$  in  $\tilde{F}$ . In particular,  $r_{p,i}(\pi') \cong \text{Ind}_{\Gamma_L}^{\Gamma_{\tilde{F}}} \psi$ .

*Step 5:* We use the automorphic representation  $\pi'$  to define an irreducible admissible  $\overline{\mathbb{F}}_p$ -representation  $\sigma$  of  $G(\tilde{F}_v^+) \cong \text{GL}_n(D)$ .

Fix a maximal compact open subgroup  $K_p$  of  $\prod_{v' \mid p} G(\tilde{F}_{v'}^+)$ . If  $U$  is any compact open subgroup of  $K_p G(\mathbb{A}_{\tilde{F}^+}^{\infty,p})$  and  $\mathcal{W}$  is any  $K_p$ -module we let  $S(U, \mathcal{W})$  be the space of functions  $f : G(\tilde{F}^+) \backslash G(\mathbb{A}_{\tilde{F}^+}^{\infty,p}) \rightarrow \mathcal{W}$  such that  $f(gu) = u_p^{-1}f(g)$  for all  $g \in G(\mathbb{A}_{\tilde{F}^+}^{\infty,p})$  and  $u \in U$ . Similarly, if  $U^v$  is any compact open subgroup of  $G(\mathbb{A}_{\tilde{F}^+}^{\infty,v})$  we let  $S(U^v, \overline{\mathbb{F}}_p) := \varinjlim_{U^v} S(U^v U_v, \overline{\mathbb{F}}_p)$ . It is an admissible smooth representation of  $G(\tilde{F}_v^+)$ .

We see as in [EGH13, Lemma 7.1.6] that there exists a  $\mathbb{Q}_p$ -algebraic representation  $\mathcal{W}_{\text{alg}}$  over  $\overline{\mathbb{Q}}_p$  of  $\prod_{v' \mid p} G(\tilde{F}_{v'}^+)$  such that  $\varinjlim_U S(U, \mathcal{W}_{\text{alg}})$  contains  $\iota^{-1}\pi'^\infty$  as  $G(\mathbb{A}_{\tilde{F}^+}^{\infty,p})$ -representation. Pick a compact open subgroup  $U = U_p U^p \leq K_p G(\mathbb{A}_{\tilde{F}^+}^{\infty,p})$  with  $(\pi'^\infty)^U \neq 0$ . We may assume moreover, without loss of generality, that  $U^p$  is sufficiently small, i.e. that there exists a finite place  $v'$  of  $\tilde{F}^+$  the projection of  $U$  to  $G(\tilde{F}_{v'}^+)$  contains no element of finite order other than the identity.

Let  $\mathcal{P}$  denote the set of places  $w' \nmid p$  of  $\tilde{F}$  that split over a place  $v'$  of  $\tilde{F}^+$ , are not contained in  $\Delta(G)$ , and are such that  $U = U_{v'} U^{v'}$  for compact open subgroups of  $G(\tilde{F}_{v'}^+)$  and  $G(\mathbb{A}_{\tilde{F}^+}^{\infty,v'})$ . For each such  $w'$  we conjugate the isomorphism  $\iota_{w'}$  of Step 2 so that  $\iota_{w'}(U_{v'}) = \text{GL}_{nd}(\mathcal{O}_{\tilde{F}_{w'}})$ . Let  $\mathbb{T}^{\mathcal{P}}$  denotes the commutative polynomial  $\overline{\mathbb{Z}}_p$ -algebra generated by variables  $T_{w'}^{(i)}$  for  $w' \in \mathcal{P}$  and  $0 \leq i \leq nd$ , acting on any  $S(U, \mathcal{W})$  as in [EGH13, §7.1.2].

Let  $\mathfrak{m}$  denote the maximal ideal of  $\mathbb{T}^{\mathcal{P}}$  with residue field  $\overline{\mathbb{F}}_p$  defined by demanding that the characteristic polynomial of  $\bar{r}(\text{Frob}_{w'})$  equals  $\sum_{i=0}^{nd} (-1)^i (\mathbf{N}w')^{i(i-1)/2} T_{w'}^{(i)} X^{n-i}$

modulo  $\mathfrak{m}$  for all  $w' \in \mathcal{P}$ , where  $\text{Frob}_{w'}$  denotes a geometric Frobenius element at  $w'$ . By local-global compatibility and [CHT08, Cor. 3.1.2] it follows that  $S(U, \mathcal{W}_{\text{alg}})_{\mathfrak{m}} \neq 0$ . Picking an invariant lattice in  $\mathcal{W}_{\text{alg}}$  and reducing modulo the maximal ideal of  $\overline{\mathbb{Z}}_p$  we see that  $S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}} \neq 0$ , where  $U^v = U^p \prod_{v'|p, v' \neq v} U_{v'}$  and  $U_{v'} \leq G(\widetilde{F}_{v'}^+)$  are (small) compact open subgroups. Let  $\sigma$  be an irreducible (admissible)  $G(\widetilde{F}_v^+)$ -subrepresentation of  $S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}}$ , which exists by the proof of Lemma 9.9 in [Her11] or [HV12, Lemma 7.10]). We will consider it as representation of  $\text{GL}_n(D)$  via the isomorphism  $\iota_w$  of Step 2.

*Step 6:* We show that  $\sigma$  is supersingular, or equivalently, supercuspidal.

By [OV, Thm. 3] it suffices to show that the  $\mathcal{H}_{\overline{\mathbb{F}}_p}$ -module  $\sigma^{I(1)}$  is supersingular, where  $I(1)$  denotes the pro- $p$  Iwahori subgroup of  $\text{GL}_n(D) \cong G(\widetilde{F}_v^+)$  defined in §6.3. In fact, we will even show that  $(S(U^v, \overline{\mathbb{F}}_p)_{\mathfrak{m}})^{I(1)} \cong S(U^v \cdot I(1), \overline{\mathbb{F}}_p)_{\mathfrak{m}}$  is supersingular. Assume by contradiction that this is false. As  $U^p$  is sufficiently small we have  $S(U^v \cdot I(1), \overline{\mathbb{Z}}_p)_{\mathfrak{m}} \otimes_{\overline{\mathbb{Z}}_p} A \cong S(U^v \cdot I(1), A)_{\mathfrak{m}}$  for  $A = \overline{\mathbb{F}}_p$  or  $\overline{\mathbb{Q}}_p$ . It follows by our assumption that there exists an automorphic representation  $\pi$  of  $G(\mathbb{A}_{\widetilde{F}^+})$  satisfying

- (i)  $v^{-1}\pi_v^{I(1)} \neq 0$  is an integral  $\mathcal{H}_{\overline{\mathbb{Q}}_p}$ -module whose reduction is non-supersingular;
- (ii) the associated Galois representation  $r_{p,\iota}(\pi)$  lifts  $\bar{r}$ ;
- (iii)  $\pi_{\infty}$  is trivial.

By Proposition 6.5.1 we obtain an automorphic representation  $\Pi$  of  $\text{GL}_{nd}(\mathbb{A}_{\widetilde{F}})$  with associated Galois representation  $r_{p,\iota}(\Pi)$  lifting  $\bar{r}$  such that  $|\text{LJ}_{G(\widetilde{F}_{w'})}(\Pi_{w'}) \cong \pi_{v'}$  for all finite places  $v'$  of  $\widetilde{F}^+$  that split as  $v' = w'w'^c$  in  $\widetilde{F}$ . As  $\bar{r}$  is irreducible by construction we know that  $\Pi$  is cuspidal. In particular,  $\Pi_{w'}$  is essentially unitarisable and generic for each finite place  $w'$  of  $\widetilde{F}$ . Let  $v_1 \leq \dots \leq v_{nd}$  denote the valuations of the eigenvalues of a geometric Frobenius on  $\iota^{-1}(\text{rec}_F(\Pi_w))$ . From Proposition 6.3.3 (applied to  $\Pi_w$ ) we deduce that there exists  $1 \leq j \leq n-1$  such that

$$(6.6.3) \quad \sum_{i=1}^{jd} v_i = -\frac{d^2 j(n-j)}{2} \text{val}(q).$$

Note that the infinitesimal character of  $\Pi$  is the same as that of the trivial representation. By [BLGGT14b, Thm. 2.1.1] we deduce that  $\text{HT}_{\tau}(r_{p,\iota}(\Pi)|_{\Gamma_{\widetilde{F}_w}}) = \{0, 1, \dots, nd-1\}$  for all  $\tau : \widetilde{F}_w \rightarrow \overline{\mathbb{Q}}_p$  and that  $\iota \text{WD}(r_{p,\iota}(\Pi)|_{\Gamma_{\widetilde{F}_w}})^{\text{F-ss}} \cong \text{rec}_F(\Pi_w \otimes |\det|_F^{(1-nd)/2})$ . Together with (6.6.3) it follows that

$$\sum_{i=1}^{jd} v'_i = -\frac{d^2 j(n-j)}{2} \text{val}(q) + jd \text{val}(q^{(nd-1)/2}) = \binom{jd}{2} \text{val}(q),$$

where  $v'_1 \leq \dots \leq v'_{nd}$  denote the valuations of the eigenvalues of a geometric Frobenius on  $\text{WD}(r_{p,\iota}(\Pi)|_{\Gamma_{\widetilde{F}_w}})$ . By Lemma 6.4.1, noting that  $\text{val}(q) = [F_0 : \mathbb{Q}_p]$ , it follows that  $r_{p,\iota}(\Pi)|_{\Gamma_{\widetilde{F}_w}}$  is reducible, which contradicts that its reduction  $\bar{r}|_{\Gamma_{\widetilde{F}_w}}$  is irreducible by Step 3.

*Step 7:* We fix the central character.

Suppose we are given a smooth character  $\zeta : F^{\times} \rightarrow \overline{\mathbb{F}}_p^{\times}$ . As in Step 0 it is enough to construct an irreducible admissible supercuspidal representation such that  $\mathcal{O}_F^{\times}$  acts via  $\zeta|_{\mathcal{O}_F^{\times}}$ .



Note that  $\sigma$  has a central character  $\chi_\sigma$ , as it is irreducible and admissible. We claim that  $\chi_\sigma|_{\mathcal{O}_{\tilde{F}_w}^\times} = \det(\bar{\tau}|_{I_{\tilde{F}_w}}) \cdot \varepsilon^{nd(nd-1)/2}$  under the local Artin map. The central character of the  $\mathrm{GL}_n(D)$ -representation  $\iota^{-1}\pi_v$  in Step 6 lifts  $\chi_\sigma$  and is equal to the central character of  $\iota^{-1}\Pi_w$ . (This equality follows from the definition of LJ in [Bad08, §2.7], noting that  $\Pi_w$  is generic and hence fully induced from an essentially square-integrable representation.) By local-global compatibility at  $p$  (cf. Step 6) the latter character equals  $\mathrm{WD}(\det r_{p,\iota}(\Pi)|_{\Gamma_{\tilde{F}_w}}) \cdot \iota^{-1}|_F|^{(nd-1)/2}$ . As  $\det r_{p,\iota}(\Pi)|_{\Gamma_{\tilde{F}_w}}$  has parallel Hodge–Tate weights  $0, 1, \dots, nd-1$ , we have  $\det r_{p,\iota}(\Pi)|_{I_{\tilde{F}_w}} = \varepsilon^{-nd(nd-1)/2} \cdot \mathrm{WD}(\det r_{p,\iota}(\Pi)|_{\Gamma_{\tilde{F}_w}})|_{I_{\tilde{F}_w}}$  and hence deduce the claim.

It thus suffices to show that in Step 3 above we can choose  $r$  such that  $\det(\bar{\tau}|_{I_{\tilde{F}_w}})$  is any prescribed character that is extendable to  $\Gamma_{\tilde{F}_w}$ . Let us fix any  $\sigma \in S_{k_{L_w}}$  and write  $\bar{\psi}|_{I_{L_w}} = \omega_\sigma^s$  for some  $s \in \mathbb{Z}$ . Then the condition that the  $\bar{\psi}|_{I_{L_w}}^{q^i}$  ( $i = 0, 1, \dots, nd-1$ ) are pairwise distinct means:

$$(6.6.4) \quad s \not\equiv 0 \pmod{\frac{q^{nd}-1}{q^\ell-1}} \quad \forall \ell \mid nd, \ell < nd.$$

On the other hand,  $\det(\bar{\tau}|_{I_{\tilde{F}_w}}) = \prod_{i=0}^{nd-1} \bar{\psi}|_{I_{L_w}}^{q^i} = \omega_{\sigma_0}^s$ , where  $\sigma_0 \in S_{k_{\tilde{F}_w}}$  is the restriction of  $\sigma$  to  $k_{\tilde{F}_w}$ . As any character  $\Gamma_{\tilde{F}_w} \rightarrow \overline{\mathbb{F}_p}^\times$  restricts to a power of  $\omega_{\sigma_0}$  on inertia, we can prescribe  $\det(\bar{\tau}|_{I_{\tilde{F}_w}})$  if and only if we can choose  $s$  in any residue class modulo  $q-1$ . Since  $\frac{q^{nd}-1}{q^\ell-1} \geq q+1$  for any  $\ell \mid nd, \ell < nd$ , it follows that we can pick any  $s$  in the interval  $[1, q-1]$ , completing the proof.  $\square$

*Proof of Corollary 6.6.2.* Going back to Step 5 of the proof of Theorem 6.6.1, it is clear that the representation  $S(U^v, \overline{\mathbb{F}_p})_{\mathfrak{m}} \neq 0$  is defined over a finite field (as  $\bar{\tau}$  is), and hence so is its irreducible subrepresentation  $\sigma$ . This proves the corollary when  $C$  is a sufficiently large finite field of characteristic  $p$ . We conclude by Proposition 3.2.1.  $\square$

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## APPENDIX A. BASE CHANGE

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In this appendix we will prove Propositions 6.5.1 and 6.5.2.

We need a character identity for the Jacquet–Langlands correspondence. We fix compatible Haar measures on  $\mathrm{GL}_{nd}(F)$  and  $\mathrm{GL}_n(D)$  in the sense of [Kot88]. We say that  $f \in C_c^\infty(\mathrm{GL}_n(D))$  and  $f^* \in C_c^\infty(\mathrm{GL}_{nd}(F))$  are associated, or that  $f^*$  is a transfer of  $f$ , if the orbital integral identity  $O_\delta(f) = O_{\delta^*}(f^*)$  holds for every regular semisimple elements  $\delta \in \mathrm{GL}_n(D)$  and  $\delta^* \in \mathrm{GL}_{nd}(F)$  with the same characteristic polynomial. (We use the same Haar measures on the centralizers of  $\delta$  and  $\delta^*$  in  $\mathrm{GL}_n(D)$  and  $\mathrm{GL}_{nd}(F)$ , respectively, to compute the orbital integrals.) A well known fact, proven in [DKV84], is that every  $f \in C_c^\infty(\mathrm{GL}_n(D))$  admits a transfer in  $C_c^\infty(\mathrm{GL}_{nd}(F))$ . Let  $e(\mathrm{GL}_n(D)) \in \{\pm 1\}$  denote the Kottwitz sign [Kot83].

**Proposition A.0.1.** *Let  $\pi^*$  be an irreducible unitarisable representation of  $\mathrm{GL}_{nd}(F)$ . For every associated pair  $f \in C_c^\infty(\mathrm{GL}_n(D))$  and  $f^* \in C_c^\infty(\mathrm{GL}_{nd}(F))$ , we have*

$$\mathrm{tr} \pi^*(f^*) = e(\mathrm{GL}_n(D)) \cdot \mathrm{tr} (|\mathrm{LJ}_{\mathrm{GL}_n(D)}|(\pi^*)) (f).$$

*Proof.* This follows from [Bad07, Prop 3.3] and the Weyl integration formula for  $\mathrm{GL}_n(D)$  and  $\mathrm{GL}_{nd}(F)$ .  $\square$

We assume that the CM extension  $\tilde{F}/\tilde{F}^+$  and the unitary group  $G$  over  $\tilde{F}^+$  are as in Section 6.5.

Write  $G^*$  for a quasi-split inner twist of  $G$  over  $\tilde{F}^+$  (with an isomorphism between  $G^*$  and  $G$  over an algebraic closure of  $\tilde{F}^+$ ). Every character considered on  $p$ -adic or adelic points of  $G^*$  over  $\tilde{F}$  will be the twisted trace in what follows without further comments.

*Proof of Proposition 6.5.1.* This proposition is implied by [Lab11, Cor. 5.3] except possibly the relation  $|\mathrm{LJ}_{G(\tilde{F}_w)}|(\Pi_w) \cong \pi_v$ .<sup>4</sup> We elaborate on this point. Thus we assume  $v = ww^c$  as in the proposition. We will omit the subscript for  $|\mathrm{LJ}|$  when there is little danger of confusion.

Let  $S$  be a finite set of places of  $\tilde{F}^+$  containing all infinite places as well as all finite place where either  $\pi$  or  $G$  is ramified. Denote by  $S_f$  the subset of finite places in  $S$ . In particular  $S_f \supset \Delta^+(G)$ . For an irreducible admissible representation  $\sigma$  of  $G(\mathbb{A}_{\tilde{F}^+})$  unramified outside  $S$ , we write  $BC(\sigma^S) = \Pi^S$  to mean that the local unramified base change of  $\sigma_u$  is  $\Pi_u$  at all places  $u \notin S$ . (The unramified base change is defined via Satake transform.) Using the Langlands parametrization at archimedean places, we write  $BC(\sigma_\infty) = \Pi_\infty$  to mean that the local base change of  $\sigma_\infty$  is  $\Pi_\infty$ .

For each  $f_v \in C_c^\infty(G(\tilde{F}_v^+))$  let  $f_v^* \in C_c^\infty(G^*(\tilde{F}_v^+))$  denote a transfer. There exists  $\phi_v \in C_c^\infty(G^*(\tilde{F} \otimes_{\tilde{F}^+} \tilde{F}_v^+))$  whose base change transfer is  $f_v^*$  by [Lab11, Lem. 4.1].

Let  $\Pi_v = \Pi_w \otimes \Pi_{w^c}$  be the  $v$ -component of  $\Pi$ , which is a representation of  $G^*(\tilde{F} \otimes_{\tilde{F}^+} \tilde{F}_v^+)$ . Let  $\pi_w^* := \Pi_w$  via the isomorphism  $G^*(\tilde{F}_v^+) \cong G^*(\tilde{F}_w)$ . Then we have the following character identities, where  $\mathrm{tr} \Pi_v(\phi_v)$  means the twisted trace by abuse of

<sup>4</sup>In fact this assertion is implicit in [Lab11, Cor. 5.3] where it reads “Aux places non ramifiées ou décomposées la correspondance  $\sigma_v \mapsto \pi_v$  est donnée par le changement de base local.” However when  $v = ww^c$  the author introduced the notion of local base change (§4.10 of *op. cit.*) only when  $\mathbf{U}$  is a general linear group at  $v$  (in his notation). We need the case when  $\mathbf{U}$  is a nontrivial inner form of a general linear group at  $v$ .

notation:

$$(A.0.2) \quad \mathrm{tr} \Pi_v(\phi_v) = \mathrm{tr} \pi_v^*(f_v^*) = e(G(\tilde{F}_v^+)) \cdot \mathrm{tr} (|\mathrm{LJ}|(\pi_v^*)) (f_v).$$

The first equality holds by the same computation as for [Rog90, Prop. 4.13.2 (a)]. The second equality is Proposition A.0.1. On the other hand, the trace formula argument of [Lab11, Thm. 5.1] shows

$$(A.0.3) \quad \sum_{\sigma} m(\sigma) \mathrm{tr} \sigma_{S_f}(f_{S_f}) = c \cdot \mathrm{tr} \Pi_{S_f}(\phi),$$

with a nonzero constant  $c$  and the automorphic multiplicity  $m(\sigma) \in \mathbb{Z}_{\geq 0}$ , where the sum runs over  $\sigma$  such that  $BC(\sigma^S) = \Pi^S$  and  $BC(\sigma_{\infty}) = \Pi_{\infty}$ . Again the trace on the right hand side is the twisted trace. Since (A.0.3) holds for each  $f^{\infty} = \prod_{u \neq \infty} f_u$  (and  $f_u^*$  and  $\phi_u$  constructed from  $f_u$  at each  $u$  as above), we choose  $f_u$  to be the characteristic function on a sufficiently small compact open subgroup of  $G(\tilde{F}_u^+)$  at  $u \in S_f \setminus \{v\}$ . Then  $\mathrm{tr} \sigma_u(f_u) \geq 0$ , so we obtain

$$\sum_{\sigma} n(\Pi_v, \sigma) \mathrm{tr} \sigma_v(f_v) = c \cdot \mathrm{tr} \Pi_v(\phi), \quad \text{with } n(\Pi_v, \sigma) \geq 0,$$

where the sum runs over  $\sigma$  such that  $BC(\sigma^S) = \Pi^S$ ,  $BC(\sigma_{\infty}) = \Pi_{\infty}$ , and  $\mathrm{tr} \sigma_u(f_u) \neq 0$  at every  $u \in S_f \setminus \{v\}$ . Notice that  $\sigma = \pi$  contributes to the sum with  $n(\Pi_v, \pi) > 0$ , by choice of  $f_u$  at  $u \in S_f \setminus \{v\}$ . Substituting (A.0.2) we obtain

$$\sum_{\sigma} n(\Pi_v, \sigma) \mathrm{tr} \sigma_v(f_v) = c \cdot e(G(\tilde{F}_v^+)) \cdot \mathrm{tr} (|\mathrm{LJ}|(\pi_v^*)) (f_v),$$

with the sum running over the same set of  $\sigma$ . Since the sum is not identically zero,  $|\mathrm{LJ}|(\pi_v^*)$  is irreducible (rather than 0). By linear independence of characters of  $G(\tilde{F}_v^+)$ , we deduce that the coefficients on the left hand side are zero unless  $\sigma_v \cong |\mathrm{LJ}|(\pi_v^*)$ . Since  $n(\Pi_v, \pi) > 0$ , we must have  $\pi_v \cong |\mathrm{LJ}|(\pi_v^*)$ , noting that no cancellation takes place in the sum as the coefficients are non-negative.  $\square$

*Proof of Proposition 6.5.2.* The proposition would follow from [Lab11, Thm. 5.4] but we need some care since our  $G$  is not quasi-split<sup>5</sup>; we also need some more information at split places. Thus we sketch the trace formula argument. Again we drop the subscript from  $|\mathrm{LJ}|$ .

The argument of [Lab11, Thm. 5.4] shows the identity (adapted to our notation)

$$(A.0.4) \quad \sum_{\sigma} m(\sigma) \mathrm{tr} \sigma(f) = \mathrm{tr} \Pi(\phi)$$

with the functions  $\phi = \prod_u \phi_u$  on  $G^*(\mathbb{A}_{\tilde{F}})$  and  $f = \prod_u f_u$  on  $G(\mathbb{A}_{\tilde{F}^+})$  as in the proof there, where the sum runs over automorphic representations  $\sigma$  of  $G(\mathbb{A}_{\tilde{F}^+})$  with multiplicity  $m(\sigma)$  whose weak base change is  $\Pi$ . The right hand side is interpreted as the twisted trace by the convention mentioned earlier.

<sup>5</sup>Contrary to the assumption on  $\mathbf{U}$  above [Lab11, Thm. 5.4] that  $U$  is quasi-split at all *inert* places, it seems the assumption ought to be that  $\mathbf{U}$  is quasi-split at *all* finite places. We believe that “Le second membre étant non identiquement nul” (in the proof of [Lab11, Thm. 5.4], between the second and third displays) is not always true, e.g. if  $\Pi_w$  is a principal series representation at a non-quasi-split place that splits in  $\tilde{F}$ . (See the third paragraph of the current proof.) If it were true, we could deduce Proposition 6.5.2 directly from [Lab11, Thm. 5.4].

The key point to show is that the right hand side does not always vanish. There is subtlety when  $G$  is not quasi-split, because not every test function  $\phi$  may be allowed in (A.0.4). The potential problem is that a base change transfer of  $\phi_u$  at  $u$  from  $G^*(\tilde{F}_u)$  to  $G^*(\tilde{F}_u^+)$  is not in the image of endoscopic transfer from  $G(\tilde{F}_u^+)$  to  $G^*(\tilde{F}_u^+)$ . We make a choice of test functions avoiding this problem.

At  $\infty$  one does the same as in Labesse's proof so that  $\mathrm{tr} \Pi_\infty(\phi_\infty) \neq 0$ . At finite places  $u$ , we recall that  $f_u$  and  $\phi_u$  are related as follows: writing  $f_u^*$  for a transfer of  $f_u$  from  $G(\tilde{F}_u^+)$  to  $G^*(\tilde{F}_u^+)$ , the functions  $f_u^*$  and  $\phi_u$  are associated in the sense of [Lab11, 4.5]. There is no problem when  $u \notin \Delta^+(G)$  as  $G$  and  $G^*$  are isomorphic outside  $\Delta^+(G)$ ; more precisely we choose  $\phi_u$  on  $G(\tilde{F} \otimes_{\tilde{F}^+} \tilde{F}_u^+)$  such that

$$\mathrm{tr} \Pi_u(\phi_u) \neq 0$$

and choose  $f_u$  to be a base change transfer to  $G(\tilde{F}_u^+)$  (which exists by [Lab11, Lem. 4.1], where it is called an ‘‘associated’’ function). At each  $v = ww^c \in \Delta^+(G)$ , choose  $f_v$  and let  $f_v^*$  be a transfer. Write  $\pi_v^* := \Pi_w$  via the chosen isomorphism  $G^*(\tilde{F}_w) \cong G^*(\tilde{F}_v^+)$ . Then by Proposition A.0.1,

$$\mathrm{tr} \pi_v^*(f_v^*) = e(G(\tilde{F}_v^+)) \cdot \mathrm{tr} (|\mathrm{LJ}|(\pi_v^*)) (f_v).$$

Note that  $|\mathrm{LJ}|(\pi_v^*)$  is irreducible (i.e. nonzero) since  $\pi_v^*$  is supercuspidal by assumption. If we choose  $f_v$  such that  $\mathrm{tr} (|\mathrm{LJ}|(\pi_v^*)) (f_v) \neq 0$  then the above identity tells us that  $\mathrm{tr} \pi_v^*(f_v^*) \neq 0$ . Choosing  $\phi_v$  to be a function associated with  $f_v^*$  (such a  $\phi_v$  exists by either [Lab11, Lem. 4.1]), we have as in (A.0.2),

$$\mathrm{tr} \Pi_v(\phi_v) = \mathrm{tr} \pi_v^*(f_v^*) \neq 0.$$

We have exhibited a choice of  $f$  and  $\phi$  above such that (A.0.4) is valid with the right hand side non-vanishing. Therefore there exists some  $\pi$  on the left hand side contributing with positive multiplicity. Let  $S$  be the set of places of  $\tilde{F}^+$  containing all infinite places and the finite places where  $G$  and  $\Pi$  are ramified. Write  $S_f$  for the subset of finite places in  $S$ . As we are free to choose  $\phi_u$  in the unramified Hecke algebra at each  $u \in S_f$ , we may assume that  $\pi^S$  is unramified with  $BC(\pi^S) = \Pi^S$ . The nonvanishing of  $\mathrm{tr} \pi_\infty(f_\infty)$  tells us that  $BC(\pi_\infty) = \Pi_\infty$ . Thus (A.0.4) is reduced to a formula of the form (A.0.3), with  $\pi$  contributing nontrivially to the sum. Arguing as in the proof of the preceding proposition, we deduce that  $|\mathrm{LJ}|(\pi_v^*) \cong \pi_v$ .  $\square$

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