Linear Algebraic Groups

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Contents

Introduction.

Algebraic group: a group that is also an algebraic variety such that the group operations are maps of varieties.

Example. $G = \mathrm{GL}_n(k)$, $k = \overline{k}$

Goal: to understand the structure of reductive/semisimple affine algebraic groups over algebraically closed fields k (not necessarily of characteristic 0). Roughly, they are classified by their Dynkin diagrams, which are associated graphs.

Within G are maximal, connected, solvable subgroups, called the Borel subgroups.

Example. In $G = \mathrm{GL}_n(k)$, a Borel subgroup B is given by the upper triangular matrices.

A fundamental fact is that the Borels are conjugate in G, and much of the structure of G is grounded in those of the B. (Thus, it is important to study solvable algebraic groups). B decomposes as

$$B = T \ltimes U$$

where $T \cong \mathbf{G}_m^n$ is a maximal torus and U is unipotent.

Example. With $G = \mathrm{GL}_n(k)$, we can take T consisting of all diagonal matrices with U the upper triangular matrices with 1's along the diagonal.

G acts on its Lie algebra $\mathfrak{g} = T_1 G$. This action restricts to a semisimple action of T on $\mathfrak{g}$. From the nontrivial eigenspaces, we get characters $T \to k^\times$ called the roots. The roots give a root system, which allows us to define the Dynkin diagrams.

Example. $G = \mathrm{GL}_n(k)$. $\mathfrak{g} = M_n(k)$ and the action of G on $\mathfrak{g}$ is by conjugation. The roots are given by

$$\mathrm{diag}(x_1, \ldots, x_n) \mapsto x_i x_j^{-1}$$

for $1 \leqslant i \neq j \leqslant n$.

Main References:
- Springer's *Linear Algebraic Groups*, second edition
- Polo's course notes at `www.math.jussieu.fr/~polo/M2`
- Borel's *Linear Algebraic Groups*

0. Algebraic geometry (review).

We suppose $k = \overline{k}$. Possible additional references for this section: Milne's notes on Algebraic Geometry, Mumford's Red Book.

0.1 Zariski topology on k^n.

If $I \subset k[x_1, \ldots, x_n]$ is an ideal, then $V(I) := \{x \in k^n \mid f(x) = 0 \ \forall f \in I\}$. Closed subsets are defined to be the $V(I)$. We have

$$\bigcap_\alpha V(I_\alpha) = V(\sum I_\alpha)$$
$$V(I) \cup V(J) = V(I \cap J)$$

Note: this topology is not T_2 (i.e., Hausdorff). For example, when $n = 1$ this is the finite complement topology.

0.2 Nullstellensatz.

Theorem 1 (Nullstellensatz)**.**

(i)
$$\{\textit{radical ideals } I \textit{ in } k[x_1, \ldots, x_n]\} \underset{I}{\overset{V}{\rightleftarrows}} \{\textit{closed subsets in } k^n\}$$

are inverse bijections, where $I(X) = \{f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \ \ \forall x \in X\}$

(ii) I, V *are inclusion-reversing*

(iii) *If* $I \leftrightarrow X$*, then* I *prime* $\iff$ X *irreducible.*

It follows that the maximal ideals of $k[x_1, \ldots, x_n]$ are of the form

$$\mathfrak{m}_a = I(\{a\}) = (x_1 - a_1, \ldots, x_n - a_n), \quad a \in k.$$

0.3 Some topology.

X is a topological space.

$$
\begin{aligned}
X \text{ is } \textbf{irreducible} \quad & \text{if } X = C_1 \cup C_2, \text{ for closed sets } C_1, C_2 \text{ implies that } C_i = X \text{ for some } i. \\
& \iff \text{any two non-empty open sets intersect} \\
& \iff \text{any non-empty open set is dense}
\end{aligned}
$$

Facts.
• X irreducible $\implies$ X connected.
• If $Y \subset X$, then Y irreducible $\iff$ $\overline{Y}$ irreducible.

X is **noetherian** if any chain of closed subsets $C_1 \supset C_2 \supset \cdots$ stabilises. If X is noetherian, any irreducible subset is contained in a maximal irreducible subset (which is automatically closed), an **irreducible component**. X is the union of its finitely many irreducible components:

$$X = X_1 \cup \cdots \cup X_n$$

Fact. The Zariski topology on k^n is noetherian and compact (a consequence of Nullstellensatz).

0.4 Functions on closed subsets of k^n

$X \subset k^n$ is a closed subset.

$$X = \{a \in k^n \mid \{a\} \subset X \iff \mathfrak{m}_a \supset I(X)\} \leftrightarrow \{ \text{ maximal ideals in } k[x_1, \ldots, x_n]/I(X)\}$$

Define the **coordinate ring** of X to be $k[X] := k[x_1, \ldots, x_n]/I(X)$. The coordinate ring is a reduced, finitely-generated k-algebra and can be regarded as the restriction of polynomial functions on k^n to X.

• X irreducible $\iff$ $k[X]$ integral domain
• The closed subsets of X are in bijection with the radical ideals of $k[X]$.

Definition 2. *For a non-empty open $U \subset X$, define*

$$\mathcal{O}_X(U) := \{f : U \to k \mid \forall x \in U, \ \exists x \in V \subset U, V \text{ open, and } \exists p, q \in k[x_1, \ldots, x_n] \\ \text{such that } f(y) = \frac{p(y)}{q(y)} \ \ \forall y \in V\}$$

$\mathcal{O}_X$ is a *sheaf of k-valued functions on X*:

- for all U, $\mathcal{O}_X(U)$ is a k-subalgebra of {set-theoretic functions $U \to k$}
- $U \subset V$, then $f \in \mathcal{O}_X(V) \implies f|_U \in \mathcal{O}_X(U)$;
- if $U = \bigcup U_\alpha$, $f : U \to k$ function, then $f|_{U_\alpha} \in \mathcal{O}_X(U_\alpha) \ \forall \alpha \implies f \in \mathcal{O}_X(U)$.

Facts.
• $\mathcal{O}_X(X) \cong k[X]$
• If $f \in \mathcal{O}_X(X)$, $D(f) := \{x \in X \mid f(x) \neq 0\}$ is open and these sets form a basis for the topology. $\mathcal{O}_X(D(f)) \cong k[X]_f$.

Definitions 3. *A* **ringed space** *is a pair* $(X, \mathcal{F}_X)$ *of a topological space* X *and a sheaf of* k*-valued functions on* X*. A morphism* $(X, \mathcal{F}_X) \to (Y, \mathcal{F}_Y)$ *of ringed spaces is a continuous map* $\phi : X \to Y$ *such that*

$$\forall V \subset Y \text{ open }, \forall f \in \mathcal{F}_Y(V), \quad f \circ \phi \in \mathcal{F}_X(\phi^{-1}(V))$$

An **affine variety** *(over* k*) is a pair* $(X, \mathcal{O}_X)$ *for a closed subset* $X \subset k^n$ *for some* n *(with* $\mathcal{O}_X$ *as above).* **Affine** n**-space** *is defined as* $\mathbf{A}^n := (k^n, \mathcal{O}_{k^n})$*.*

Theorem 4. $X \mapsto k[X]$*,* $\phi \mapsto \phi^*$ *gives an equivalence of categories*

$$\{ \text{ affine varieties over } k \ \}^{\text{op}} \to \{\text{reduced finitely-generated } k\text{-algebras}\}$$

If $\phi : X \to Y$ is a morphism of varieties, then $\phi^* : k[Y] \to k[X]$ here is $f + I(Y) \mapsto f \circ \phi + I(X)$. The inverse functor is given by mapping A to m-Spec(A), the spectrum of maximal ideals of A, along with the appropriate topology and sheaf.

0.5 Products.

Proposition 5. A, B *finitely-generated* k*-algebras. If* A, B *are reduced (resp. integral domains), then so is* $A \otimes_k B$*.*

From the above theorem and proposition, we get that if X, Y are affine varieties, then m-Spec$(k[X] \otimes_k k[Y])$ is a product of X and Y in the category of affine varieties.

Remark 6. $X \times Y$ *is the usual product as a set, but not as topological spaces (the topology is finer).*

Definition 7. *A* **prevariety** *is a ringed space* $(X, \mathcal{F}_X)$ *such that* $X = U_1 \cup \cdots \cup U_n$ *with the* U_i *open and the* $(U_i, \mathcal{F}|_{U_i})$ *isomorphic to affine varieties.* X *is compact and noetherian. (This is too general of a construction. Gluing two copies of* $\mathbf{A}^1$ *along* $\mathbf{A}^1 - \{0\}$ *(a pathological space) gives an example of a prevariety.)*

Products in the category of prevarieties exist: if $X = \bigcup_{i=1}^{n}$, $Y = \bigcup_{j=1}^{m} V_j$ (U_i, V_j affine open), then $X \times Y = \bigcup_{i,j}^{n,m} U_i \times V_j$, where each $U_i \times V_j$ is the product above. As before, this gives the usual products of sets but not topological spaces.

Definition 8. *A prevariety is a* **variety** *if the diagonal* $\Delta_X \subset X \times X$ *is a closed subset. (This is like being* T_2*!)*

- Affine varieties are varieties; X, Y varieties $\implies$ $X \times Y$ variety.
- If is Y a variety, then the graph of a morphism $X \to Y$ is closed in $X \times Y$.
- If Y is a variety, $f, g : X \to Y$, then $f = g$ if f, g agree on a dense subset.
- If X, Y are irreducible, then so is $X \times Y$.

0.6 Subvarieties.

Let X be a variety and $Y \subset X$ a **locally closed** subset (i.e., Y is the intersection of a closed and an open set, or, equivalently, Y is open in $\overline{Y}$). There is a unique sheaf $\mathcal{O}_Y$ on Y such that $(Y, \mathcal{O}_Y)$ is a prevariety and the inclusion $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism such that

$$\text{for all morphisms } f : Z \to X \text{ such that } f(Z) \subset Y, \quad f \text{ factors through the inclusion } Y \to X.$$

Concretely,

$$\mathcal{O}_Y(V) = \{f : V \to k \mid \forall x \in V,\ \exists U \subset X, x \in U \text{ open, and } \exists \tilde{f} \in \mathcal{O}_X(U) \text{ such that } f|_{U\cap V} = \tilde{f}|_{U\cap V}\}.$$

Remarks 9. *Y, X as above.*
- *If $Y \subset X$ is open, then $\mathcal{O}_Y = \mathcal{O}_X|_Y$.*
- *Y is a variety ($\Delta_Y = \Delta_X \cap (Y \times Y)$)*
- *If X is affine and Y is closed, then Y is affine with $k[Y] \cong k[X]/I(Y)$*
- *If X is affine and $Y = D(f)$ is basic open, then Y is affine with $k[Y] \cong k[X]_f$. (Note that general open subsets of affine varieties need not be affine (e.g., $\mathbf{A}^2 - \{0\} \subset \mathbf{A}^2$).)*

It's easy to see from the above definitions that if X, Y are varieties and $Z \subset X$, $W \subset Y$ are locally closed, then $Z \times W \subset X \times Y$ is locally closed and the subvariety structure on $Z \times W$ inside the product $X \times Y$ agrees with the product structure on the product of subvarieties Z, W.

Theorem 10. *Let $\phi : X \to Y$ be a morphism of affine varieties.*

(i) $\phi^* : k[Y] \to k[X]$ *surjective* $\iff$ *ϕ is a closed immersion (i.e., an isomorphism onto a closed subvariety)*

(ii) $\phi^* : k[Y] \to k[X]$ *is injective* $\iff$ $\overline{\phi(X)} = Y$ *(i.e., ϕ is* **dominant***)*

0.7 Projective varieties.

$\mathbf{P}^n = \frac{k^{n+1} - \{0\}}{k^\times}$ as a set. The *Zariski topology* on $\mathbf{P}^n$ is given by defining, for all homogeneous ideals I, $V(I)$ to be a closed set. For $U \subset \mathbf{P}^n$ open,

$$\mathcal{O}_{\mathbf{P}^n}(U) := \{f : U \to k \mid \forall x \in U \ \ \exists F, G \in k[x_0, \ldots, x_n], \text{ homogeneous of the same degree such that } f(y) = \frac{F(y)}{G(y)}, \text{ for all } y \text{ in a neighbourhood of } x.\}$$

Let $U_i = \{(x_0 : \cdots : x_n) \in \mathbf{P}^n \mid x_i \neq 0\} = \mathbf{P}^n - V((x_i))$, which is open. $\mathbf{A}^n \to U_i$ given by

$$x \mapsto (x_1 : \cdots : x_{i-1} : 1 : x_i : \cdots : x_n)$$

gives an isomorphism of ringed spaces, which implies that $\mathbf{P}^n$ is a prevariety; in fact, it is an irreducible variety.

Definitions 11. *A* **projective variety** *is a closed subvariety of $\mathbf{P}^n$. A* **quasi-projective variety** *is a locally closed subvariety of $\mathbf{P}^n$.*

Facts.
- The natural map $\mathbf{A}^{n+1} - \{0\} \to \mathbf{P}^n$ is a morphism
- $\mathcal{O}_{\mathbf{P}^n}(\mathbf{P}^n) = k$

0.8 Dimension.

X here is an irreducible variety. The **function field** of X is $k(X) := \varinjlim_{U \neq \varnothing \text{ open}} \mathcal{O}_X(U)$, the *germs of regular functions.*

Facts.
- For $U \subset X$ open, $k(U) = k(X)$.
- For $U \subset X$ irreducible affine, $k(U)$ is the fraction field of $k[U]$.
- $k(X)$ is a finitely-generated field extension of k.

Definition 12. *The* **dimension** *of X is* $\dim X := tr.deg_k\, k(X)$.

Theorem 13. *If X is affine, then* $\dim X =$ *Krull dimension of* $k[X]$ *(which is the maximum length of chains of $C_0 \subsetneq \cdots \subsetneq C_n$ of irreducible closed subsets).*

Facts.
- $\dim \mathbf{A}^n = n = \dim \mathbf{P}^n$
- If $Y \subsetneq X$ is closed and irreducible, then $\dim Y < \dim X$
- $\dim(X \times Y) = \dim X + \dim Y$

For general varieties X, define $\dim X := \max\{\dim Y \mid Y \text{ is an irreducible component}\}$.

0.9 Constructible sets.

A subset $A \subset X$ of a topological space is **constructible** if it is the union of finitely many locally closed subsets. Constructible sets are stable under finite unions and intersection, taking complements, and taking inverse images under continuous maps.

Theorem 14 (Chevalley)**.** *Let $\phi : X \to Y$ be a morphism of varieties.*

(i) $\phi(X)$ *contains a nonempty open subset of its closure.*

(ii) $\phi(X)$ *is constructible.*

0.10 Other examples.

- A finite dimensional k-vector space is an affine variety: fix a basis to get a bijection $V \xrightarrow{\sim} k^n$, giving V the corresponding structure (which is actually independent of the basis chosen). Intrinsically, we can define the topology and functions using polynomials in linear forms of V, that is, from $\mathrm{Sym}(V^*) = \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V^*)$: $k[V] := \mathrm{Sym}(V^*)$.

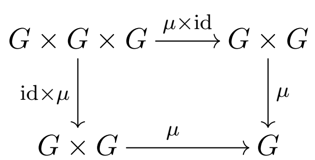
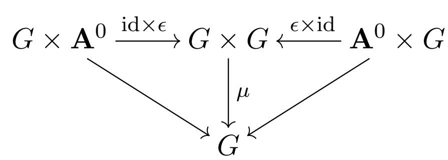
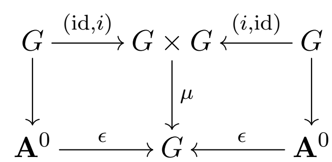
- Similarly, $\mathbf{P}V = \frac{V-\{0\}}{k^\times}$. As above, use a linear isomorphism $V \xrightarrow{\sim} k^{n+1}$ to get the structure of a projective space; or, intrinsically, use homogeneous elements of $\mathrm{Sym}(V^*)$.

1. Algebraic groups: beginnings.

1.1 Preliminaries.

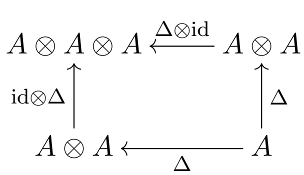
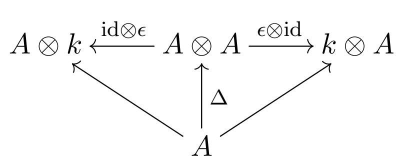
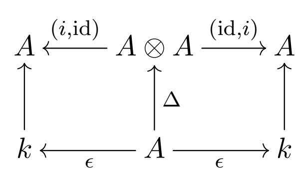
We will only consider the category of *affine* algebraic groups, a.k.a. **linear algebraic groups**. In future, by "algebraic group" we will mean "affine algebraic group". There are three descriptions of the category:

(1)
Objects: affine varieties G over k with morphisms $\mu : G \times G \to G$ (multiplication), $i : G \to G$ (inversion), and $\epsilon : \mathbf{A}^0 \to G$ (i.e., a distinguished point $e \in G$) such that the *group axioms* hold, i.e., that the following diagrams commute.

Maps: morphisms of varieties compatible with the above structure maps.

(2)
Objects: commutative Hopf k-algebras, which are reduced, commutative, finitely-generated k-algebras A with morphisms $\Delta : A \to A \otimes A$ (co-multiplication), $i : A \to A$ (co-inverse, also called antipode), and $\epsilon : A \to k$ (co-unit) such that the *co-group axioms* hold, i.e., that the following diagrams commute:

Maps: k-algebra morphisms compatible with the above structure maps.

(3)
Objects: functors

$$\Big(\text{reduced finitely-generated (commutative) } k\text{-algebras}\Big) \to \Big(\text{groups}\Big)$$

that are representable as set-valued functors;
Maps: natural transformations.

Here are the relationships:
(1) $\leftrightarrow$ (2) : $G \mapsto A = k[G]$ gives an equivalence of categories. Note that $k[G \times G] = k[G] \otimes k[G]$.
(2)$\leftrightarrow$(3) : $A \mapsto \mathrm{Hom}_{\mathrm{alg}}(A, -)$ gives an equivalence of categories by Yoneda's lemma.

Examples.
• $G = \mathbf{A}^1 =: \mathbf{G}_a$
In (1): $\mu : (x, y) \mapsto x + y$ (sum of projections), $i : x \mapsto -x$, $\epsilon : * \mapsto 0$
In (2): $A = k[T]$, $\Delta(T) = T \otimes 1 + 1 \otimes T$, $i(T) = -T$, $\epsilon(T) = 0$
In (3): the functor $\mathrm{Hom}_{\mathrm{alg}}(k[T], -)$ sends an algebra R to its additive group $(R, +)$.

• $G = \mathbf{A}^1 - \{0\} =: \mathbf{G}_m = \mathrm{GL}_1$
In (1): $\mu : (x, y) \mapsto xy$ (product of projections), $i : x \mapsto x^{-1}$, $\epsilon : * \mapsto 1$
In (2): $A = k[T, T^{-1}]$, $\Delta(T) = T \otimes T$, $i(T) = T^{-1}$, $\epsilon(T) = 1$
In (3): the functor $\mathrm{Hom}_{\mathrm{alg}}(k[T, T^{-1}], -)$ sends an algebra R to its group of units $(R, \times)$.

• $G = \mathrm{GL}_n$
In (1): $\mathrm{GL}_n(k) \subset M_n(k) \cong k^{n^2}$ with the usual operations is the basic open set given by $\det \neq 0$
In (2): $A = k[T_{ij}, \det(T_{ij})^{-1}]_{1 \leqslant i,j \leqslant n}$, $\Delta(T_{ij}) = \sum_k T_{ik} \otimes T_{kj}$
In (3): the functor $R \mapsto \mathrm{GL}_n(R)$

• $G = V$ finite-dimensional k-vector space
Given by the functor $R \mapsto (V \otimes_k R, +)$

• $G = \mathrm{GL}(V)$, for a finite-dimensional k-vector space V
Given by the functor $R \mapsto \mathrm{GL}(V \otimes_k R)$

Examples of morphisms.
• For $\lambda \in k^\times$, $x \mapsto \lambda x$ is an automorphism of $\mathbf{G}_a$

Exercise. Show that $\mathrm{Aut}(\mathbf{G}_a) \cong k^\times$. Note that $\mathrm{End}(\mathbf{G}_a)$ can be larger, as we have the Frobenius $x \mapsto x^p$ when $\mathrm{char}\, k = p > 0$.

• For $n \in \mathbf{Z}$, $x \mapsto x^n$ gives an automorphism of $\mathbf{G}_m$.
• $g \mapsto \det g$ gives a morphism $\mathrm{GL}_n \to \mathbf{G}_m$.

Note that if G, H are algebraic groups, then so is $G \times H$ (in the obvious way).

1.2 Subgroups.

A *locally closed subgroup* $H \leqslant G$ is a locally closed subvariety that is also a subgroup. H has a unique structure as an algebraic group such that the inclusion $H \to G$ is a morphism (it is given by restricting the multiplication and inversion maps of G).

Examples. Closed subgroups of GL_n:
- $G = \mathrm{SL}_n$, ($\det = 1$)
- $G = D_n$, diagonal matrices ($T_{ij} = 0 \ \ \forall\, i \neq j$)
- $G = B_n$, upper-triangular matrices ($T_{ij} = 0 \ \ \forall\, i > j$)
- $G = U_n$, unipotent matrices (upper-triangular with 1's along the diagonal)
- $G = \mathrm{O}_n$ or Sp_n, for a particular $J \in \mathrm{GL}_n$ with $J^t = \pm J$, these are the matrices g with $g^t J g = J$
- $G = \mathrm{SO}_n = \mathrm{O}_n \cap \mathrm{SL}_n$

Exercise. $D_n \cong \mathbf{G}_m^n$. Multiplication $(d, n) \mapsto dn$ gives an isomorphism $D_n \times U_n \to B_n$ as varieties. (Actually, B_n is a semidirect product of the two, with $U_n \trianglelefteq B_n$.)

Remark 15. *$\mathbf{G}_a$, $\mathbf{G}_m$, and GL_n are irreducible (latter is dense in $\mathbf{A}^{n^2}$). SL_n is irreducible, as it is defined by the irreducible polynomial $\det - 1$. In fact, $\mathrm{SO}_n, \mathrm{Sp}_n$ are also irreducible.*

Lemma 16.

(a) *If $H \leqslant G$ is an (abstract) subgroup, then $\overline{H}$ is a (closed) subgroup.*

(b) *If $H \leqslant G$ is a locally closed subgroup, then H is closed.*

(c) *If $\phi : G \to H$ is a morphism of algebraic groups, then $\ker\phi$, $\operatorname{im}\phi$ are closed subgroups.*

Proof.
(a). Multiplication by g is an isomorphism of varieties $G \to G$: $g\overline{H} = \overline{gH}$ and $\overline{H}g = \overline{Hg}$ $\implies$ $\overline{H} \cdot \overline{H} \subset \overline{H}$. Inversion is an isomorphism of varieties $G \to G$: $(\overline{H})^{-1} = \overline{H^{-1}} = \overline{H}$.

(b). $H \subset \overline{H}$ is open and $\overline{H} \subset G$ is closed, so without loss of generality suppose that $H \subset G$ is open. Since the complement of H is a union of cosets of H, which are open since H is, it follows that H is closed.

(c). $\ker\phi$ is clearly a closed subgroup. $\operatorname{im}\phi = \phi(G)$ contains a nonempty open subset $U \subset \overline{\phi(G)}$ by Chevalley; hence, $\phi(G) = \bigcup_{h \in \phi(G)} hU$ is open in $\overline{\phi(G)}$ and so $\phi(G)$ is closed by (b). $\square$

Lemma 17. *The connected component G^0 of the identity $e \in G$ is irreducible. The irreducible and connected components of G^0 coincide and they are the cosets of G^0. G^0 is an open normal subgroup (and thus has finite index).*

Proof. Let X be an irreducible component containing e (which must be closed). Then $X \cdot X^{-1} = \mu(X \times X^{-1})$ is irreducible and contains X; hence, $X = X \cdot X^{-1}$ is a subgroup as it is closed under inverse and multiplication. So $G = \coprod_{gX \in G/X} gX$ gives a decomposition of G into its irreducible components. Since G has a finite number of irreducible components, it follows that $(G : X) < \infty$ and X is open. Hence, the cosets gX are the connected components: $X = G^0$. Moreover, G^0 is normal since gG^0g^{-1} is another connected component containing e. $\square$

Corollary 18. *G connected $\iff$ G irreducible*

Exercise. $\phi : G \to H \implies \phi(G^0) = \phi(G)^0$

1.3 Commutators.

Proposition 19. *If H, K are closed, connected subgroups of G, then*

$$[H,K] = \langle [h,k] = hkh^{-1}k^{-1} \mid h \in H, k \in K \rangle$$

is closed and connected. (Actually, we just need one of H, K to be connected. Moreover, without any of the connected hypotheses, Borel shows that $[H,K]$ is closed.)

Lemma 20. *Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of irreducible varieties and $\{\phi_\alpha : X_\alpha \to G\}$ a collection of morphisms into G such that $e \in Y_\alpha := \phi_\alpha(X_\alpha)$ for all α. Then the subgroup H of G generated by the Y_α is connected and closed. Furthermore, $\exists\, \alpha_1, \ldots, \alpha_n \in I$, $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$ such that $H = Y_{\alpha_1}^{\epsilon_1} \cdots Y_{\alpha_n}^{\epsilon_n}$.*

Proof of Lemma. Without loss of generality suppose that $\phi_\alpha^{-1} = i \circ \phi_\alpha : X_\alpha \to G$ is also among the maps for all α. For $n \geqslant 1$ and $a \in I^n$, write $Y_a := Y_{\alpha_1} \cdots Y_{\alpha_n} \subset G$. Y_a is irreducible, and so $\overline{Y}_a$ is as well. Choose n, a such that $\dim \overline{Y}_a$ is maximal. Then for all $m, b \in I^m$,

$$\overline{Y}_a \subset \overline{Y}_a \cdot \overline{Y}_b \subset \overline{Y_a \cdot Y_b} = \overline{Y}_{(a,b)}$$

(second inclusion as in Lemma 16(a)) which by maximality implies that $\overline{Y}_a = \overline{Y_{(a,b)}}$ and $\overline{Y}_b \subset \overline{Y}_a$. In particular, this gives that

$$\overline{Y}_a \cdot \overline{Y}_a \subset \overline{Y_{(a,a)}} = \overline{Y}_a \quad \text{and} \quad \overline{Y}_a^{-1} \subset \overline{Y}_a$$

$\overline{Y}_a$ is a subgroup. By Chevalley, there is a nonempty $U \subset Y_a$ open in $\overline{Y}_a$.

Claim: $\overline{Y}_a = U \cdot U$ $\quad (\implies \overline{Y}_a = Y_a \cdot Y_a = Y_{(a,a)} \implies$ done.)

$$g \in \overline{Y}_a \implies gU^{-1} \cap U \neq \varnothing \implies g \in U \cdot U$$

□

Proof of Proposition. For $k \in K$, consider the morphisms $\phi_k : H \to G$, $h \mapsto [h,k]$. Note that $\phi_k(e) = e$. □

Corollary 21. *If $\{H_\alpha\}$ are connected closed subgroups, then so is the subgroup generated by them.*

Corollary 22. *If G is connected, then its derived subgroup $\mathfrak{D}G := [G,G]$ is closed and connected.*

Definitions 23. *Inductively define $\mathfrak{D}^n G := \mathfrak{D}(\mathfrak{D}^{n-1}G) = [\mathfrak{D}^{n-1}G, \mathfrak{D}^{n-1}G]$ with $\mathfrak{D}^0 G = G$.*

$$G \supset \mathfrak{D}G \supset \mathfrak{D}^2 G \supset \cdots$$

is the **derived series** *of G, with each group a normal subgroup in the previous (even in G). G is* **solvable** *if $\mathfrak{D}^n G = 1$ for some $n \geqslant 0$. Now, inductively define $\mathcal{C}^n G := [G, \mathcal{C}^{n-1}G]$ with $\mathcal{C}^0 G = G$.*

$$G \supset \mathcal{C}G \supset \mathcal{C}^2 G \supset \cdots$$

is the **descending central series** *of G, with each group normal in the previous (even in G). G is* **nilpotent** *if $\mathcal{C}^n G = 1$ for some $n \geqslant 0$.*

Recall the following facts of group theory:
• nilpotent $\implies$ solvable
• G solvable (resp. nilpotent) $\implies$ subgroups, quotients of G are solvable (resp. nilpotent)
• If $N \trianglelefteq G$, then N and G/N solvable $\implies$ G solvable.

Examples.
• B_n is solvable. ($\mathfrak{D}B_n = U_n$)
• U_n is nilpotent.

1.4 G-spaces.

A **G-space** is a variety X with an action of G on X (as a set) such that $G \times X \to X$ is a morphism of varieties. For each $x \in X$ we have a morphism $f_x : G \to X$ be given by $g \mapsto gx$, and for each $g \in G$ we have an isomorphism $t_g : X \to X$ given by $x \mapsto gx$. $\text{Stab}_G(x) = f_x^{-1}(\{x\})$ is a closed subgroup.

Examples.
• G acts on itself by $g * x = gx$ or xg^{-1} or gxg^{-1}. (Note that in the case of the last action, $\text{Stab}(x) = \mathcal{Z}_G(x)$ is closed and so the center $\mathcal{Z}_G = \bigcap_{x \in G} \mathcal{Z}_G(x)$ is closed.)
• $\text{GL}(V) \times V \to V$, $(g, x) \mapsto g(x)$
• $\text{GL}(V) \times \mathbf{P}V \to \mathbf{P}V$ (exercise)

Proposition 24.

(a) *Orbits are locally closed (so each orbit is a subvariety and is itself a G-space).*

(b) *There exists a closed orbit.*

Proof.
(a). Let Gx be an orbit, which is the image of f_x. By Chevalley, there is an nonempty $U \subset Gx$ open in $\overline{Gx}$. Then $Gx = \bigcup_{g \in G} gU$ is open in $\overline{Gx}$.

(b). Since X is noetherian, we can choose an orbit Gx such that $\overline{Gx}$ is minimal (with respect to inclusion). We will show that Gx is closed. Suppose otherwise. Then $\overline{Gx} - Gx$ is nonempty, closed in $\overline{Gx}$ by (a), and G-stable (by the usual argument); let y be an element in the difference. But then $\overline{Gy} \subsetneq \overline{Gx}$. Contradiction. Hence, Gx is closed. □

Lemma 25. *If G is irreducible, then G preserves all irreducible components of X.*

Exercise.

Suppose $\theta : G \times X \to X$ gives an affine G-space. Then G acts linearly on $k[X]$ by

$$(g \cdot f)(x) := f(g^{-1}x), \quad \text{i.e.,} \quad g \cdot f = t_{g^{-1}}^*(f)$$

Definitions 26. *Suppose a group G acts linearly on a vector space W. Say the action is* **locally finite** *if W is the union of finite-dimensional G-stable subspaces. If G is an algebraic group, say the action is* **locally algebraic** *if it is locally finite and, for any finite-dimensional G-stable subspace V, the action $\theta : G \times V \to V$ is a morphism.*

Proposition 27. *The action of G on $k[X]$ is locally algebraic. Moreover, for all finite-dimensional G-stable $V \subset k[X]$, then $\theta^*(V) \subset k[G] \otimes V$.*

Proof. $t_{g^{-1}}$ factors as

$$\begin{aligned} t_{g^{-1}} : \ & X \to G \times X \xrightarrow{\theta} X \\ & x \mapsto (g^{-1}, x) \\ t^*_{g^{-1}} : \ & k[X] \xrightarrow{\theta^*} k[G] \otimes k[X] \xrightarrow{(\mathrm{ev}_{g^{-1}}, \mathrm{id})} k[X] \end{aligned}$$

Fix $f \in k[X]$ and write $\theta^*(f) = \sum_{i=1}^n h_i \otimes f_i$, so

$$g \cdot f = t^*_{g^{-1}}(f) = \sum_{i=1}^n h_i(g^{-1}) f_i$$

Hence, the G-orbit of f is contained in $\sum_{i=1}^n k f_i$, implying local finiteness.

Let $V \subset k[X]$ be finite-dimensional and G-stable, and pick basis $(e_i)_{i=1}^n$. Extend the e_i to a basis $\{e_i\}_i \cup \{e'_\alpha\}_\alpha$ of $k[X]$. Write

$$\begin{aligned} \theta^* e_i &= \sum_j h_{ij} \otimes e_j + \sum_\alpha h'_{i\alpha} \otimes e'_\alpha \\ \implies \ & g \cdot e_i = \sum_j h_{ij}(g^{-1}) e_j + \sum_\alpha h'_{i\alpha}(g^{-1}) e'_\alpha \in V \\ \implies \ & h'_{i\alpha}(g^{-1}) = 0 \ \ \forall\, g, i, \alpha \\ \implies \ & h'_{i\alpha} = 0 \ \ \forall\, i, \alpha \end{aligned}$$

Hence, $\theta^*(V) \subset k[G] \otimes V$. Moreover, we see that $G \times V \to V$ is a morphism, as it is given by

$$(g, \sum_i \lambda_i e_i) \mapsto \sum_{i,j} \lambda_j h_{ij}(g^{-1}) e_j$$

It follows that the action of G on $k[X]$ is locally algebraic. ☐

Theorem 28 (Analogue of Cayley's Theorem)**.** *Any algebraic group is isomorphic to a closed subgroup of some* GL_n.

Proof. G acts on itself by right translation, so $(g \cdot f)(\gamma) = f(\gamma g)$. By Proposition 27 we know that this gives a locally algebraic action on $k[G]$. Let $f_1, \ldots, f_n$ be generators of $k[G]$. Without loss of generality, the f_i are linearly independent and $V = \sum_{i=1}^n k f_i$ is G-stable. Write

$$g \cdot f_i = \sum_j h_{ji}(g^{-1}) f_j = \sum_j h'_{ji}(g) f_j$$

where $h_{ji} \in k[G]$ and $h'_{ji} : g \mapsto h_{ji}(g^{-1})$. It follows that $\phi : G \to \mathrm{GL}(V)$ given by $g \mapsto (h'_{ij}(g))$ is a morphism of algebraic groups. It remains to show that ϕ is a closed immersion.

We have $h'_{ij} \in \operatorname{im} \phi^*$ for all i, j, as they are the image of projections. Moreover,

$$f_i(g) = (g \cdot f_i)(e) = \sum_j h'_{ji}(g) f_j(e) \implies f_i \in \sum_j k h'_{ji} \subset \operatorname{im} \phi^*$$

Since the f_i generate $k[G]$, it follows that ϕ^* is surjective; that is, ϕ is a closed immersion. □

1.5 Jordan Decomposition.

Let V be a finite-dimensional k-vector space. $\alpha \in \mathrm{GL}(V)$ is **semisimple** if it is diagonalisable, and is **unipotent** if 1 is its only eigenvalue. If α, β commute then

$$\alpha \text{ and } \beta \text{ semisimple (resp. unipotent)} \implies \alpha\beta \text{ semisimple (resp. unipotent)}$$

Proposition 29. $\alpha \in \mathrm{GL}(V)$

(i) $\exists!\, \alpha_s$ (semisimple), α_u (unipotent) $\in \mathrm{GL}(V)$ *such that* $\alpha = \alpha_s \alpha_u = \alpha_u \alpha_s$.

(ii) $\exists\, p_s(x), p_u(x) \in k[X]$ *such that* $\alpha_s = p_s(\alpha)$, $\alpha_u = p_u(\alpha)$.

(iii) *If* $W \subset V$ *is an* α*-stable subspace, then*

$$(\alpha|_W)_s = \alpha_s|_W, \quad (\alpha|_{V/W})_s = \alpha_s|_{V/W}$$
$$(\alpha|_W)_u = \alpha_u|_W, \quad (\alpha|_{V/W})_u = \alpha_u|_{V/W}$$

(iv) *If* $f : V_1 \to V_2$ *linear with* $\alpha_i \in \mathrm{GL}(V_i)$ *for* $i = 1, 2$, *then*

$$f \circ \alpha_1 = \alpha_2 \circ f \implies \begin{cases} f \circ (\alpha_1)_s = (\alpha_2)_s \circ f \\ f \circ (\alpha_1)_u = (\alpha_2)_u \circ f \end{cases}$$

(v) *If* $\alpha_i \in \mathrm{GL}(V_i)$ *for* $i = 1, 2$, *then*

$$(\alpha_1 \otimes \alpha_2)_s = (\alpha_1)_s \otimes (\alpha_2)_s$$
$$(\alpha_1 \otimes \alpha_2)_u = (\alpha_1)_u \otimes (\alpha_2)_u$$

Proof sketch.
(i) – existence:
A Jordan block for an eigenvalue λ decomposes as

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \lambda \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} & & \\ & \ddots & \ddots & \\ & & \ddots & \lambda^{-1} \\ & & & 1 \end{pmatrix}$$

The left factor is semisimple and the right is unipotent, and so they both commute.
(i) – uniqueness:
If $\alpha = \alpha_s\alpha_u = \alpha'_s\alpha'_u$, then $\alpha_s^{-1}\alpha'_s = \alpha_u^{-1}\alpha'_s$ is both unipotent and semisimple, and thus is the identity.

(ii): This follows from the Chinese Remainder Theorem.

(iii): Use (ii) + uniqueness.

(iv): Since $f : V_1 \twoheadrightarrow \operatorname{im} f \hookrightarrow V_2$, it suffices to consider the cases where f is injective or surjective, in which we can invoke (iii).

(v): Exercise. □

Definition 30. *An* **(algebraic) G-representation** *is a linear G-action on a finite-dimension k-vector space such that $G \times V \to V$ is a morphism of varieties, which is equivalent to $G \to \mathrm{GL}(V)$ being a morphism of algebraic groups. Note that if $G \to \mathrm{GL}(V)$ is given by $g \mapsto (h_{ij}(g))$, then $G \times V \to V$ is given by $(g, \sum_i \lambda_i e_i) \mapsto \sum_{i,j} \lambda_i h_{ji}(g) e_j$.*

Lemma 31. *Suppose $\rho : G \to \mathrm{GL}(V)$ is an algebraic representation. Then there is a unique G-linear map $\eta : V \to V \otimes k[G]$ such that $(1 \otimes \mathrm{ev}_g) \circ \eta = \rho(g)$ for all $g \in G$. Moreover, η is injective and $\eta \circ h = (1 \otimes h) \circ \eta$ for all $h \in G$, i.e. as map of G-representations $\eta : V \hookrightarrow V_0 \otimes k[G]$, where V_0 is V with the trivial G-action and G acts on $k[G]$ by right translation.*

Proof. Suppose $\eta(e_i) = \sum_j e_j \otimes f_{ji}$ for some $f_{ij} \in k[G]$. Then $(1 \otimes \mathrm{ev}_g) \circ \eta = \rho(g)$ for all g implies that $f_{ij} = h_{ij}$ in the notation above, so η is unique, and conversely it shows that η exists. Moreover, η is injective since $\rho(g)$ is injective.
To see that $\eta \circ h = (1 \otimes h) \circ \eta$ holds, it suffices to check it after evaluating it at any $v \in V$ and then applying $1 \otimes \mathrm{ev}_g$ on both sides. We get equality, since $\rho(g)\rho(h)(v) = \rho(gh)(v)$. □

Proposition 32. *Suppose that for all algebraic G-representations V, there is a $\alpha_V \in \mathrm{GL}(V)$ such that*

(i) *$\alpha_{k_0} = \mathrm{id}_{k_0}$, where k_0 is the one-dimensional trivial representation.*

(ii) *$\alpha_{V \otimes W} = \alpha_V \otimes \alpha_W$*

(iii) *If $f : V \to W$ is a map of G-representations, then $\alpha_W \circ f = f \circ \alpha_V$.*

Then $\exists!\, g \in G$ such that $\alpha_V = g_V$ for all V.

Proof. From (iii), if $W \hookrightarrow V$ is a G-stable subspace, then $\alpha_V|_W = \alpha_W$. If V is a local algebraic G-representation, then $\exists!\, \alpha_V$ such that $\alpha_V|_W = \alpha_W$ for all finite-dimensional G-stable $W \subset V$. Note that (ii), (iii) still hold for locally algebraic representations. Also note that from (iii) it follows that $\alpha_{V \oplus W} = \alpha_V \oplus \alpha_W$. Define $\alpha = \alpha_{k[G]} \in \mathrm{GL}(k[G])$, where G acts on $k[G]$ by $(gf)(\lambda) = f(\lambda g)$.

Claim. *α is a ring automorphism.*
$m : k[G] \otimes k[G] \to k[G]$ is a map of locally algebraic G-representations: $f_1(\,\cdot\, g) f_2(\,\cdot\, g) = (f_1 f_2)(\cdot g)$. Thus, by (ii) and (iii), $\alpha \circ m = m \circ (\alpha \otimes \alpha)$, and so $\alpha(f_1 f_2) = \alpha(f_1)\alpha(f_2)$.

Therefore, the composition $k[G] \xrightarrow{\alpha} k[G] \xrightarrow{\text{ev}_e} k$ is a ring homomorphism and is equal to ev_g for some unique g.

Claim. $\alpha(f) = gf \;\; \forall f$, *i.e.*, $\alpha = g_{k[G]}$.
By above $\alpha(f)(e) = f(g)$. Also, if $\ell(\lambda)(f) := f(\lambda^{-1}\cdot)$, then $\ell(\lambda) : k[G] \to k[G]$ is G-linear by (iii):

$$\alpha \circ \ell(\lambda) = \ell(\lambda) \circ \alpha \implies \alpha(f)(\lambda^{-1}) = f(\lambda^{-1}g) \implies \alpha(f) = gf$$

Now if V is a G-rep, $\eta : V \hookrightarrow V_0 \otimes k[G]$ is G-linear, by Lemma 31, and so

$$\alpha_{V_0\otimes k[G]} \circ \eta = \eta \circ \alpha_V$$

Since

$$\alpha_{V_0\otimes k[G]} = \alpha_{V_0} \otimes \alpha_{k[G]} = \text{id}_{V_0} \otimes g_{k[G]} = g_{V_0\otimes k[G]}$$

and

$$g_{V_0\otimes k[G]} \circ \eta = \eta \circ g_V$$

and the fact that η is injective, it follows that $\alpha_V = g_V$. (g is unique, as $G \to \text{GL}(k[G])$ is injective. Exercise!) □

Theorem 33. *Let G be an algebraic group.*

(i) $\forall\, g \in G \;\; \exists!\, g_s, g_u \in G$ *such that for all representations* $\rho : G \to \text{GL}(V)$

$$\rho(g_s) = \rho(g)_s \quad \textit{and} \quad \rho(g_u) = \rho(g)_u$$

and $g = g_s g_u = g_u g_s$.

(ii) *For all* $\phi : G \to H$

$$\phi(g_s) = \phi(g)_s \quad \textit{and} \quad \phi(g_u) = \phi(g)_u$$

Proof.
(i). Fix $g \in G$. For all G-representations V, let $\alpha_V := (g_V)_s$. If $f : V \to W$ is G-linear, then $f \circ g_V = g_W \circ f$ implies that $f \circ \alpha_V = \alpha_W \circ f$ by Proposition 29. Also, $\alpha_{k_0} = \text{id}_s = \text{id}$, and

$$\alpha_{V\otimes W} = (g_{V\otimes W})_s = (g_V \otimes g_W)_s = \alpha_V \otimes \alpha_W$$

(the last equality following from Proposition 29). By Proposition 32, there is a unique $g_s \in G$ such that $\alpha_V = (g_s)_V$ for all V, i.e., $\rho(g_s) = \rho(g)_s$. Similarly for g_u. From a closed immersion $G \hookrightarrow \text{GL}(V)$, from Theorem 28, we see that $g = g_s g_u = g_u g_s$.

(ii). Given $\phi : G \to H$, let $\rho : H \to \text{GL}(V)$ be a closed immersion. Then

$$\rho(\phi(g_*)) = \rho(\phi(g))_* = \rho(\phi(g)_*)$$

where the first equality is by (i) for G (as $\phi \circ \rho$ makes V into a G-representation) and the second equality is by (i) for H. □

Exercise. What is the Jordan decomposition in $\mathbf{G}_a$? How about in a finite group?

Remark 34. F : (G-representations) $\to$ (k-vector spaces) *denotes the forgetful functor, then Proposition 32 says that*

$$G \cong \mathrm{Aut}^{\otimes}(F)$$

where the left side is the group of natural isomorphisms $F \to F$ *respecting* $\otimes$*.*

2. Diagonalisable and elementary unipotent groups.

2.1 Unipotent and semisimple subsets.

Definitions 35.

$$G_s := \{g \in G \mid g = g_s\}$$
$$G_u := \{g \in G \mid g = g_u\}$$

Note that $G_s \cap G_u = \{e\}$ and G_u is a closed subset of G (embedding G into a GL_n, G_u is the closed subset consisting of g such that $(g-I)^n = 0$. G_s, however, need not be closed (as in the case $G = B_2$)).

Corollary 36. *If $gh = hg$ and $g, h \in G_*$, then $gh, g^{-1} \in G_*$, where $* = s, u$.*

Proposition 37. *If G is commutative, then G_s, G_u are closed subgroups and $\mu : G_s \times G_u \to G$ is an isomorphism of algebraic groups.*

Remark 38. *This will be generalised to connected nilpotent groups in Proposition 131.*

Proof. G_s, G_u are subgroups by Corollary 36 and G_u is closed by a remark above. Without loss of generality, $G \subset \mathrm{GL}(V)$ is a closed subgroup for some V. As G is commutative, $V = \bigoplus_{\lambda : G_s \to k^\times} V_\lambda$ (a direct sum of eigenspaces for G_s) and G preserves each V_λ. Hence, we can choose a basis for each V_λ such that the G-action is upper-triangular (commuting matrices are simultaneously upper-triangular-isable), and so $G \subset B_n$ and $G_s = G \cap D_n$. Then $G \hookrightarrow B_n$ followed by projecting to the diagonal D_n gives a morphism $G \to G_s$, $g \mapsto g_s$; hence, $g \mapsto (g_s, g_s^{-1}g)$ gives a morphism $G \to G_s \times G_u$, one inverse to μ. □

Definition 39. *G is* **unipotent** *if $G = G_u$.*

Example. U_n is unipotent, and so is $\mathbf{G}_a$ (as $\mathbf{G}_a \cong U_2$).

Proposition 40. *If G is unipotent and $\phi : G \to \mathrm{GL}_n$, then there is a $\gamma \in \mathrm{GL}_n$ such that $\mathrm{im}\,(\gamma\phi\gamma^{-1}) \subset U_n$.*

Proof. We prove this by induction on n. Suppose that this true for $m < n$, let V be an n-dimensional vector space, and $\phi : G \to \mathrm{GL}(V)$. Suppose that there is a G-invariant subspace $0 \subsetneq W_1 \subsetneq V$. Let W_2 is complementary to W_1, so that $V = W_1 \oplus W_2$, and let $\phi_i : G \to \mathrm{GL}(V_i)$ be the induced morphisms for $i = 1, 2$, so that $\phi = \phi_1 \oplus \phi_2$. Since $n > \dim W_1, \dim W_2$, there are $\gamma_1, \gamma_2 \in \mathrm{GL}(V)$

such that $\mathrm{im}\,(\gamma_i\phi_i\gamma_i^{-1})$ consists of unipotent elements for $i = 1, 2$. If $\gamma = \gamma_1 \oplus \gamma_2$, then it follows that $\mathrm{im}\,(\gamma\phi\gamma^{-1})$ consists of unipotent elements as well.

Now, suppose that there does not exists such a W_1, so that V is irreducible. For $g \in G$

$$\begin{aligned}
\mathrm{tr}\,(\phi(g)) = n \implies & \forall\, h \in G \;\; \mathrm{tr}\,((\phi(g) - 1)\phi(h)) = \mathrm{tr}\,(\phi(gh)) - \mathrm{tr}\,(\phi(h)) = n - n = 0 \\
\implies & \forall\, x \in \mathrm{End}(V) \;\; \mathrm{tr}\,((\phi(g) - 1)x) = 0, \text{ by Burnside's theorem} \\
\implies & \phi(g) - 1 = 0 \\
\implies & \phi(g) = 1 \\
\implies & \mathrm{im}\,\phi = 1
\end{aligned}$$

(Recall that Burnside's Theorem says that G spans $\mathrm{End}(V)$ as a vector space.) ☐

Remark 41. *Here's a sketch proof of Burnside's theorem, which works for any* abstract *subgroup G of* $\mathrm{GL}(V)$ *even: let A be the k-span of G insider* $\mathrm{End}(V)$*. This is a k-subalgebra of* $\mathrm{End}(V)$ *acting irreducibly on V.*
We'll prove more generally that any (possibly non-commutative) k-algebra A with $\dim_k A < n^2$ *cannot have an irreducible module of k-dimension n. By replacing A by* $A/\mathrm{rad}(A)$*, where* $\mathrm{rad}(A)$ *is the Jacobson radical of A, we may assume WLOG that A is semisimple. Then* $A \cong \prod_{i=1}^{r} M_{n_i}(k)$ *by the Artin-Wedderburn theorem (since k is algebraically closed!). Now the irreducible modules of this ring are precisely the modules k^{n_i} with A acting naturally via the i-th projection. Hence any irreducible module has dimension* $n_i \leq \sqrt{\dim_k A} < n$.

Corollary 42. *Any irreducible representation of a unipotent group is trivial.*

Corollary 43. *Any unipotent G is nilpotent.*

Proof. U_n is nilpotent. ☐

Remark 44. *The converse is not true; any torus is nilpotent (the definition of a torus to come immediately). More generally we will see that any connected nilpotent group is a product of a torus and a connected unipotent group.*

2.2 Diagonalisable groups and tori.

Definitions 45. *G is* **diagonalisable** *if G is isomorphic to a closed subgroup of* $D_n \cong \mathbf{G}_m^n$ $(n \geqslant 0)$*. G is a* **torus** *if* $G \cong D_n$ $(n \geqslant 0)$*. The* **character group** *of G is*

$$X^*(G) := \mathrm{Hom}(G, \mathbf{G}_m) \quad \text{(morphisms of algebraic groups)}$$

It is an abelian group under multiplication ($(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$) *and is a subgroup of* $k[G]^\times$.

Recall the following result:

Proposition 46 (Dedekind)**.** *Suppose* $X^*(G)$ *is a linearly independent subset of* $k[G]$.

The proof shows in fact that characters are linearly independent for *any* (abstract) group.

Proof. Suppose that $\sum_{i=1}^n \lambda_i \chi_i = 0$ in $k[G]$, $\lambda_i \in k$. Without loss of generality, $n \geqslant 2$ is minimal among all possible nontrivial linear combinations (so that $\lambda_i \neq 0 \ \ \forall i$). Then

$$\forall\, g, h, \quad \begin{cases} 0 = \sum \lambda_i \chi_i(g) \chi_i(h) \\ 0 = \sum \lambda_i \chi_i(g) \chi_n(h) \end{cases}$$
$$\implies \forall\, h, \quad 0 = \sum_{i=1}^{n-1} \lambda_i [\chi_i(h) - \chi_n(h)] \chi_i$$

By the minimality of n, we must have that the coefficients are are all 0; that is, $\forall\, i, h \ \ \chi_i(h) = \chi_n(h) \implies \chi_i = \chi_n$. We still arrive at a contradiction. □

Proposition 47. *The following are equivalent:*

(i) *G is diagonalisable.*

(ii) *$X^*(G)$ is a basis of $k[G]$ and $X^*(G)$ is finitely-generated.*

(iii) *G is commutative and $G = G_s$.*

(iv) *Any G-representation is a direct sum of 1-dimensional representations.*

Proof.
(i) ⇒ (ii): Fix an embedding $G \hookrightarrow D_n$. $k[D_n] = k[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]$ – as seen from restricting $T_{ij}, \det(T_{ij})^{-1} \in k[\mathrm{GL}_n]$ – has a basis of monomials $T_1^{a_1} \cdots T_n^{a_n}$, $a_i \in \mathbf{Z}$, each of which is in $X^*(G)$:

$$\mathrm{diag}(x_1, \ldots, x_n) \mapsto x_1^{a_1} \cdots x_n^{a_n}$$

Hence, $X^*(D_n) \cong \mathbf{Z}^n$ (by Proposition 46). The closed immersion $G \to D_n$ gives a surjection $k[D^n] \to k[G]$, inducing a map $X^*(D_n) \to X^*(G)$, $\chi \mapsto \chi|_G$. $\mathrm{im}\,(X^*(D_n) \to X^*(G))$ spans $k[G]$ and is contained in $X^*(G)$, which is linearly independent. Hence, $X^*(G)$ is a basis of $k[G]$ and we have the surjection

$$\mathbf{Z}^n \cong X^*(D_n) \twoheadrightarrow X^*(G)$$

implying the finite-generation.

(ii) ⇒ (iii): Say $\chi_1, \ldots, \chi_n$ are generators of $X^*(G)$. Define a morphism $\phi : G \to \mathrm{GL}_n$ by $g \mapsto \mathrm{diag}(\chi_1(g), \ldots, \chi_n(g))$.

$$\begin{aligned} g \in \ker \phi &\implies \chi_i(g) = 1 \ \forall\, i \\ &\implies \chi(g) = 1 \ \forall\, \chi \in X^*(G) \\ &\implies f(g) = 0 \ \forall f \in M_e = \{g = \sum_\chi \lambda_\chi \chi \in k[X] \mid 0 = g(e) = \sum_\chi \lambda_\chi\} \\ &\implies M_e \subset M_g \\ &\implies M_e = M_g \\ &\implies g = e \end{aligned}$$

So ϕ is injective, which implies that G is commutative and $G = G_s$.

(iii) $\Rightarrow$ (iv): Let $\phi : G \to \mathrm{GL}_n$ be a representation. $\mathrm{im}\,\phi$ is a commuting set of diagonalisable elements, which means we can simultaneously diagonalise them.

(iv) $\Rightarrow$ (i): Pick $\phi : G \hookrightarrow \mathrm{GL}_n$ (Theorem 28). By (iii), without loss of generality, suppose that $\mathrm{im}\,\phi \subset D_n$. Hence, $\phi : G \hookrightarrow D_n$. □

Corollary 48. *Subgroups and images under morphisms of diagonalisable groups are diagonalisable.*

Proof. (iii). □

Observations:
- $\mathrm{char}\, k = p \implies X^*(G)$ has no p-torsion.
- $k[G] \cong k[X^*(G)]$ as algebras ($k[X^*(G)]$ being a group algebra).
- For $\chi \in X^*(G)$,

$$\Delta(\chi) = \chi \otimes \chi, \quad i(\chi) = \chi^{-1}, \quad \epsilon(\chi) = 1$$

Indeed,

$$\begin{aligned} \Delta(\chi)(g_1, g_2) &= \chi(g_1 g_2) = \chi(g_1)\chi(g_2) = (\chi \otimes \chi)(g_1, g_2) \\ i(\chi)(g) &= \chi(g^{-1}) = \chi(g)^{-1} = \chi^{-1}(g) \\ \epsilon(\chi) &= \chi(e) = 1 \end{aligned}$$

Theorem 49. *Let $p = \mathrm{char}\, k$.*

$$\Big(\text{diagonalisable algebraic groups}\Big) \xrightarrow{X^*} \Big(\text{finitely-generated abelian groups (with no } p\text{-torsion if } p > 0)\Big)$$

$$\begin{array}{ccc} G & \longmapsto & X^*(G) \\ \big\downarrow & & \big\uparrow \\ H & \longmapsto & X^*(H) \end{array}$$

is a (contravariant) equivalence of categories.

Proof. It is well-defined by the above. We will define an inverse functor F. Given $X \cong \mathbf{Z}^{\oplus} \bigoplus_{i=1}^{s} \mathbf{Z}/n_i\mathbf{Z}$ from the category on the right, we have that its group algebra $k[X]$ is finitely-generated and reduced:

$$k[X] \cong k[\mathbf{Z}]^{\otimes r} \otimes \bigotimes_{i=1}^{s} k[\mathbf{Z}/n_i\mathbf{Z}] \cong k[T^{\pm 1}]^{\otimes r} \otimes \bigotimes_{i=1}^{s} k[T]/(T^{n_i} - 1)$$

Moreover, $k[X]$ is a Hopf algebra, which is easily checked, defining

$$\Delta : e_x \mapsto e_x \otimes e_x, \quad i : e_x \mapsto e_{x^{-1}} = e_x^{-1}, \quad \epsilon : e_x \mapsto 1$$

where X has been written multiplicatively and $k[X] = \bigoplus_{x \in X} k e_x$. Define F by $F(X) = \text{m-Spec}(k[X])$. Above, we saw that $FX^*(G)) \cong G$ as algebraic groups.

$$\begin{aligned} X^*(F(X)) &= \mathrm{Hom}(F(X), \mathbf{G}_m) \\ &= \mathrm{Hom}_{\text{Hopf-alg}}(k[T,T^{-1}], k[X]) \\ &= \{\lambda \in k[X]^\times (\text{corresponding to the images of } T) \mid \Delta(\lambda) = \lambda \otimes \lambda\} \end{aligned}$$

For an element above, write $\lambda = \sum_{x \in X} \lambda_x e_x$ (almost all of the $\lambda_x \in k$ of course being zero). Then

$$\Delta(\lambda) = \sum_x \lambda_x (e_x \otimes e_x) \quad \text{and} \quad \lambda \otimes \lambda = \sum_{x,x'} \lambda_x \lambda_{x'} (e_x \otimes e'_x)$$

Hence,

$$\lambda_x \lambda_{x'} = \begin{cases} \lambda_x, & x = x' \\ 0, & x \neq x' \end{cases}$$

So, $\lambda_x \neq 0$ for an *unique* $x \in X$, and

$$\lambda_x^2 = \lambda \implies \lambda_x = 1 \implies \lambda = e_x \in X$$

Thus we have $X^*(F(X)) \cong X$ as abelian groups. The two functors are inverse on maps as well, as is easily checked. □

Corollary 50.

(i) *The diagonalisable groups are the groups $\mathbf{G}_m^r \times H$, where H is a finite group of order prime to p.*

(ii) *For a diagonalisable group G,*

$$G \text{ is a torus} \iff G \text{ is connected} \iff X^*(G) \text{ is free abelian}$$

Proof. Define $\mu_n := \ker(\mathbf{G}_m \xrightarrow{n} \mathbf{G}_m)$, which is diagonalisable. If $(n,p) = 1$, then $k[\mu_n] = k[T]/(T^n - 1)$ ($T^n - 1$ is separable) and $X^*(\mu_n) \cong \mathbf{Z}/n\mathbf{Z}$. Since $X^*(\mathbf{G}_m) \cong \mathbf{Z}$ and $X^*(G \times H) \cong X^*(G) \oplus X^*(H)$, the result follows from Theorem 49. □

Corollary 51. $\mathrm{Aut}(D_n) \cong \mathrm{GL}_n(\mathbf{Z})$

Fact/Exercise. If G is diagonalisable, then

$$G \times X^*(G) \to \mathbf{G}_m, \quad (g, \chi) \mapsto \chi(g)$$

is a *"perfect bilinear pairing"*, i.e., it induces isomorphisms $X^*(G) \xrightarrow{\sim} \mathrm{Hom}(G, \mathbf{G}_m)$ and $G \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}}(X^*(G), \mathbf{G}_m)$ (as abelian groups). Moreover, it induces inverse bijections

$$\begin{aligned} \{\text{ closed subgroups of } G\} &\longleftrightarrow \{\text{ subgroups } Y \text{ of } X^*(G) \text{ such that } X^*(G)/Y \text{ has no } p\text{-torsion}\} \\ H &\longmapsto H^\perp \\ Y^\perp &\longleftarrow\!\shortmid\, Y \end{aligned}$$

□

Fact. Say

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

is **exact** if the sequence is set-theoretically exact and the induced sequence of lie algebras

$$0 \to \operatorname{Lie} G_1 \to \operatorname{Lie} G_2 \to \operatorname{Lie} G_3 \to 0$$

is exact. (See Definition 92.) Suppose the G_i are diagonalisable, so that $\operatorname{Lie} G_i \cong \operatorname{Hom}_{\mathbf{Z}}(X^*(G_i), k)$. Then the sequence of the G_i is exact if and only if

$$0 \to X^*(G_3) \to X^*(G_2) \to X^*(G_1) \to 0$$

Remark 52.

$$1 \to \mu_p \to \mathbf{G}_m \xrightarrow{p} \mathbf{G}_m \to 1$$

is set-theoretically exact, but

$$0 \to X^*(\mathbf{G}_m) \xrightarrow{p} X^*(\mathbf{G}_m) \to X^*(\mu_p) \to 0$$

is not if $\operatorname{char} k = p$ *(in which case* $X^*(\mu_p) = 0$*).*

Definition. *The group of* **cocharacters** *of G are*

$$X_*(G) := \operatorname{Hom}(\mathbf{G}_m, G)$$

If G is abelian, then $X_*(G)$ is an abelian group.

Proposition 53. *If T is a torus, then $X_*(T), X^*(T)$ are free abelian and*

$$X^*(T) \times X_*(T) \to \operatorname{Hom}(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbf{Z}, \quad (\chi, \lambda) \mapsto \chi \circ \lambda$$

is a perfect pairing.

Proof.

$$X_*(T) = \operatorname{Hom}(\mathbf{G}_m, T) \cong \operatorname{Hom}(X^*(T), \mathbf{Z}).$$

The isomorphism follows from Theorem 49. Since $X^*(T)$ is finitely-generated free abelian by Corollary 50, we have that $X_*(T) \cong \operatorname{Hom}(X^*(T), \mathbf{Z})$ is free abelian as well. Moreover, since

$$\operatorname{Hom}(X, \mathbf{Z}) \times X \to \mathbf{Z}, \quad (\alpha, x) \mapsto \alpha(x)$$

is a perfect pairing for any finitely-generated free abelian X, it follows from the isomorphism above that the pairing in question is also perfect. □

Proposition 54 (Rigidity of diagonalisable groups)**.** *Let G, H be diagonalisable groups and V a connected affine variety. If $\phi : G \times V \to H$ is a morphism of varieties such that $\phi_v : G \to H$, $g \mapsto \phi(g, v)$ is a morphism of algebraic groups for all $v \in V$, then ϕ_v is independent of v.*

Proof. Under $\phi^* : k[H] \to k[G] \otimes k[V]$, for $\chi \in X^*(H)$, write

$$\phi^*(\chi) = \sum_{\chi' \in X^*(G)} \chi' \otimes f_{\chi\chi'}$$

Then

$$\begin{aligned}
\phi_v^*(\chi) = \sum_{\chi'} f_{\chi\chi'}(v)\chi \in X^*(G) &\implies \forall\, \chi', v \quad f_{\chi\chi'}(v) \in \{0, 1\} \\
&\implies \forall\, \chi' \quad f_{\chi\chi'}^2 = f_{\chi\chi'} \\
&\implies \forall\, \chi' \quad V = V(f_{\chi\chi'}) \sqcup V(1 - f_{\chi\chi'}) \\
&\implies \forall\, \chi' \quad f_{\chi\chi'} \text{ is constant, since } V \text{ is connected} \\
&\implies \forall \phi_v \text{ is independent of } v
\end{aligned}$$

□

Corollary 55. *Suppose that $H \subset G$ is a closed diagonalisable subgroup. Then $N_G(H)^0 = \mathcal{Z}_G(H)^0$ and $N_G(H)/\mathcal{Z}_G(H)$ is finite. ($N_G(H), \mathcal{Z}_G(H)$ are easily seen to be closed subgroups.)*

Proof. Applying the above proposition to the morphism

$$H \times N_G(H)^0 \to H, \quad (h, n) \mapsto nhn^{-1}$$

we get that $nhn^{-1} = h$ for all h, n. Hence

$$N_G(H)^0 \subset \mathcal{Z}_G(H) \subset N_G(H)$$

and the corollary immediately follows. □

2.3 Elementary unipotent groups.

Define $\mathcal{A}(G) := \mathrm{Hom}(G, \mathbf{G}_a)$, which is an abelian group under addition of maps; actually, it is an R-module, where $R = \mathrm{End}(\mathbf{G}_a)$. Note that $\mathcal{A}(\mathbf{G}_a^n) \cong R^n$. $R = \mathrm{End}(\mathbf{G}_a)$ can be identified with

$$\{f \in k[\mathbf{G}_a] = k[x] \mid f(x+y) = f(x) + f(y) \text{ in } k[x,y]\} = \begin{cases} \{\lambda x \mid \lambda \in k\}, & \text{char } k = p = 0 \\ \{\sum \lambda_i x^{p^i} \mid \lambda_i \in k\}, & \text{char } k = p > 0 \end{cases}$$

Accordingly,

$$R \cong \begin{cases} k, & p = 0 \\ \text{noncommutative polynomial ring over } k, & p > 0 \end{cases}$$

Proposition 56. *G is an algebraic group. The following are equivalent:*

(i) *G is isomorphic to a closed subgroup of $\mathbf{G}_a^n$ ($n \geqslant 0$).*

(ii) *$\mathcal{A}(G)$ is a finitely-generated R-module and generates $k[G]$ as a k-algebra.*

(iii) *G is commutative and $G = G_u$ (and $G^p = 1$ if $p > 0$).*

Definition 57. *If one of the above conditions holds, then G is* **elementary unipotent***. Note that (iii) rules out $\mathbf{Z}/p^n\mathbf{Z}$ as elementary unipotent when $n > 1$.*

Theorem 58.

$$(\text{ elementary unipotent groups }) \xrightarrow{\mathcal{A}} (\text{ finitely-generated } R\text{-modules })$$

is an equivalence of categories.

Proof. For the inverse functor, see Springer 14.3.6. ☐

Corollary 59.

(i) *The elementary unipotent groups are $\mathbf{G}_a^n$ if $p = 0$, and $\mathbf{G}_a^n \times (\mathbf{Z}/p\mathbf{Z})^s$ if $p > 0$*

(ii) *For an elementary unipotent group G,*

$$G \text{ is isomorphic to a } \mathbf{G}_a^n \iff G \text{ is connected} \iff \mathcal{A}(G) \text{ is free}$$

Theorem 60. *Suppose G is a connected algebraic group of dimension 1, then $G \cong \mathbf{G}_a$ or $\mathbf{G}_m$.*

Proof.
Claim: G is commutative.
Fix $\gamma \in G$ and consider $\phi : G \to G$ given by $g \mapsto g\gamma g^{-1}$. Then $\overline{\phi(G)}$ is irreducible and closed, which implies that $\overline{\phi(G)} = \{\gamma\}$ or $\overline{\phi(G)} = G$. Now, either $\overline{\phi(G)} = \{\gamma\}$ for all $\gamma \in G$, in which case G is commutative and the claim is true, or $\overline{\phi(G)} = G$ for at least one γ. Suppose the second case holds with a particular γ and fix an embedding $G \hookrightarrow \mathrm{GL}_n$. Consider the morphism $\psi : G \to \mathbf{A}^{n+1}$ which takes g to the coefficients of the characteristic polynomial of g, $\det(T \cdot \mathrm{id} - g)$. ψ is constant on the conjugacy class $\phi(G)$, implying that ψ is constant. Hence, every $g \in G$, e included, has the same characteristic polynomial: $(T-1)^n$. Thus

$$G = G_u \implies G \text{ is nilpotent} \implies G \supsetneq [G,G] \implies [G,G] = 1 \implies G \text{ is commutative}$$

Now, by Proposition 37,

$$G \cong G_s \times G_u \implies G = G_s \text{ or } G = G_u$$

as dimension is additive. In the former case, $G \cong \mathbf{G}_m$ by Corollary 50. In the latter, if we can prove that G is elementary unipotent, then $G \cong \mathbf{G}_a$ by Corollary 59; we must show that $G^p = 1$ when $p > 0$ by Proposition 56. Suppose that $G^p \neq 1$, so that $G^p = G$. Then $G = G^p = G^{p^2} = \cdots$. But $(g-1)^n = 0$ in GL_n and so for $p^r \geqslant n$,

$$0 = (g-1)^{p^r} = g^{p^r} - 1 \implies g^{p^r} = 1 \implies \{e\} = G^{p^r} = G$$

which is a contradiction. ☐

3. Lie algebras.

If X is a variety and $x \in X$, then the **local ring** at x is

$$\mathcal{O}_{X,x} := \varinjlim_{\substack{U \text{ open} \\ U \ni x}} \mathcal{O}_X(U) = \text{ germs of functions at } x = \frac{\{(f,U) \mid f \in \mathcal{O}_X(U)\}}{\sim}$$

where $(f,U) \sim (f',U')$ if there is an open neighbourhood $V \subset U \cap U'$ of x for which $f|_V = f'|_V$. There is a well-defined ring morphism $\mathrm{ev}_x : \mathcal{O}_{X,x} \to k$ given by evaluating at x: $[(f,U)]] \mapsto f(x)$. $\mathcal{O}_{X,x}$ is a local ring (hence the name) with unique maximal ideal

$$\mathfrak{m}_x =: \ker \mathrm{ev}_x = \{[(f,U)]. \mid f(x) = 0\}$$

for if $f \notin \mathfrak{m}_x$, then f^{-1} is defined near x, implying that $f \in \mathcal{O}_{X,x}^\times$.

Fact. If X is affine and x corresponds to the maximal ideal $\mathfrak{m} \subset k[X]$ (via Nullstellensatz), then $\mathcal{O}_{X,x} \cong k[X]_\mathfrak{m}$. By choosing an affine chart in X at x, we see in general that $\mathcal{O}_{X,x}$ is noetherian.

3.1 Tangent Spaces.

Analogous to the case of manifolds, the **tangent space** to a variety X at a point x is

$$T_x X := \mathrm{Der}_k(\mathcal{O}_{X,x}, k) = \{\delta : \mathcal{O}_{X,x} \to k \mid \delta \text{ is } k\text{-linear, } \delta(fg) = f(x)\delta(g) + g(x)\delta(f)\}$$

(so k is viewed as a $\mathcal{O}_{X,x}$-module via ev_x.) $T_x X$ is a k-vector space.

Lemma 61. *Let A be a k-algebra, $\epsilon : A \to k$ a k-algebra morphism, and $\mathfrak{m} = \ker \epsilon$. Then*

$$\mathrm{Der}_k(A,k) \xrightarrow{\sim} (\mathfrak{m}/\mathfrak{m}^2)^*, \quad \delta \mapsto \delta|_\mathfrak{m}$$

Proof. An inverse map is given by sending λ to a derivation defined by $x \mapsto \begin{cases} 0, & x = 1 \\ \lambda(x), & x \in \mathfrak{m} \end{cases}$. Checking this is an exercise. □

Hence, $T_x X \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is finite-dimensional.

Examples.
- If $X = \mathbf{A}^n$, then $T_x X$ has basis

$$\left.\frac{\partial}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}\right|_x$$

.

• For a finite-dimensional k-vector space V, $T_x(V) \cong V$.

Definition 62. *X is* **smooth** *at x if $\dim T_xX = \dim X$. Moreover, X is* **smooth** *if it is smooth at every point. From the above example, we see that $\mathbf{A}^n$ is smooth.*

If $\phi : X \to Y$ we get $\phi^* : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ and hence

$$d\phi : T_xX \to T_{\phi(x)}Y, \quad \delta \mapsto \delta \circ \phi^*$$

Remark 63. *If $U \subset X$ is an open neighbourhood of x, then $d(U \hookrightarrow X) : T_xU \xrightarrow{\sim} T_xX$. More generally, if $X \subset Y$ is a locally closed subvariety, then T_xX embeds into T_xY.*

Theorem 64.

$$\dim T_xX \geqslant \dim X$$

with equality holding for all x in some open dense subset.

Note that if X is affine and x corresponds to $\mathfrak{m} \subset k[X]$, then the natural map $k[X] \to k[X]_\mathfrak{m} = \mathcal{O}_{X,x}$ induces an isomorphism

$$T_xX \xrightarrow{\sim} \mathrm{Der}_k(k[X], k), \quad (k \text{ being viewed as a } k[X]\text{-modules via } \mathrm{ev}_x)$$

which is isomorphic to $(\mathfrak{m}/\mathfrak{m}^2)^*$ by Lemma 61. So, we can work without localising. □

Remark 65. *If G is an algebraic group, then G is smooth by Theorem 64 since*

$$d(\ell_g : x \mapsto gx) : T_\gamma G \xrightarrow{\sim} T_{g\gamma}G$$

The same holds for **homogeneous** *G-spaces (i.e., G-spaces for which the G-action is transitive).*

3.2 Lie algebras.

Definition 66. *A* **Lie algebra** *is a k-vector space L together with a bilinear map $[,] : L \times L \to L$ such that*

(i) $[x, x] = 0 \;\; \forall x \in L \;\; (\implies [x, y] = -[y, x])$

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \;\; \forall x, y, z \in L$

Examples.
• If A is an associative k-algebra (maybe non-unital), then $[a, b] := ab - ba$ gives A the structure of a Lie algebra.
• Take $A = \mathrm{End}(V)$ and as above define $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$.
• For L an arbitrary k-vector space, define $[,] = 0$. When $[,] = 0$ a Lie algebra is said to be **abelian**.

We will construct a functor

$$(\text{ algebraic groups }) \xrightarrow{\text{Lie}} (\text{ Lie algebras })$$

As a vector space, $\operatorname{Lie} G = T_eG$. $\dim \operatorname{Lie} G = \dim G$ by above remarks.

The following is another way to think about T_eG. Recall that we can identify G with the functor

$$R \mapsto \operatorname{Hom}_{\mathrm{alg}}(k[G], R) := G(R)$$

(where $k[G]$ is a reduced finite-dimensional commutative Hopf k-algebra). The Hopf (i.e., co-group) structure on R induces a group structure on $G(R)$, even when R is not reduced..

Lemma 67.

$$\operatorname{Lie} G \cong \ker\left(G(k[\epsilon]/(\epsilon^2)) \to G(k)\right)$$

as abelian groups.

Proof. Write the algebra morphism $\theta : k[G] \to k[\epsilon]/(\epsilon^2)$ as given by $f \mapsto \mathrm{ev}_e(f) + \delta(f) \cdot \epsilon$ for some $\delta : k[G] \to k$. δ is a derivation. □

Examples.
• For $G = \mathrm{GL}_n$, $G(R) = \mathrm{GL}_n(R)$, and we have

$$\operatorname{Lie} G = \ker\left(\mathrm{GL}_n(k[\epsilon]/(\epsilon^2) \to \mathrm{GL}_n(k)\right) = \{I + A\epsilon \mid A \in M_n(k)\} \xrightarrow{\sim} M_n(k)$$

Explicitly, the isomorphism $\operatorname{Lie} \mathrm{GL}_n \to M_n(k)$ is given by $\delta \mapsto (\partial(T_{ij}))$.
• Intrinsically, for a finite-dimensional k-vector space V: Since $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$, we have

$$\operatorname{Lie} \mathrm{GL}(V) \xrightarrow{\sim} T_I(\mathrm{End}\ V) \xrightarrow{\sim} \mathrm{End\ V}$$

Definition 68. *A* **left-invariant vector field** *on G is an element $D \in \mathrm{Der}_k(k[G], k[G])$ such that the*

$$\begin{array}{ccc} k[G] & \xrightarrow{D} & k[G] \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ k[G] \otimes k[G] & \xrightarrow{\mathrm{id} \otimes D} & k[G] \otimes k[G] \end{array}$$

commutes.

For a fixed D, for $g \in G$, define $\delta_g := \mathrm{ev}_g \circ D \in T_gG$.

Evaluating $\Delta \circ D$ at (g_1, g_2) gives $\delta_{g_1g_2}$
Evaluating $(\mathrm{id} \otimes D) \circ \Delta$ at (g_1, g_2) gives $\delta_{g_2} \circ \ell_{g_1}^* = d\ell_{g_1}(\delta_{g_2})$

Hence $D \in \mathrm{Der}_k(k[G], k[G])$ being left-invariant is equivalent to $\delta_{g_1g_2} = d\ell_{g_1}(\delta_{g_2})$ for all $g_1, g_2 \in G$. Define

$$\mathcal{D}_G := \text{ vector space of left-invariant vector fields on } G.$$

Theorem 69.

$$\mathcal{D}_G \to \operatorname{Lie} G, \quad D \mapsto \delta_e = \mathrm{ev}_e \circ D$$

is a linear isomorphism.

Proof. We shall prove that $\delta \mapsto (\mathrm{id} \otimes \delta) \circ \Delta$ is an inverse morphism. Fix $\delta \in \operatorname{Lie} G$, set $D = (\mathrm{id}, \delta) \circ \Delta : k[G] \to k[G]$, and check that (id, δ) is a k-derivation $k[G] \otimes k[G] \to k[G]$, where $k[G]$ is viewed as a $k[G] \otimes k[G]$-module via $\mathrm{id} \otimes \mathrm{ev}_e$. First, we shall check that $D \in \mathcal{D}_G$:

$$\begin{aligned} D(fh) &= (\mathrm{id} \otimes \delta)(\Delta(fh)) \\ &= (\mathrm{id} \otimes \delta)(\Delta(f) \cdot \Delta(h)) \\ &= (\mathrm{id} \otimes \mathrm{ev}_e)(\Delta f) \cdot (\mathrm{id} \otimes \delta)(\Delta h) + (\mathrm{id} \otimes \mathrm{ev}_e)(\Delta h) \cdot (\mathrm{id} \otimes \delta)(\Delta f) \\ &= f \cdot D(h) + h \cdot D(f). \end{aligned}$$

Next, we show that D is left-invariant:

$$\begin{aligned} (\mathrm{id} \otimes D) \circ \Delta &= (\mathrm{id} \otimes ((\mathrm{id} \otimes \delta) \circ \Delta)) \circ \Delta \\ &= (\mathrm{id} \otimes (\mathrm{id} \otimes \delta)) \circ (\mathrm{id} \otimes \Delta) \circ \Delta \\ &= (\mathrm{id} \otimes (\mathrm{id} \otimes \delta)) \circ (\Delta \otimes \mathrm{id}) \circ \Delta \quad (\text{“co-associativity”}) \\ &= \Delta \circ (\mathrm{id} \otimes \delta) \circ \Delta \quad (\text{easily checked}) \\ &= \Delta \circ D. \end{aligned}$$

Lastly, we show that the maps are inverse:

$$\begin{aligned} \delta \mapsto (\mathrm{id} \otimes \delta) \otimes \Delta &\mapsto \mathrm{ev}_e \circ (\mathrm{id} \otimes \delta) \circ \Delta = \delta \circ (\mathrm{ev}_e \otimes \mathrm{id}) \circ \Delta = \delta \\ D \mapsto \mathrm{ev}_e \circ D &\mapsto (\mathrm{id} \otimes \mathrm{ev}_e) \circ (\mathrm{id} \otimes D) \circ D = (\mathrm{id} \otimes \mathrm{ev}_e) \circ \Delta \circ D = D. \end{aligned}$$

□

Since $\operatorname{Hom}_k(k[G], k[G])$ is an associative algebra, there is a natural candidate for a Lie bracket on $\mathcal{D}_G \subset \operatorname{Hom}_k(k[G], k[G])$: $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. We must check that $[\mathcal{D}_G, \mathcal{D}_G] \subset \mathcal{D}_G$. Let $D_1, D_2 \in \mathcal{D}_G$. Since

$$\begin{aligned} [D_1, D_2](fh) &= D_1(D_2(fh)) - D_2(D_1(fh)) \\ &= D_1(f \cdot D_2(h) + h \cdot D_2(f)) - D_2(f \cdot D_1(h) + h \cdot D_1(f)) \\ &= D_1(f \cdot D_2(h)) + D_1(h \cdot D_2(f)) - D_2(f \cdot D_1(h)) - D_2(h \cdot D_1(f)) \\ &= \Big(fD_1(D_2(h)) + D_2(h)D_1(f)\Big) + \Big(hD_1(D_2(f)) + D_2(f)D_1(h)\Big) \\ &\quad - \Big(fD_2(D_1(h)) + D_1(h)D_2(f)\Big) - \Big(hD_2(D_1(f)) + D_1(f)D_2(h)\Big) \\ &= f\Big(D_1(D_2(h)) - fD_2(D_1(h))\Big) + h\Big(D_1(D_2(f)) - hD_2(D_1(f))\Big) \\ &= f \cdot [D_1, D_2](h) + h \cdot [D_1, D_2](f) \end{aligned}$$

we have that $[D_1, D_2]$ is a derivation. Moreover,

$$\begin{aligned} (\mathrm{id} \otimes [D_1, D_2]) \circ \Delta &= (\mathrm{id} \otimes (D_1 \circ D_2)) \circ \Delta - (\mathrm{id} \otimes (D_2 \circ D_1)) \circ \Delta \\ &= (\mathrm{id} \otimes D_1) \circ (\mathrm{id} \otimes D_2) \circ \Delta - (\mathrm{id} \otimes D_2) \circ (\mathrm{id} \otimes D_1) \circ \Delta \\ &= (\mathrm{id} \otimes D_1) \circ \Delta \circ D_2 - (\mathrm{id} \otimes D_2) \circ \Delta \circ D_1 \\ &= \Delta \circ D_1 \circ D_2 - \Delta \circ D_2 \circ D_1 \\ &= \Delta \circ [D_1, D_2] \end{aligned}$$

and so $[D_1, D_2]$ is left-invariant. Accordingly, $[\mathcal{D}_G, \mathcal{D}_G] \subset \mathcal{D}_G$, and thus by the above theorem $\operatorname{Lie} G$ becomes a Lie algebra.

Remark 70. *If $p > 0$, then $\mathcal{D}_G$ is also stable under $D \mapsto D^p$ (composition with itself p-times).*

Proposition 71. *If $\delta_1, \delta_2 \in \operatorname{Lie} G$, then $[\delta_1, \delta_2] : k[G] \to k$ is given by*

$$[\delta_1, \delta_2] = ((\delta_1, \delta_2) - (\delta_2, \delta_1)) \circ \Delta$$

Proof. Let $D_i = (\mathrm{id} \otimes \delta_i) \circ \Delta$ for $i = 1, 2$. Then

$$\begin{aligned}
[\delta_1, \delta_2] &= \mathrm{ev}_e \circ [D_1, D_2] \\
&= \mathrm{ev}_e \circ D_1 \circ D_2 - \mathrm{ev}_e \circ D_2 \circ D_1 \\
&= \delta_1 \circ (\mathrm{id} \otimes \delta_2) \circ \Delta - \delta_2 \circ (\mathrm{id} \otimes \delta_1) \circ \Delta \\
&= (\delta_1 \otimes \delta_2) \circ \Delta - (\delta_2 \otimes \delta_1) \circ \Delta \\
&= ((\delta_1 \otimes \delta_2) - (\delta_2 \otimes \delta_1)) \circ \Delta.
\end{aligned}$$

□

Corollary 72. *If $\phi : G \to H$ is a morphism of algebraic groups, then $d\phi : \operatorname{Lie} G \to \operatorname{Lie} H$ is a morphism of Lie algebras (i.e., brackets are preserved).*

Proof.

$$\begin{aligned}
d\phi([\delta_1, \delta_2]) &= [\delta_1, \delta_2] \circ \phi^* \\
&= (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1) \circ \Delta \circ \phi^*, \quad \text{(by the above Prop.)} \\
&= (\delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1) \circ (\phi^* \otimes \phi^*) \circ \Delta \\
&= (\delta_1 \circ \phi^*, \delta_2 \circ \phi^*) \circ \Delta - (\delta_2 \circ \phi^*, \delta_1 \circ \phi^*) \circ \Delta \\
&= (d\phi(\delta_1), d\phi(\delta_2)) \circ \Delta - (d\phi(\delta_2), d\phi(\delta_1)) \circ \Delta \\
&= [d\phi(\delta_1), d\phi(\delta_2)].
\end{aligned}$$

□

Corollary 73. *If G is commutative, then so too is $\operatorname{Lie} G$ (i.e., $[\cdot, \cdot] = 0$).*

Example. We have that $\phi : \operatorname{Lie} \mathrm{GL}_n \cong M_n(k)$ is given by $\phi : \delta \mapsto (\delta(T_{ij}))$. Since

$$\begin{aligned}
[\delta_1, \delta_2](T_{ij}) &= (\delta_1, \delta_2)(\Delta T_{ij}) - (\delta_2, \delta_1)(\Delta T_{ij}) \\
&= \sum_{l=1}^{n} \delta_1(T_{il})\delta_2(T_{lj}) - \sum_{l=1}^{n} \delta_2(T_{il})\delta_1(T_{lj}) \\
&= (\phi(\delta_1)\phi(\delta_2))_{ij} - (\phi(\delta_2)\phi(\delta_1))_{ij}
\end{aligned}$$

Hence,

$$\phi([\delta_1, \delta_2]) = \phi(\delta_1)\phi(\delta_2) - \phi(\delta_2)\phi(\delta_1)$$

and so in identifying $\operatorname{Lie} \mathrm{GL}_n$ with $M_n(k)$, we can also identify the Lie bracket with the usual one on $M_n(k)$: $[A, B] = AB - BA$. Similarly, the Lie bracket on $\operatorname{Lie} \mathrm{GL}(V) \cong \mathrm{End}(V)$ can be identified with the commutator. □

Remark 74. *If $\phi : G \to H$ is a closed immersion, then ϕ^* is surjective, and so $d\phi : \mathrm{Lie}\, G \to \mathrm{Lie}\, H$ is injective. Hence, if $G \hookrightarrow \mathrm{GL}_n$, then the above example determines $[\cdot,\cdot]$ on $\mathrm{Lie}\, G$.*

Examples.
- $\mathrm{Lie}\,\mathrm{SL}_n =$ trace 0 matrices in $M_n(k)$
- $\mathrm{Lie}\, B_n =$ upper-triangular matrices in $M_n(k)$
- $\mathrm{Lie}\, U_n =$ upper-triangular matrices in $M_n(k)$ with 1's along diagonal
- $\mathrm{Lie}\, D_n =$ diagonal matrices in $M_n(k)$

Exercise. If G is diagonal, show that $\mathrm{Lie}\, G \cong \mathrm{Hom}_{\mathbf{Z}}(X^*(G), k)$.

3.3 Adjoint representation.

G acts on itself by conjugation: for $x \in G$,

$$c_x : G \to G, \quad g \mapsto xgx^{-1}$$

is a morphism. $\mathrm{Ad}(x) := dc_x : \mathrm{Lie}\, G \to \mathrm{Lie}\, G$ is a Lie algebra endomorphism such that

$$\mathrm{Ad}(e) = \mathrm{id}, \quad \mathrm{Ad}(xy) = \mathrm{Ad}(x) \circ \mathrm{Ad}(y)$$

Hence, we have a morphism of groups

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}\, G)$$

Proposition 75. Ad *is an algebraic representation of G.*

Proof. We must show that

$$\theta : G \times \mathrm{Lie}\, G \to \mathrm{Lie}\, G, \quad (x, \delta) \mapsto \mathrm{Ad}(x)(\delta) = dc_x(\delta) = \delta \circ c_x^*$$

is a morphism of varieties. It is enough to show that $\lambda \circ \theta$ is a morphism for all $\lambda \in (\mathrm{Lie}\, G)^*$. Given such a λ, since $(\mathrm{Lie}\, G)^* \cong \mathfrak{m}/\mathfrak{m}^2$ we must have $\lambda(\delta) = \delta(f)$ for some $f \in \mathfrak{m}$. Accordingly, for any $f \in \mathfrak{m}$ we must show that

$$(x, \delta) \mapsto \delta(c_x^* f)$$

is a morphism. Recall from the proof of Proposition 27 that $c_x^* f = \sum_i h_i(x) f_i$ for some $f_i, h_i \in k[G]$, which implies that

$$(x, \delta) \mapsto \delta(c_x^* f) = \sum_i h_i(x) \delta(f_i)$$

is a morphism as $x \mapsto h_i(x)$ and $\delta \mapsto \delta(f_i)$ are morphisms. □.

Exercises.
- Show that $\mathrm{ad} := d(\mathrm{Ad}) : \mathrm{Lie}\, G \to \mathrm{End}(\mathrm{Lie}\, G)$ is

$$\delta_1 \mapsto (\delta_2 \mapsto [\delta_1, \delta_2])$$

This is hard, but is easiest to manage in reducing to the case of GL_n using an embedding $G \hookrightarrow \mathrm{GL}_n$.
- Show that $d(\det : \mathrm{GL}_n \to \mathrm{GL}_1) : M_n(k) \to k$ is the trace map.

3.4 Some derivatives.

If X_1, X_2 are varieties with points $x_1 \in X_1$ and $x_2 \in X_2$, then the morphisms

$$\begin{array}{ccccc} & & X_1 & & \\ & \overset{\pi_1}{\nearrow} & & \overset{i_{x_1}: x \mapsto (x_1, x)}{\searrow} & \\ X_1 \times X_2 & & & & X_1 \times X_2 \\ & \underset{\pi_2}{\searrow} & & \underset{i_{x_2}: x \mapsto (x, x_2)}{\nearrow} & \\ & & X_2 & & \end{array}$$

induce inverse isomorphisms $T_{x_1}X_1 \oplus T_{x_2}X_2 \leftrightarrows T_{(x_1,x_2)}(X_1 \times X_2)$. In particular, for algebraic groups G_1, G_2 we have inverse isomorphisms

$$\operatorname{Lie} G_1 \oplus \operatorname{Lie} G_2 \leftrightarrows \operatorname{Lie}(G_1 \times G_2)$$

Proposition 76.

(i) $d(\mu : G \times G \to G) = (\operatorname{Lie} G \oplus \operatorname{Lie} G \xrightarrow{(X,Y) \mapsto X+Y} \operatorname{Lie} G)$

(ii) $d(i : G \to G) = (\operatorname{Lie} G \xrightarrow{X \mapsto -X} \operatorname{Lie} G)$

Proof.
(i). It is enough to show that $d\mu$ is the identity on each factor. Since id_G can be factored as

$$G \xrightarrow{i_e} G \times G \xrightarrow{\mu} G$$

where $i_e : x \mapsto (e, x)$ or $x \mapsto (x, e)$, we are done.

(ii). Since $x \mapsto e$ can be factored $G \xrightarrow{(\mathrm{id}, i)} G \times G \xrightarrow{\mu} G$. From (i) we have that $0 : \operatorname{Lie} G \to \operatorname{Lie} G$ can factored as

$$\operatorname{Lie} G \xrightarrow{(\mathrm{id}, di)} \operatorname{Lie} G \oplus \operatorname{Lie} G \xrightarrow{+} \operatorname{Lie} G$$

□

Remark 77. *The open immersion $G^0 \hookrightarrow G$ induces an isomorphism $\operatorname{Lie} G^0 \xrightarrow{\sim} \operatorname{Lie} G$.*

Proposition 78 (Derivative of a linear map)**.** *If V, W be vector spaces and $f : V \to W$ a linear map (hence a morphism), then, for all $v \in V$, we have the commutative diagram*

$$\begin{CD} T_v V @>{T_v(f)}>> T_{f(v)} W \\ @V{\wr}VV @VV{\wr}V \\ V @>>{f}> W \end{CD}$$

Proof. Exercise. □

Proposition 79. *Suppose that $\sigma : G \to \mathrm{GL}(V)$ is a representation and $v \in V$. Define $o_v : G \to V$ by $g \mapsto \sigma(g)v$. Then*

$$do_v(X) = d\sigma(X)(v)$$

in $T_v V \cong V$, for all $X \in \operatorname{Lie} G$.

Proof. Factor o_v as

$$\begin{array}{ccccc} G & \xrightarrow{\phi} & \mathrm{GL}(V) \times V & \xrightarrow{\psi} & V \\ g & \mapsto & (\sigma(g), v) & & \\ & & (A, w) & \mapsto & Aw \end{array}$$

$d\phi = (d\sigma, 0) : \operatorname{Lie} G \to \operatorname{End} V \oplus V$. By Proposition 78, under the identification $V \cong T_v V$, we have that the derivative at (e, v) of the first component of ψ, which sends $A \to Av$, is the same map. The result follows. ☐

Proposition 80. *Suppose that $\rho_i : G \to \mathrm{GL}(V_i)$ are algebraic representations for $i = 1, 2$. Then the derivative of $\rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$ is*

$$d(\rho_1 \otimes \rho_2)X = d\rho_1(X) \otimes \mathrm{id} + \mathrm{id} \otimes d\rho_2(X)$$

(i.e., $X(v_1 \otimes v_2) = (Xv_1) \otimes v_2 + v_1 \otimes (Xv_2)$.) Similarly for $V_1 \otimes \cdots \otimes V_n$, $\mathrm{Sym}^n V$, $\Lambda^n V$.

Proof. We have the commutative diagram

$$\begin{array}{ccccc} \rho_1 \otimes \rho_2 : G & \longrightarrow & \mathrm{GL}(V_1) \times \mathrm{GL}(V_2) & \longrightarrow & \mathrm{GL}(V_1 \otimes V_2) \\ & & \Big\downarrow \text{open} & & \Big\downarrow \text{open} \\ & & \mathrm{End}(V_1) \times \mathrm{End}(V_2) & \xrightarrow{\phi} & \mathrm{End}(V_1 \otimes V_2) \end{array}$$

where $\phi : (A, B) \mapsto A \otimes B$. (Note that ϕ being a morphism implies that $\rho_1 \otimes \rho_2$ is.) Computing $d\phi$ component-wise at $(1, 1)$, we get that $d\phi|_{\mathrm{End}(V_1)}$ is the derivative of the linear map $\mathrm{End}(V_1) \to \mathrm{End}(V_1 \otimes V_2)$ given by $A \mapsto A \otimes 1$, which is the same map; likewise for $d\phi|_{\mathrm{End}(V_2)}$. Hence,

$$d\phi(A, B) = A \otimes 1 + 1 \otimes B$$

and we are done. ☐

Exercise. If $\rho : G \to \mathrm{GL}(V)$ is an algebraic representation, then so is $\rho^\vee : G \to \mathrm{GL}(V^*)$, given by $\rho^\vee(g) = \rho(g^{-1})^*$. (Here, V^* is the dual vector space.) Moreover, $d\rho^\vee(X) = -d\rho(X)^*$.

Proposition 81 (Adjoint representation for $\mathrm{GL}(V)$)**.** *For $g \in \mathrm{GL}(V)$, $A \in \operatorname{Lie} \mathrm{GL}(V) \cong \mathrm{End}(V)$,*

$$\mathrm{Ad}(g)A = gAg^{-1}$$

Proof. This follows from Proposition 78 by considering the linear map $f : \mathrm{End}(V) \to \mathrm{End}(V)$ given by $A \mapsto gAg^{-1}$ and noting that $\mathrm{GL}(V)$ is open in $\mathrm{End}(V)$. ☐

Exercise. Deduce that, for $\mathrm{GL}(V)$, $\mathrm{ad}(A)(B) = AB - BA$.

3.5 Separable morphisms.

Let $\phi : X \to Y$ be a *dominant* morphism of irreducible varieties (i.e., $\overline{\phi(X)} = Y$). From the induced maps $\mathcal{O}_Y(V) \to \mathcal{O}_X(\phi^{-1}(V))$ – note that $\phi^{-1}(V) \neq \varnothing$, as ϕ is dominant – given by $f \mapsto f \circ \phi$, we get a morphism of fields $\phi^* : k(Y) \to k(X)$. That is, $k(X)$ is a finitely-generated field extension of $k(Y)$.

Remark 82. *This field extension has transcendence degree* $\dim X - \dim Y$*, and hence is algebraic if and only if* $\dim X = \dim Y$*.*

Definition 83. *A dominant ϕ is* **separable** *if $\phi^* : k(Y) \to k(X)$ is a separable field extension.*

Recall.
• An algebraic field extension E/F being separable means that every $\alpha \in E$ has a minimal polynomial without repeated roots.
• A finitely-generated field extension E/F is separable if it is of the form

$$\begin{array}{c} E \\ \Big| \ \text{finite separable} \\ F(x_1, \ldots, x_n) \\ \Big| \ x_1, \ldots, x_n \text{ algebraically independent} \\ F \end{array}$$

Facts.
• If E'/E and E/F are separable then E'/F is separable.
• If char $k = 0$, all extensions are separable; in characteristic 0 being dominant is equivalent to being separable. (As an example, if char $k = p > 0$, then $F(t^{1/p})/F(t)$ is never separable.)
• The composition of separable morphisms is separable.

Example. If $p > 0$, then $\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m$ is not separable.

Theorem 84. *Let $\phi : X \to Y$ be a morphism between irreducible varieties. The following are equivalent:*

(i) *ϕ is separable.*

(ii) *There is a dense open set $U \subset X$ such that $d\phi_x : T_x X \to T_{\phi(x)} Y$ is surjective for all $x \in U$.*

(iii) *There is an $x \in X$ such that X is smooth at x, Y is smooth at $\phi(x)$, and $d\phi_x$ is surjective.*

Corollary 85. *If X, Y are irreducible, smooth varieties, then $\phi : X \to Y$*

$$\text{is separable} \iff d\phi_x \text{ is surjective for all } x \iff d\phi_x \text{ is surjective for one } x$$

Remark 86. *The corollary applies in particular if X, Y are connected algebraic groups or homogeneous spaces.*

3.6 Fibres of morphisms.

Theorem 87. *Let $\phi : X \to Y$ be a dominant morphism between irreducible varieties and let $r := \dim X - \dim Y \geqslant 0$.*

(i) *For all $y \in \phi(X)$, $\dim \phi^{-1}(y) \geqslant r$.*

(ii) *There is a nonempty open subset $V \subset Y$ such that for all irreducible closed $Z \subset Y$ and for all irreducible components $Z' \subset \phi^{-1}(Z)$ with $Z' \cap \phi^{-1}(V) \neq \varnothing$, $\dim Z' = \dim Z + r$ (which implies that $\dim \phi^{-1}(y) = r$ for all $y \in V$). If $r = 0$, $|\phi^{-1}(y)| = [k(X) : k(Y)]_s$ for all $y \in V$.*

Theorem 88. *If $\phi : X \to Y$ is a dominant morphism between irreducible varieties, then there is a nonempty open $V \subset Y$ such that $\phi^{-1}(V) \xrightarrow{\phi} V$ is universally open, i.e., for all varieties Z*

$$\phi^{-1}(V) \times Z \xrightarrow{\phi \times \mathrm{id}_Z} V \times Z$$

is an open map.

Corollary 89. *If $\phi : X \to Y$ is a G-equivariant morphism of homogeneous G-spaces,*

(i) *For all varieties Z, $\phi \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is an open map.*

(ii) *For all closed, irreducible $Z \subset Y$ and for all irreducible components $Z' \subset \phi^{-1}(Z)$, $\dim Z' = \dim Z + r$. (In particular, all fibres are equidimensional of dimension r.)*

(iii) *ϕ is an isomorphism if and only if ϕ is bijective and $d\phi_x$ is an isomorphism for one (or, equivalently, all) x.*

(In this statement it's easy to reduce to the irreducible case.)

Corollary 90. *For all G-spaces, $\dim \mathrm{Stab}_G(x) + \dim(Gx) = \dim G$.*

Proof. Apply the above to $G \to Gx$. □

Corollary 91. *Let $\phi : G \to H$ be a surjective morphism of algebraic groups.*

(i) *ϕ is open*

(ii) $\dim G = \dim H + \dim \ker \phi$

(iii)

$$\phi \text{ is an isomorphism} \iff \phi \text{ and } d\phi \text{ are bijective} \iff \phi \text{ is bijective and separable}$$

Proof. They are homogeneous G-spaces by left-translation, H via ϕ. □

Definition 92. *A sequence of algebraic groups*

$$1 \to K \xrightarrow{\phi} G \xrightarrow{\psi} H \to 1$$

is **exact** *if*
(i) it is exact as sequence of abstract groups and
(ii)

$$0 \to \operatorname{Lie} K \xrightarrow{d\phi} \operatorname{Lie} G \xrightarrow{d\psi} \operatorname{Lie} H \to 0$$

is an exact sequence of Lie algebras (i.e., of vector spaces).

Exercise.
(a) Show that ϕ is a closed immersion if and only if ϕ is injective and $d\phi$ injective.
(b) Suppose that G is connected. Show that ψ is separable if and only if ψ is surjective and $d\psi$ surjective.
(c) Suppose that G is connected. Deduce that the sequence is exact if and only if (i) as above and (ii′) ϕ is a closed immersion and ψ is separable.
(d) If the characteristic of k is 0, show that (i) implies (ii). (Hint: reduce to the case when G is connected.)

Theorem 93 (Weak form of Zariski's Main Theorem). *If $\phi : X \to Y$ is a morphism between irreducible varieties such that Y is smooth, and ϕ is birational (i.e., $k(Y) = k(X)$) and bijective, then ϕ is an isomorphism.*

3.7 Semisimple automorphisms.

Our goal is to show that semisimple conjugacy classes are closed, and to deduce some related results. The following definition is introduced purely for this purpose.

Definition 94. *An automorphism $\sigma : G \to G$ is* **semisimple** *if there is a $G \hookrightarrow \mathrm{GL}_n$ and a semisimple element $s \in \mathrm{GL}_n$ such that $\sigma(g) = sgs^{-1}$ for all $g \in G$.*

Example. If $s \in G_s$, then the inner automorphism $g \mapsto sgs^{-1}$ is semisimple.

Example. Here's an example that is not inner. Consider $G = \mathbb{G}_m^n \cong D_n \leq \mathrm{GL}_n$. Then any "permutation automorphism" $\mathbb{G}_m^n \to \mathbb{G}_m^n$ is semisimple, at least provided the characteristic is 0 or $p > n$.

Definitions 95. *Given a semisimple automorphism of G, define*

$$G_\sigma := \{g \in G \mid \sigma(g) = g\}, \text{ which is a closed subgroup}$$
$$\mathfrak{g}_\sigma := \{X \in \mathfrak{g} := \operatorname{Lie} G \mid d\sigma(X) = X\}$$

Let $\tau : G \to G$, $g \mapsto \sigma(g)g^{-1}$. Then $G_\sigma = \tau^{-1}(e)$ and $d\tau = d\sigma - \mathrm{id}$ by Proposition 76, which implies that $\ker d\tau = \mathfrak{g}_\sigma$. Since $G_\sigma \hookrightarrow G \xrightarrow{\tau} G$ is constant, we have

$$d\tau(\operatorname{Lie} G_\sigma) = 0 \implies \operatorname{Lie} G_\sigma \subset \mathfrak{g}_\sigma$$

Lemma 96.

$$\operatorname{Lie} G_\sigma = \mathfrak{g}_\sigma \iff G \xrightarrow{\tau} \tau(G) \text{ is separable} \iff d\tau : \operatorname{Lie} G \to T_e(\tau(G)) \text{ is surjective}$$

Proof. τ is a G-map of homogeneous spaces, acting by $x * g = \sigma(x)gx^{-1}$ on the codomain. $\tau(G)$ is smooth and is, by Proposition 24, locally closed. Hence, by Theorem 84

$$\begin{aligned} \tau \text{ is separable} &\iff d\tau \text{ is surjective} \\ &\iff \dim \mathfrak{g}_\sigma = \dim \ker d\tau = \dim G - \dim \tau(G) = \dim G_\sigma = \dim \operatorname{Lie} G_\sigma \\ &\iff \mathfrak{g}_\sigma = \operatorname{Lie} G_\sigma \end{aligned}$$

□

Proposition 97. *$\tau(G)$ is closed and* $\operatorname{Lie} G_\sigma = \mathfrak{g}_\sigma$.

Proof. Without loss of generality $G \subset \mathrm{GL}_n$ is a closed subgroup and $\sigma(g) = sgs^{-1}$ for some semisimple $s \in \mathrm{GL}_n$. Without loss of generality, s is diagonal with

$$s = a_1 I_{m_1} \times \cdots \times a_n I_{m_n}$$

with the a_i distinct and $n = m_1 + \cdots + m_n$. Then, extending τ, σ to GL_n, we have

$$(\mathrm{GL}_n)_\sigma = \mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_n} \quad \text{and} \quad (\mathfrak{gl}_n)_\sigma = M_{m_1} \times \cdots \times M_{m_n}$$

So, $\operatorname{Lie}(\mathrm{GL}_n)_\sigma = (\mathfrak{gl}_n)_\sigma$. Hence

$$\begin{array}{ccc} \mathfrak{gl}_n & \xrightarrow{d\tau} & T_e(\tau(\mathrm{GL}_n)) \\ \uparrow & & \uparrow \\ \mathfrak{g} & \xrightarrow[d\tau]{} & T_e(\tau(G)) \end{array}$$

So, if $X \in T_e(\tau(G))$, there is $Y \in \mathfrak{gl}_n$ such that $X = d\tau(Y) = (d\sigma - 1)Y$. But, since $d\sigma : A \mapsto sAs^{-1}$ acts semisimply on $\mathfrak{gl}_n$ and preserves $\mathfrak{g}$, we can write $\mathfrak{gl}_n = \mathfrak{g} \oplus V$, with V a $d\sigma$-stable complement. Without loss of generality, $Y \in \mathfrak{g}$, so $d\tau$ is surjective and $\operatorname{Lie} G_\sigma = \mathfrak{g}_\sigma$.

Consider $S := \{x \in \mathrm{GL}_n \mid \text{(i), (ii), (iii)}\}$ where

(i) $xGx^{-1} = G$, which implies that $\mathrm{Ad}(x)$ preserves $\mathfrak{g}$

(ii) $m(x) = 0$, where $m(T) = \prod_i (T - a_i)$ is the minimal polynomial of s on k^n

(iii) $\mathrm{Ad}(x)$ has the same characteristic polynomial on $\mathfrak{g}$ as $\mathrm{Ad}(s)$

Note that $s \in S$,S is closed (check), and if $x \in S$ then (ii) implies that x is semisimple. G acts on S by conjugation. Define $G_x, \mathfrak{g}_x$ as $G_\sigma, \mathfrak{g}_\sigma$ were defined. Then

$$\mathfrak{g}_x = \{X \in \mathfrak{g} \mid \mathrm{Ad}(x)X = X\}$$

and

$$\dim \mathfrak{g}_x = \text{ multiplicity of eigenvalue 1 in } \mathrm{Ad}(x) \text{ on } \mathfrak{g} \overset{(iii)}{=} \dim g_\sigma$$

and

$$\dim G_x = \dim G_\sigma$$

by what we proved above. The stabilisers of the G-action on S (conjugation) all $G_x, x \in S$, and have the same dimension. This implies that the orbits of G on S all have the same dimension, which further gives that all orbits are closed (Proposition. 24) in S and hence in G. We have

$$\text{orbit of } s = \{gsg^{-1} \mid g \in G\} = \{g\sigma(g^{-1})s \mid g \in G\}$$

and that the map from the orbit to $\tau(G)$ given by $z \mapsto sz^{-1}$ is an isomorphism. □

Corollary 98. *If $s \in G_s$, then $\mathrm{cl}_G(s)$, the conjugacy class of s, is closed and*

$$G \to \mathrm{cl}_G(s), \quad g \mapsto gsg^{-1}$$

is separable.

Remark 99. *The conjugacy class of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in B_2 is not closed!*

Proposition 100. *If a torus D is a closed subgroup of a connected G, then $\mathrm{Lie}\,\mathcal{Z}_G(D) = \mathfrak{z}_{\mathfrak{g}}(D)$, where*

$$\mathcal{Z}_G(D) = \{g \in G \mid dgd^{-1} = g \ \ \forall\, d \in D\} \text{ is the centraliser of } D \text{ in } G\text{, and}$$
$$\mathfrak{z}_{\mathfrak{g}}(D) = \{X \in \mathfrak{g} \mid \mathrm{Ad}(d)(X) = X \ \ \forall\, d \in D\}$$

Note: $\mathcal{Z}_G(D) = \bigcap_{d\in D} G_d$ and $\mathfrak{z}_{\mathfrak{g}}(D) = \bigcap_{d \in D} \mathfrak{g}_d$ (G_d,$\mathfrak{g}_d$ as above) since, for $d \in G_s$ and $\mathrm{Lie}\, G_d = \mathfrak{g}_d$ by above.

Proof. Use induction on $\dim G$. When $G = 1$ this is trivial.
Case 1: If $\mathfrak{z}_{\mathfrak{g}}(D) = \mathfrak{g}$, then $\mathfrak{g}_d = \mathfrak{g}$ for all $d \in D$ so $G_d = G$ for all $d \in D$, implying that $\mathcal{Z}_G(D) = G$.
Case 2: Otherwise, there exists $d \in D$ such that $\mathfrak{g}_d \subsetneq \mathfrak{g}$. Hence, $G_d \subsetneq G$. Also have $D \subset G_d^0$, as D is connected. Note that $\mathcal{Z}_{G_d^0}(D) = \mathcal{Z}_G(D) \cap G_d^0$ has finite index in $\mathcal{Z}_G(D) \cap G_d = \mathcal{Z}_G(D)$ and so their Lie algebras coincide. By induction,

$$\mathrm{Lie}\,\mathcal{Z}_G(D) = \mathrm{Lie}\,\mathcal{Z}_{G_d^0}(D) = \mathfrak{z}_{\mathrm{Lie}\, G_d^0}(D) = \mathfrak{z}_{\mathfrak{g}_d}(D) = \mathfrak{z}_{\mathfrak{g}}(D) \cap \mathfrak{g}_d = \mathfrak{z}_{\mathfrak{g}}(D)$$

□

Proposition 101. *If G is connected, nilpotent, then $G_s \subset \mathcal{Z}_G$ (which implies that G_s is a subgroup).*

Proof. Pick $s \in G_s$ and set $\sigma : g \mapsto sgs^{-1}$ and $\tau : g \mapsto \sigma(g)g^{-1} = [s,g]$. Since G is nilpotent, there is an $n > 0$ such that $\tau^n(g) = [s,[s,\dots,[s,g]\cdots]] = e$ for al $g \in G$ and so

$$\begin{aligned} \tau^n = e &\implies d\tau^n = 0 \\ &\implies d\tau = d\sigma - 1 \text{ is nilpotent, but is also semisimple by above, since } d\sigma \text{ is semisimple} \\ &\implies d\tau = 0 \\ &\implies \tau(G) = \{e\} \text{ as } G \xrightarrow{\tau} \tau(G) \text{ is separable} \\ &\implies sgs^{-1} = g \quad \text{for all } g \in G \end{aligned}$$

□

4. Quotients.

4.1 Existence and uniqueness as a variety.

Given a closed subgroup $H \subset G$, we want to give the coset space G/H the structure of a variety such that $\pi : G \to G/H$, $g \mapsto gH$ is a morphism satisfying a natural universal property.

Proposition 102. *There is a G-representation V and a subspace $W \subset V$ such that*

$$H = \{g \in G \mid gW \subset W\} \quad \text{and} \quad \mathfrak{h} = \operatorname{Lie} H = \{X \in \mathfrak{g} \mid XW \subset W\}$$

(We only need the characterisation of $\mathfrak{h}$ when $\operatorname{char} k > 0$*.)*

Proof. Let $I = I_G(H)$, so that $0 \to I \to k[G] \to k[H] \to 0$. Since $k[G]$ is noetherian, I is finitely-generated; say, $I = (f_1, \dots, f_n)$. Let $V \supset \sum k f_i$ be a finite-dimensional G-stable subspace of $k[G]$ (with G acting by right translation). This gives a G-representation $\rho : G \to \mathrm{GL}(V)$. Let $W = V \cap I$. If $g \in H$, then $\rho(g)I \subset I \implies \rho(g)W \subset W$. Conversely,

$$\begin{aligned} \rho(g)W \subset W &\implies \rho(g)(f_i) \in I \ \forall\, i \\ &\implies \rho(g)I \subset I, \quad \text{as } \rho(g) \text{ is a ring morphism } k[G] \to k[G] \\ &\implies g \in H \quad (\text{easy exercise. Note that } \rho(g)I = I_G(Hg^{\pm 1})) \end{aligned}$$

Moreover, if $X \in \mathfrak{h}$, then $d\rho(X)W \subset W$ from the above. For the converse $d\rho(X)W \subset W \implies X \in \mathfrak{h}$, we first need a lemma.

Lemma 103. $d\rho(X)f = D_X(f) \ \ \forall X \in \mathfrak{g}, f \in V$

Proof. We know (Proposition 79) that $d\rho(X)f = d\mathfrak{o}_f(X)$, identifying V with $T_f V$, where

$$\mathfrak{o}_f : G \to V, \quad g \mapsto \rho(g)f$$

That is, for all $f^\vee \in V^*$

$$\langle d\rho(X)f, f^\vee \rangle = \langle d\mathfrak{o}_f(X), f^\vee \rangle$$

Extend any $f^\vee$ to $k[G]^*$ arbitrarily. We need to show that

$$\langle d\mathfrak{o}_f(X), f^\vee \rangle = \langle D_X(f), f^\vee \rangle$$

or, equivalently,

$$X(\mathfrak{o}_f^*(f^\vee)) = \langle d\mathfrak{o}_f(X), f^\vee \rangle = \langle D_X(f), f^\vee \rangle = (1, X)\Delta f, f^\vee \rangle = (f^\vee, X)\Delta f.$$

We have

$$\mathfrak{o}_f^*(f^\vee) = f^\vee \circ \mathfrak{o}_f : g \mapsto \langle \rho(g)f, f^\vee \rangle = \langle f(\,\cdot\, g), f^\vee \rangle = \langle (\mathrm{id}, \mathrm{ev}_g)\Delta f, f^\vee \rangle = (f^\vee, \mathrm{ev}_g)\Delta f$$

and so

$$\mathfrak{o}_f^*(f^\vee) = (f^\vee, \mathrm{id})\Delta f \implies X(\mathfrak{o}_f^*(f^\vee)) = (f^\vee, X)\Delta f$$

□

Now,

$$\begin{aligned} d\rho(X)W \subset W &\implies D_X(f_i) \in I \quad \forall\, i \\ &\implies D_X(I) \subset I \qquad (\text{as } D_X \text{ is a derivation}) \\ &\implies X(I) = 0 \qquad \text{easy exercise} \end{aligned}$$

which implies that X factors through $k[H]$:

$$\begin{array}{ccc} k[G] & \twoheadrightarrow & k[H] \\ & \searrow^{X} & \downarrow{\scriptstyle \overline{X}} \\ & & k \end{array}$$

It is easy to see that $\overline{X}$ is a derivation, which means that $X \in \mathfrak{h}$.

Corollary 104. *We can even demand* $\dim W = 1$ *in Proposition 102 above.*

Proof. Let $d = \dim W$, $V' = \Lambda^d V$, and $W' = \Lambda^d W$, which has dimension 1 and is contained in V'. We have actions

$$\begin{aligned} g(v_1 \wedge \cdots \wedge v_d) &= gv_1 \wedge \cdots \wedge gv_d \\ X(v_1 \wedge \cdots \wedge v_d) &= (Xv_1 \wedge \cdots \wedge v_d) + (v_1 \wedge Xv_2 \wedge \cdots \wedge v_d) + \cdots + (v_1 \wedge \cdots \wedge Xv_d) \end{aligned}$$

We need to show that

$$\begin{aligned} gW' \subset W' &\iff gW \subset W \\ XW' \subset W' &\iff XW \subset W \end{aligned}$$

which is just a lemma in linear algebra (see Springer). □

Corollary 105. *There is a quasiprojective homogeneous space X for G and $x \in X$ such that*

(i) $\mathrm{Stab}_G(x) = H$

(ii) *If* $\mathfrak{o}_x : G \to X$, $g \mapsto gx$, *then*

$$0 \to \mathrm{Lie}\, H \to \mathrm{Lie}\, G \xrightarrow{d\mathfrak{o}_x} T_x X \to 0$$

is exact.

Note that (ii) follows from (i) if char $k = 0$ (use Corollaries 85 and 89.)

Proof. Take a line $W \subset V$ as in the corollary above. Let $x = [W] \in \mathbf{P}V$ and let $X = Gx \subset \mathbf{P}V$. X is a subvariety and is a quasiprojective homogeneous space. Then (i) is clear. $\square$

Exercise. The natural map $\phi : V - \{0\} \to \mathbf{P}V$ induces an isomorphism

$$V/x \cong T_v V/x \cong T_x(\mathbf{P}V)$$

for all $x \in \mathbf{P}V$ and $v \in \phi^{-1}(x)$. (Hint:

$$k^\times \xrightarrow{\lambda \mapsto \lambda v} V - \{0\} \xrightarrow{\phi} \mathbf{P}V$$

is constant. Use an affine chart in $\mathbf{P}V$ to prove that $d\phi$ is surjective.)

Claim. $\ker(d\mathfrak{o}_x) = \mathfrak{h}$ *(then (ii) follows by dimension considerations.)*

Fix $v \in \phi^{-1}(x)$.

$$\begin{aligned} \phi \circ \mathfrak{o}_x : G &\xrightarrow{g \mapsto (\rho(g), v)} \mathrm{GL}(V) \times (V - \{0\}) \xrightarrow{(\rho(g),v) \mapsto \rho(g)v} V - \{0\} \xrightarrow{\phi : \rho(g)v \mapsto [\rho(g)v]} \mathbf{P}V \\ d\phi \circ d\mathfrak{o}_x : \mathfrak{g} &\xrightarrow{X \mapsto (d\rho(X), 0)} \mathrm{End}(V) \oplus V \xrightarrow{(d\rho(X),0) \mapsto d\rho(X)v} V \xrightarrow{d\phi : d\rho(X)v \mapsto [d\rho(X)v]} V/x. \end{aligned}$$

We have

$$[d\phi(X)v] = 0 \iff XW \subset W \iff X \in \mathfrak{h}$$

$\square$

Definition 106. *If $H \subset G$ is a closed subgroup (not necessarily normal). A* **quotient** *of G by H is a variety G/H together with a morphism $\pi : G \to G/H$ such that*
(i) π is constant on H-cosets, i.e., $\pi(g) = \pi(gh)$ for all $g \in G, h \in H$, and
(ii) if $G \to X$ is a morphism that is constant on H-cosets, then there exists a unique morphism $G/H \to X$ such that

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H \\ \downarrow & \swarrow & \\ X & & \end{array}$$

commutes. Hence, if a quotient exists, it is unique up to unique isomorphism.

Theorem 107. *A quotient of G by H exists; it is quasiprojective. Moreover,*

(i) *$\pi : G \to G/H$ is surjective whose fibers are the H-cosets.*

(ii) *G/H is a homogeneous G-space under*

$$G \times G/H \to G/H, \quad (g, \pi(\gamma)) \mapsto \pi(g\gamma)$$

Proof. Let $G/H = \{\text{cosets } gH\}$ as a set with natural surjection $\pi : G \to G/H$ and give it the quotient topology (so that G/H is the quotient in the category of topological spaces). π is open. For $U \subset G/H$ let $\mathcal{O}_{G/H}(U) := \{f : U \to k \mid f \circ \pi \in \mathcal{O}_G(\pi^{-1}(U))\}$. Easy check: $\mathcal{O}_{G/H}$ is a sheaf of k-valued functions on G/H and so $(G/H, \mathcal{O}_{G/H})$ is a ringed space.

If $\phi : G \to X$ is a morphism constant on H-cosets, then we get

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H \\ {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle \exists!} & \\ X & & \end{array}$$

in the category of *ringed spaces* .

By the second corollary 105 to Proposition 102 there is a quasiprojective homogeneous space X of G and $x \in X$ such that

(i) $\mathrm{Stab}_G(x) = H$

(ii) If $\mathfrak{o}_x : G \to X$, $g \mapsto gx$, then

$$0 \to \mathrm{Lie}\, H \to \mathrm{Lie}\, G \xrightarrow{d\mathfrak{o}_x} T_x X \to 0$$

is exact.

Since $\mathfrak{o}_x$ is constant on H-cosets, we get a map $\psi : G/H \to X$ of ringed spaces (from the above universal property). ψ is necessarily given by $gH \mapsto gx$ and is bijective. If we show that ψ is an isomorphism of ringed spaces and that $(G/H, \mathcal{O}_{G/H})$ is a variety, then the theorem follows.

ψ is a homeomorphism:
We need only show that ψ is open. If $U \subset G/H$ is open then

$$\psi(U) = \psi(\pi(\pi^{-1}(U))) = \phi(\pi^{-1}(U))$$

is open, as ϕ is an open map (by Corollary 89).

ψ gives an isomorphism of sheaves:
We must show that for $V \subset X$ open

$$\mathcal{O}_X(V) \to \mathcal{O}_{G/H}(\psi^{-1}(V))$$

is an isomorphism of rings. Clearly it is injective. To get surjectivity we need that for all $f : V \to k$

$$f \circ \phi : \phi^{-1}(V) \to k \quad \text{regular} \implies f \text{ regular}$$

Since

$$\begin{array}{ccc} G & \xrightarrow{\pi} & G/H \\ {\scriptstyle\phi}\downarrow & \swarrow{\scriptstyle \psi} & \\ X & & \end{array}$$

and ψ is a homeomorphism, we need only focus on (X, ϕ). A lemma:

Lemma 108. *Let X, Y be irreducible varieties and $f : X \to Y$ a map of sets. If f is a morphism, then the graph $\Gamma_f \subset X \times Y$ is closed. The converse is true if X is smooth if Γ_f is irreducible, and $\Gamma_f \to X$ is separable.*

Proof.

($\Rightarrow$:) If f is a morphism, then $\Gamma_f = \theta^{-1}(\Delta_Y)$ is closed, where

$$\theta : X \times Y \to Y \times Y, \quad (x, y) \mapsto (f(x), y).$$

($\Leftarrow$:) We have

$$\begin{array}{ccc} \Gamma_f & \hookrightarrow & X \times Y \\ {\scriptstyle\eta}\downarrow & \swarrow & \\ X & & \end{array}$$

with $\Gamma_f \hookrightarrow X \times Y$ the closed immersion.

$$\eta \text{ bijective} \overset{87}{\Longrightarrow} \dim \Gamma_f = \dim X \text{ and } 1 = [k(\Gamma_f) : k(X)]_s = [k(\Gamma_f) : k(X)]$$

as η is separable. Hence η is birational and bijective with X smooth, meaning that η is an isomorphism by Theorem 93 and

$$f : X \xrightarrow{\eta^{-1}} \Gamma_f \to Y$$

is a morphism. □

Now, for simplicity, assume that G is connected, which implies that $X, V, \phi^{-1}(V)$ are irreducible. (For the general case, see Springer.) Suppose that $f \circ \phi$ is regular. It follows from the lemma that $\Gamma_{f\circ\phi} \subset \phi^{-1}(V) \times \mathbf{A}^1$ is closed, surjecting onto Γ_f via $\phi \times \mathrm{id}$. By Corollary 89 , $\phi : G \to X$ is "universally open" and so

$$V \times \mathbf{A}^1 - \Gamma_f = (\phi \times \mathrm{id})(\phi^{-1}(V) \times \mathbf{A}^1 - \Gamma_{f\circ\phi})$$

is open: Γ_f is closed. (The point is that $\Gamma_{f\circ\phi}$ is a union of fibers of $\phi \times \mathrm{id}$.)

Also, $\Gamma_{f\circ\phi} \cong \phi^{-1}(V)$ is irreducible, implying that Γ_f is irreducible, and

$$\begin{array}{ccc} \Gamma_{f\circ\phi} & \xrightarrow[\mathrm{pr}_1]{\sim} & \phi^{-1}(V) \\ \downarrow & & \downarrow \\ \Gamma_f & \xrightarrow[\mathrm{pr}_1]{} & V \end{array}$$

and

$$d\phi \text{ surjective} \implies d(\mathrm{pr}_1) \text{ surjective} \implies \Gamma_f \to V \text{ separable and } V \text{ smooth.}$$

By Lemma 108, f is a morphism. □

Corollary 109. (i) $\dim(G/H) = \dim G - \dim H$

(ii)

$$0 \to \mathrm{Lie}\, H \to \mathrm{Lie}\, G \xrightarrow{d\pi} T_e(G/H) \to 0$$

is exact.

Proof.
(i): G/H is a homogeneous with stabilisers equal to H.
(ii): Implied by Corollary 105. $\square$

Lemma 110. *Let $H_1 \subset G_1$, $H_2 \subset G_2$ be closed subgroups. The natural map*

$$(G_1 \times G_2)/(H_1 \times H_2) \to G_1/H_1 \times G_2/H_2$$

is an isomorphism.

Proof. This is a bijective map of homogeneous $G_1 \times G_2$ spaces, which is bijective on tangent spaces by the above. The rest follows from Corollary 91. $\square$

4.2 Quotient algebraic groups.

Proposition 111. *Suppose that $N \trianglelefteq G$ is a closed normal subgroup. Then G/N is an algebraic group that is affine (and $\pi : G \to G/N$ is a morphism of algebraic groups).*

Proof. Inversion $G/N \to G/N$ is a morphism, along with multiplication $G/N \times G/N \to G/N$ by Lemma 110, which gives that G/N is an algebraic group.

By Corollary 104, there exists a G-representation $\rho : G \to \mathrm{GL}(V)$ and a line $L \subset V$ such that $N = \mathrm{Stab}_G(L)$ and $\mathrm{Lie}\, N = \mathrm{Stab}_{\mathfrak{g}}(L)$. For $\chi \in X^*(N) = \mathrm{Hom}(N, \mathbb{G}_m)$, let V_χ be the χ-eigenspace of V. (Note that $L \subset V_\chi$ for some χ.) Let $V' = \sum_{\chi \in X^*(N)} V_\chi = \bigoplus_\chi V_\chi$ (by linear independence of characters). As $N \trianglelefteq G$, G permutes the V_χ. Define

$$W = \{f \in \mathrm{End}(V) \mid f(V_\chi) \subset V_\chi \ \ \forall\, \chi\} \subset \mathrm{End}(V).$$

Let $\sigma : G \to \mathrm{GL}(W)$ by

$$\sigma(g)f := \rho(g) f \rho(g)^{-1}$$

which is an algebraic representation.

Claim. *σ induces a closed immersion $G/N \hookrightarrow \mathrm{GL}(W)$.*
It is enough to show that $\ker \sigma = N$ and $\ker(d\sigma) = \mathrm{Lie}\, N$.

$$\begin{aligned} g \in \ker \sigma &\iff \rho(g) f = f \rho(g) \\ &\iff \rho(g) \text{ acts as a scalar on each } V_\chi \\ &\implies \rho(g) L = L \quad \text{as } L \subset V_\chi \text{ for some } \chi \\ &\implies g \in N \end{aligned}$$

The converse is trivial: $\ker \sigma = N$.

By Proposition 79, $\phi_f : G \to W$, $g \mapsto \sigma(g) f$ has derivative

$$d\phi_f : \mathfrak{g} \to W, \quad X \mapsto d\sigma(X) f.$$

Check that $d\sigma(X)f = d\rho(X)f - fd\rho(X)$. We have

$$\begin{aligned} d\sigma(X) = 0 &\iff d\rho(X)f = fd\rho(X) \quad \text{for all } f \in W \\ &\iff d\rho(X) \text{ acts as a scalar on each } V_\chi \\ &\implies X \in \operatorname{Lie} N \text{ (as above).} \end{aligned}$$

$\square$

Corollary 112. *Suppose $\phi : G \to H$ is a morphism of algebraic groups with $\phi(N) = 1$, $N \trianglelefteq G$ closed. Then we have a unique factorisation in the category of algebraic groups,*

$$\begin{array}{ccc} G & \longrightarrow & G/N \\ & \searrow^{\phi} & \downarrow \\ & & H \end{array}$$

In particular, we get that $G/\ker\phi \to \operatorname{im}\phi$ is bijective and is an isomorphism when in characteristic 0.

(Note that in characteristic p, $\mathbf{G}_m \xrightarrow{p} \mathbf{G}_m$ is bijective and not an isomorphism.)

Remark 113.

$$1 \to N \to G \to G/N \to 1$$

is exact by Corollary 109.

Exercise. If $N \subset H \subset G$ are closed subgroups with $N \trianglelefteq G$, then the natural map $H/N \to G/N$ is a closed immersion (so we can think of H/N as a closed subgroup of G/N) and we have a canonical isomorphism $(G/N)/(H/N) \xrightarrow{\sim} G/H$ of homogeneous G-spaces.

Exercise. Assume that $\operatorname{char} k = 0$. Suppose $N, H \subset G$ are closed subgroups such that H normalises N. Show that HN is a closed subgroup of G and that we have a canonical isomorphism $HN/N \cong H/(H \cap N)$ of algebraic groups. Find a counterexample when $\operatorname{char} k > 0$.

Exercise. Suppose H is a closed subgroup of an algebraic group G. Show that if both H and G/H are connected, then G is connected. (Use, for example, Exercise 5.5.9(1) in Springer.) Variant: Show that if $\varphi : G \to H$ is a homomorphism such that $\ker\varphi$ and $\operatorname{im}\varphi$ are connected, then G is connected. (Hint: show that $\varphi(G^0) = \operatorname{im}\varphi$.)

Exercise. Assume that $\operatorname{char} k = 0$. Suppose $\phi : G \to H$ is a surjective morphism of algebraic groups. If $H_1 \subset H_2 \subset H$ are closed subgroups, show that the map ϕ induces a canonical isomorphism $\phi^{-1}(H_2)/\phi^{-1}(H_1) \xrightarrow{\sim} H_2/H_1$. Find a counterexample when $\operatorname{char} k > 0$.

Example. The group $\mathbf{PGL}_2$:
Let $Z = \{\left(\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right) \mid x \in \mathbf{G}_m\}$. GL_2/Z is affine and the composition

$$\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2 \twoheadrightarrow \mathrm{GL}_2/Z$$

is surjective, inducing the inclusion of Hopf algebras

$$k[\mathrm{GL}_2]^Z = k[\mathrm{GL}_2/Z] \hookrightarrow k[\mathrm{SL}_2].$$

Check that the image is generated by the elements $\frac{T_i T_j}{\det}$, $1 \leq i, j \leq 4$. (See Springer Exercise 2.1.5(3).)

5. Parabolic and Borel subgroups.

5.1 Complete varieties.

Recall: A variety X is **complete** if for all varieties Z, $X \times Z \xrightarrow{\text{pr}_2} Z$ is a closed map. In the category of locally compact Hausdorff topological spaces, the analogous property is equivalent to compactness.

Proposition 114. *Let X be complete.*

(i) $Y \subset X$ *closed* $\implies$ Y *complete.*

(ii) Y *complete* $\implies$ $X \times Y$ *complete*

(iii) $\phi : X \to Y$ *morphisms* $\implies$ $\phi(X) \subset Y$ *is closed and complete, which implies that if* $X \subset Z$ *is a subvariety, then* X *is closed in* Z

(iv) X *irreducible* $\implies$ $\mathcal{O}_X(X) = k$

(v) X *affine* $\implies$ X *finite*

Proof. An exercise (or one can look in Springer). □

Theorem 115. X *projective* $\implies$ X *complete*

Note: The converse is not true.

Lemma 116. *Let X, Y be homogeneous G-spaces with $\phi : X \to Y$ a bijective G-map. Then X is complete* $\iff$ *Y is complete.*

Note that such a map is an isomorphism if the characteristic of k is 0.

Proof. For all varieties Z, then projection $X \times Z \to Z$ can be factored as

$$X \times Z \xrightarrow{\phi \times \text{id}} Y \times Z \xrightarrow{\text{pr}_2} Z$$

$\phi \times \text{id}$ is bijective and open (by Corollary 89) and is thus a homeomorphism: Y being complete implies that in X. Applying the same reasoning to $\phi^{-1} : Y \to X$ gives the converse. □

Definition 117. *A closed subgroup $P \subset G$ is* **parabolic** *if G/P is complete.*

Remark 118. *For a closed subgroup $P \subset G$, G/P is quasi-projective by Theorem 107 and so*

$$G/P \text{ projective} \iff G/P \text{ complete} \iff P \text{ parabolic.}$$

The implication of G/P being complete implying that G/P being projective follows from Proposition 114 (iii) applying to the embedding of G/P into some projective space.

Proposition 119. *If $Q \subset P$ and $P \subset G$ are parabolic, then $Q \subset G$ is parabolic.*

Proof. For all varieties Z we need to show that $G/Q \times Z \xrightarrow{\mathrm{pr}_2} Z$ is closed. Fix a closed subset $C \subset G/Q \times Z$. Letting $\pi : G \to G/P$ denote the natural projection, set $D = (\pi \times \mathrm{id}_Z)^{-1}(C) \subset G \times Z$, which is closed. For all $q \in Q$, note that $(g, z) \in D \implies (gq, z) \in D$. It is enough to show that $\mathrm{pr}_2(D) \subset Z$ is closed.

Let

$$\theta : P \times G \times Z \to G \times Z, \quad (p, g, z) \mapsto (gp, z)$$

Then $\theta^{-1}(D)$ is closed for all $q \in Q$

$$(*) \qquad (p, g, z) \in \theta^{-1}(D) \implies (pq, g, z) \in \theta^{-1}(D)$$

Let $\alpha : P \times G \times Z \to P/Q \times G \times Z$ be the natural map.

$$\begin{array}{ccc} P \times G \times Z & \xrightarrow{\alpha} & P/Q \times G \times Z \\ & \searrow^{\mathrm{pr}_{23}} & \downarrow \mathrm{pr}_{23} \\ & & G \times Z \end{array}$$

By Corollary 89, α is open. By passing to complements, $(*)$ implies that $\alpha(\theta^{-1}(D))$ is closed. P/Q being complete implies that

$$\mathrm{pr}_{23}(\theta^{-1}(D)) = \{(gp^{-1}, z) \mid (g, z) \in D, p \in P\}$$

is closed. Now,

$$\begin{array}{ccc} G \times Z & \xrightarrow{\beta} & G/P \times Z \\ & \searrow^{\mathrm{pr}_{2}} & \downarrow \mathrm{pr}_{2} \\ & & G \times Z \end{array}$$

Similarly β is open, and so $\beta(\mathrm{pr}_{23}(\theta^{-1}(D)))$ is closed. G/P being complete implies

$$\mathrm{pr}_2(\beta(\mathrm{pr}_{23}(\theta^{-1}(D)))) = \mathrm{pr}_2(\mathrm{pr}_{23}(\theta^{-1}(D))) = \mathrm{pr}_2(D) = \mathrm{pr}_2(C)$$

is closed. □

5.2 Borel subgroups.

Theorem 120 (Borel's fixed point theorem)**.** *Let G be a connected, solvable algebraic group and X a (nonempty) complete G-space. Then X has a fixed point.*

Proof. We show this by inducting on the dimension of G. When $\dim G = 0 \implies G = \{e\}$ the theorem trivially holds. Now, let $\dim G > 0$ and suppose that the theorem holds for dimensions less than $\dim G$. Let $N = [G, G] \trianglelefteq G$, which is a connected normal subgroup by Proposition 19 and is a proper subgroup as G is solvable. Since N is connected and solvable, by induction

$$X^N = \{x \in X \mid nx = x \ \forall n \in N\} \neq \varnothing$$

Since $X^N \subset X$ is closed (both topologically and under the action of G, as N is normal), by Proposition 114, X^N is complete; so, without loss of generality suppose that N acts trivially on X. Pick a closed orbit $Gx \subset X$, which exists by Proposition 24 and is complete. Since $G/\mathrm{Stab}_G(x) \to Gx$ is a bijective map of homogeneous G-spaces, $G/\mathrm{Stab}_G(x)$ is complete by Proposition 116.

$$\begin{aligned} N \subset \mathrm{Stab}_G(x) &\implies \mathrm{Stab}_G(x) \text{ is normal} \\ &\implies G/\mathrm{Stab}_G(x) \text{ is affine and complete (and connected)} \\ &\implies G/\mathrm{Stab}_G(x) \text{ is a point, by Proposition 114} \\ &\implies x \in X^G \end{aligned}$$

□

Proposition 121 (Lie-Kolchin)**.** *Suppose that G is connected and solvable. If $\phi : G \to \mathrm{GL}_n$, then there exists $\gamma \in \mathrm{GL}_n$ such that $\gamma(\mathrm{im}\,\phi)\gamma^{-1} \subset B_n$.*

Proof. Induct on n. When $n = 1$, then theorem trivially holds. Let $n > 1$ and suppose that it holds for all $m < n$. Write $\mathrm{GL}_n = \mathrm{GL}(V)$ for an n-dimensional vector space V. G acts on $\mathbf{P}V$ via ϕ. By Borel's fixed point theorem, there exists $v_1 \in V$ such that G stabilises the line $V_1 := kv_1 \subset V$, implying that G acts on V/V_1. By induction there exists a flag

$$0 = V_1/V_1 \subsetneq V_2/V_1 \subsetneq \cdots \subsetneq V/V_1$$

stabilised by G; hence G stabilises the flag

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

□

Remark 122. *Both of the above results need G connected. It's easy to find counterexamples with G finite otherwise.*

Definition 123. *A* **Borel** *subgroup of G is a maximal connected solvable closed subgroup B of G.*

Remarks 124.
- *Any G has a Borel subgroup since if $B_1 \subsetneq B_2$ is irreducible $\implies \dim B_1 < \dim B_2$.*
- *$B_n \subset \mathrm{GL}_n$ is a Borel by Lie-Kolchin.*

Theorem 125.

(i) *A closed subgroup $P \subset G$ is parabolic $\iff$ P contains a Borel subgroup.*

(ii) *Any two Borel subgroups are conjugate.*

In particular, a Borel subgroup is precisely a minimal – or, equivalently, a connected, solvable – parabolic.

Remark 126. *We will soon see that any parabolic subgroup is connected (Theorem 152).*

Proof. For simplicity, assume that G is connected.
(i) ($\Rightarrow$): Suppose that B is a Borel and P is parabolic. B acts on G/P. By the Borel fixed point theorem, there is a coset gP such that $Bg \subset gP \implies g^{-1}Bg \subset P$. $g^{-1}Bg$ is Borel.

(i) ($\Leftarrow$): Let B be a Borel. We first show that B is parabolic, inducting on $\dim G$. Pick a closed immersion $G \hookrightarrow \mathrm{GL}(V)$. G acts on $\mathbf{P}V$. Let Gx be a closed – hence complete – orbit. Since $G/\mathrm{Stab}_G(x) \to Gx$ is a bijective map of homogeneous spaces, $P := \mathrm{Stab}_G(x)$ is parabolic. By above, $B \subset gPg^{-1}$, for some $g \in G$. Without loss of generality, $B \subset P$. If $P \neq G$, then B is Borel in P. Since $P \subset G$ is parabolic and $B \subset P$ is parabolic by induction, it follows that $B \subset G$ is parabolic, by Proposition 119. Suppose $P = G$. G stabilises some line $V_1 \subset V$, which gives a morphism $G \to \mathrm{GL}(V/V_1)$. By induction on $\dim V$, we either obtain a proper parabolic subgroup, in which case we are done by the above, or G stabilises some flag $0 \subset V_1 \subset \cdots V_n = V$, giving that

$$G \hookrightarrow B_n \implies G \text{ is solvable } \implies G = B \text{ is parabolic}$$

Now, suppose that P is a closed subgroup containing a Borel B. Then $G/B \twoheadrightarrow G/P$. Since G/B is complete, by Proposition 114 we get that G/P is complete $\implies$ P is parabolic.

(ii). Let B_1, B_2 be Borel subgroups, which are parabolic by (i). By (i), there is $g \in G$ such that $gB_1g^{-1} \subset B_2 \implies \dim B_1 \leqslant \dim B_2$. Similarly,

$$\dim B_2 \leqslant \dim B_1 \implies \dim B_1 = \dim B_2 \implies gB_1g^{-1} = B_2$$

□

Corollary 127. *Let $\phi : G \to G'$ be a surjective morphism of algebraic groups.*

(i) *If $B \subset G$ is Borel, then $\phi(B) \subset G'$ is Borel.*

(ii) *If $P \subset G$ is parabolic, then $\phi(P) \subset G'$ is parabolic.*

Proof. It is enough to prove (i). Since $B \twoheadrightarrow \phi(B)$, $\phi(B)$ is connected and solvable. Since G/B is complete and $G/B \twoheadrightarrow G'/\phi(B)$ it follows that $G'/\phi(B)$ is complete and $\phi(B)$ is parabolic. Now, $\phi(B)$ is connected, solvable, and contains a Borel: $\phi(B)$ is Borel by the maximality in the definition of a Borel subgroup. □

Corollary 128. *If G is connected and $B \subset G$ a Borel, then $\mathcal{Z}_G^0 \subset \mathcal{Z}_B \subset \mathcal{Z}_G$.*

Remark 129. *We will soon see that $\mathcal{Z}_B = \mathcal{Z}_G$ (see Prop. 149).*

Proof.

$$\begin{aligned} \mathcal{Z}_G^0 \text{ connected, solvable } &\implies \mathcal{Z}_G^0 \subset gBg^{-1}, \text{ for some } g \in G \\ &\implies \mathcal{Z}_G^0 = g^{-1}\mathcal{Z}_G^0 g \subset B \\ &\implies \mathcal{Z}_G^0 \subset \mathcal{Z}_B \end{aligned}$$

Now, fix $b \in \mathcal{Z}_B$ and define the morphism $\phi : G/B \to G$ of varieties by $gB \mapsto gbg^{-1}$. $\phi(G/B)$ is complete and closed – hence affine – and irreducible, hence a point:

$$\phi(G/B) = \{b\} \implies \forall\, g \in G, gbg^{-1} = b \implies b \in \mathcal{Z}_G \implies \mathcal{Z}_B \subset \mathcal{Z}_G$$

□

Proposition 130. *Let G be a connected group and $B \subset G$ a Borel. If B is nilpotent, then G is solvable; that is, B nilpotent $\implies B = G$.*

Proof. If $B = 1$, then $G = G/B$ is complete, connected, and affine, hence $G/B = 1$, so $G = B$.
If $B \neq 1$: B being nilpotent means that

$$B \supsetneq \mathcal{C}B \supsetneq \cdots \supsetneq \mathcal{C}^n B = 1$$

for some $n > 0$ (where $\mathcal{C}^i B = [B, \mathcal{C}^{i-1}B]$ is connected and closed). Let $N = \mathcal{C}^{n-1}B$, so that

$$1 = [B, N] \implies N \subset \mathcal{Z}_B \subset \mathcal{Z}_G \text{ (above corollary)} \implies N \trianglelefteq G$$

Hence we have the morphism $B/N \hookrightarrow G/N$ of algebraic groups, which is a closed immersion by the exercise after Theorems 87, 88. Also, B/N is a Borel of G/N, by the corollary above, and B/N is nilpotent.

Inducting on $\dim G$, we get that G/N is solvable, which implies that G is solvable. □

5.3 Structure of solvable groups.

Proposition 131. *Let G be connected and nilpotent. Then G_s, G_u are (connected) closed normal subgroups and $G_s \times G_u \xrightarrow{\text{mult.}} G$ is an isomorphism of algebraic groups. Moreover, G_s is a central torus.*

Remark 132. *This generalises Proposition 37 from the commutative case (at least when G is connected).*

Proof. Without loss of generality, $G \subset \mathrm{GL}(V)$ is a closed subgroup. By Proposition 101 $G_s \subset \mathcal{Z}_G$. The eigenspaces of elements G_s coincide; let $V = \bigoplus_{\lambda: G_s \to k^\times} V_\lambda$ be a simultaneous eigenspace decomposition. Since G_s is central, G preserves each V_λ. By Lie-Kolchin (Proposition 121), we can choose a basis for each V_λ such that the G-action is upper-triangular. Therefore, $G \subset B_n$, and $G_s = G \cap D_n$, $G_u = G \cap U_n$ are closed subgroups, G_u being normal. We can now show that $G_s \times G_u \xrightarrow{\sim} G$ as in the proof of Proposition 37. Moreover, G_s is a torus, being connected and commutative. □

Proposition 133. *Let G be connected and solvable.*

(i) *$[G, G]$ is a connected, normal closed subgroup and is unipotent.*

(ii) *G_u is a connected, normal closed subgroup and G/G_u is a torus.*

Proof.
(i).

$$\begin{aligned} \text{Lie-Kolchin} &\implies G \hookrightarrow B_n \\ &\implies [G,G] \hookrightarrow [B_n, B_n] \subset U_n \\ &\implies [G,G] \text{ unipotent} \end{aligned}$$

We already know that it is connected, closed, and normal.

(ii). $G_u = G \cap U_n$ is a closed subgroup. $G_u \supset [G,G]$ implies that $G_u \trianglelefteq G$ and that G/G_u is commutative. For $[g] \in G/G_u$, $[g] = [g_s] = [g]_s$: all elements of G/G_u are semisimple. Since G/G_u is furthermore connected, it follows that G/G_u is a torus. It now remains to show that G_u is connected.

$$1 \to G_u/[G,G] \to G/[G,G] \to G/G_u \to 1$$

is exact (by the exercise on exact sequences). By Proposition 37,

$$G/[G,G] \cong (G/[G,G])_s \times (G/[G,G])_u$$

Hence $(G/[G,G])_u = G_u/[G,G]$, which is connected by the above. Since $[G,G]$ is also connected, it follows from Springer 5.5.9(1) (exercise) that G_u is connected. □

Lemma 134. *Let G be connected and solvable with $G_u \neq 1$. Then there exists a closed subgroup $N \subset \mathcal{Z}_{G_u}$ such that $N \cong \mathbf{G}_a$ and $N \trianglelefteq G$.*

Proof. Since G_u is unipotent, it is nilpotent. Let $n > 0$ be such that

$$G_u \supsetneq \mathcal{C}G_u \supsetneq \cdots \supsetneq \mathcal{C}^n G_u = 1.$$

The $\mathcal{C}^i G_u$ are connected closed subgroups and are normal as G_u is normal. Let $N = \mathcal{C}^{n-1}G_u$. Then

$$1 = [G_u, N] \implies N \subset \mathcal{Z}_{G_u},$$

in particular N is commutative. If char $k = p > 0$, let $N \hookrightarrow U_m$, for some m, and let r be minimal such that $p^r \geqslant m$ so that $N^{p^r} = 1$. Then (perhaps for a different $r > 0$),

$$N \supsetneq N^p \supsetneq \cdots \supsetneq N^{p^r} = 1.$$

The N^{p^i} are connected, closed, and normal in G. Replace N by $N^{p^{r-1}}$. Then WLOG N is a connected elementary unipotent group and hence is isomorphic to $\mathbf{G}_a^r$ for some r, by Corollary 59.

G act on N by conjugation, with G_u acting trivially. This induces an action $G/G_u \times N \to N$ (use Lemma 110). G/G_u acts on $k[N]$ in a locally algebraic manner, preserving the non-zero subspace $\mathrm{Hom}(N, \mathbf{G}_a) = \mathcal{A}(N)$. Since G/G_u is a torus, there is a nonzero $f \in \mathrm{Hom}(N, \mathbf{G}_a)$ that is a simultaneous eigenvector. So, $(\ker f)^0 \subset N$ has dimension $r-1$ and is still normal in G. Induct on r. □

Definitions 135. *A* **maximal torus** *of G is a closed subgroup that is a torus and is a maximal such subgroup with respect to inclusion; they exist by dimension considerations. A temporary definition: a torus T of a connected solvable group is* **Maximal** *(versus* m*aximal) if $\dim T = \dim(G/G_u)$. (Recall that G/G_u is a torus.) It is easy to see that Maximal $\implies$ maximal. We shall soon see that the converse is true as well, after a corollary to the following theorem (so that we can then dispense with the capital M):*

Theorem 136. *Let G be connected and solvable.*

(i) *Any semisimple element lies in a Maximal torus. (In particular, Maximal tori exist.)*

(ii) *$\mathcal{Z}_G(s)$ is connected for all semisimple s.*

(iii) *Any two Maximal tori are conjugate in G.*

(iv) *If T is a Maximal torus, then $G \cong G_u \rtimes T$ (i.e., $G_u \trianglelefteq G$ and $G_u \times T \xrightarrow{\text{mult.}} G$ is an isomorphism of varieties).*

Proof.
(iv): Let T be Maximal and consider $\phi : T \to G/G_u$. Since $\ker\phi = T \cap G_u = 1$ (Jordan decomposition), we have that

$$\dim\phi(T) = \dim T - \dim\ker\phi = \dim T = \dim G/G_u \implies \phi(T) = G/G_u :$$

ϕ is surjective and so $G = TG_u$. Thus multiplication $T \times G_u \to G$ is a bijective map of homogeneous $T \times G_u$-spaces. To see that it is an isomorphism, (if $p > 0$) we need an isomorphism – just an injection by dimension considerations – on Lie algebras, which is equivalent to $\operatorname{Lie} T \cap \operatorname{Lie} G_u = 0$, as is to be shown.

Now, pick a closed immersion $G \hookrightarrow \mathrm{GL}(V)$. Picking a basis for V such that $G_u \subset U_n$ gives that

$$\operatorname{Lie} G_u \subset \operatorname{Lie} U_n = \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ & & 0 \end{pmatrix}$$

consists of nilpotent elements. Picking a basis for V such that $T \subset D_n$ gives that

$$\operatorname{Lie} T \subset \operatorname{Lie} D_n = \operatorname{diag}(*, \dots, *)$$

consist of semisimple elements. Thus, $\operatorname{Lie} T \cap \operatorname{Lie} G_u = 0$.

(i)–(iii):
If $G_u = 1$, then G is a torus and there is nothing to show. Suppose that $\dim G_u > 0$.

Case 1. $\dim G_u = 1$:
G_u is connected, unipotent and so $G_u \cong \mathbf{G}_a$ by Theorem 60. Let $\phi : \mathbf{G}_a \to G_u$ be an isomorphism. G acts on G_u by conjugation with G_u acting trivially. We have

$$\text{Aut } G_u \cong \text{Aut } \mathbf{G}_a \cong \mathbf{G}_m \text{ (exercise).}$$

Hence

$$g\phi(x)g^{-1} = \phi(\alpha(g)x)$$

for all $g \in G, x \in \mathbf{G}_a$, for some character $\alpha : G/G_u \to \mathbf{G}_m$.

$\alpha = 1$: $G_u \subset \mathcal{Z}_G$.

$$\begin{aligned} [G,G] \subset G_u \text{ (Proposition 133)} &\implies [G,[G,G]] = 1, \text{ so } G \text{ is nilpotent} \\ &\implies G \cong G_u \times G_s \text{ (Proposition 131)} \end{aligned}$$

and so G is commutative and G_s is the unique maximal torus. (i)–(iii) are immediate.

$\underline{\alpha \neq 1}$: Given $s \in G_s$, let $Z = \mathcal{Z}_G(s)$.

$$\begin{aligned} G/G_u \text{ commutative} &\implies \mathrm{cl}_G(s) \text{ maps to } [s] \in G/G_u \\ &\implies \mathrm{cl}_G(s) \subset sG_u \\ &\implies \dim \mathrm{cl}_G(s) \leqslant 1 \\ &\implies \dim Z = \dim G - \dim \mathrm{cl}_G(s) \geqslant \dim G - 1 \end{aligned}$$

$\underline{\alpha(s) \neq 1}$: For all $x \neq 0$

$$s\phi(x)s^{-1} = \phi(\alpha(s)x) \neq \phi(x)$$

which implies that $Z \cap G_u = 1$, further giving $\dim Z = \dim G - 1$ and

$$\begin{aligned} Z_u = 1 &\implies Z^0 \text{ is a torus – which is Maximal – by Proposition 133 (it is connected, solvable and } Z_u^0 = 1) \\ &\implies G = Z^0 G_u, \quad \text{by (iv)} \end{aligned}$$

If $z \in Z$, then $z = z_0 u$ for some $z_0 \in Z^0$ and $u \in G_u$. But

$$u = z_0^{-1} z \in Z \cap G_u = 1 \implies z = z_0 \in Z^0.$$

Therefore, $Z = Z^0$, giving (iii), and $s \in Z$, giving (i).

$\underline{\alpha(s) = 1}$: For all $x \neq 0$

$$s\phi(x)s^{-1} = \phi(\alpha(s)x) = \phi(x)$$

and so $G_u \subset Z$. By the Jordan decomposition, since s commutes with G_u, $sG_u \cap G_s = \{s\}$, which means that

$$\mathrm{cl}_G(s) = \{s\} \implies s \in \mathcal{Z}_G \implies Z = G.$$

(ii) follows.

Note that since $\alpha \neq 1$ there is $g = g_s g_u$ such that $\alpha(g_s) = \alpha(g) \neq 1$ and so $\mathcal{Z}(g_s)$ is a Maximal torus by the previous case. Hence, since $\mathcal{Z}_G(s) = G$, we have $s \in \mathcal{Z}_G(g_s)$: (i) follows.

Now it remains to prove (iii) in the general case in which $\alpha \neq 1$. Let s be such that T, T' be Maximal tori. With the identification $T \xrightarrow{\sim} G/G_u$ (see (iv)), let $s \in T$ be such that $\alpha(s) \neq 1$. Then $\mathcal{Z}_G(s)$ is Maximal (by the above) and

$$T \subset \mathcal{Z}_G(s) \implies T = \mathcal{Z}_G(s) \quad \text{by dimension considerations.}$$

Likewise, with the identification $T' \xrightarrow{\sim} G/G_u$, pick $s' \in T'$ with $[s] = [s']$ in G/G_u so that $T' = \mathcal{Z}_G(s')$. $s' = su$ for some $u = G_u$. The conjugacy class of s (resp. s') – which has dimension 1 by the above – is contained in $sG_u = s'G_u$, which is irreducible of dimension 1:

$$\mathrm{cl}_G(s) = sG_u = s'G_u = \mathrm{cl}_G(s')$$

since the conjugacy classes are closed (Corollary 98). Therefore, s' is conjugate to s and thus T, T' are conjugate.

Case 2. $\dim G_u > 1$: Induct on the dimension of G.
Lemma 134 implies that there exists a closed, normal subgroup $N \subset \mathcal{Z}_{G_u}$ isomorphic to $\mathbf{G}_a$. Set $\overline{G} = G/N$ and $\overline{G}_u = G_u/N$, so $\overline{G}/\overline{G}_u \cong G/G_u$. Let $\pi : G \twoheadrightarrow \overline{G}$ be the natural surjection.

(i): If $s \in G_s$, define $\overline{s} = \pi(s) \in \overline{G}_s := \pi(G_s)$. By induction, there is a Maximal torus $\overline{T}$ in $\overline{G}$ containing $\overline{s}$. Let $H = \pi^{-1}(\overline{T})$, which is connected since N and $\overline{T}$ are connected (exercise, see homework 3). Also, $H_u = N$ (consider the map $H \to \overline{T}$ with kernel N) has dimension 1. Case 1 implies that there is a torus $T \ni s$ in H (Maximal in H) of dimension $\dim H/H_u = \dim \overline{T} = \dim G/G_u$; hence, T is Maximal in G, containing s.

(iii): Let T, T' be Maximal tori. Then $\pi(T) = \pi(T')$ are Maximal tori in $\overline{G}$ and by induction are conjugate: there is $g \in G$ such that

$$\pi(T) = \pi(gT'g^{-1}) \implies T, gT'g^{-1} \in \pi^{-1}(\pi(T)) =: H.$$

As above H_u is 1-dimensional and so $T, gT'g^{-1}$ – being Maximal tori in H – are conjugate in H and hence in G.

(ii): Again, for $s \in G_s$, set $\overline{s} = \pi(s)$. $\mathcal{Z}_{\overline{G}}(\overline{s})$ is connected by induction. $H := \pi^{-1}(\mathcal{Z}_{\overline{G}}(\overline{s}))$ is connected since N and $\mathcal{Z}_{\overline{G}}(\overline{s})$ are connected (exercise, see homework 3). Since $\pi(\mathcal{Z}_G(s)) \subset \mathcal{Z}_{\overline{G}}(\overline{s})$, we have $\mathcal{Z}_G(s) = \mathcal{Z}_H(s)$. If $H \neq G$, $\mathcal{Z}_H(s)$ is connected by induction and we are done. If $H = G$, then $\mathcal{Z}_{\overline{G}}(\overline{s}) = \overline{G}$. Hence,

$$\mathrm{cl}_G(\overline{s}) = \{\overline{s}\} \implies \mathrm{cl}_G(s) \subset \pi^{-1}(\overline{s}) = sN$$

and so the conjugacy class of s (recall that it is closed!) has dimension at most 1. We can now proceed as in Case 1 to conclude. (Sketch: fix an isomorphism $\phi : N \to G_u$. There is a $\beta \in \mathbf{G}_m$ such that $s\phi(x)s^{-1} = \phi(\beta x)$ for all $x \in \mathbf{G}_a$. If $\beta \neq 1$ we deduce $Z \cap N = 1$, so $\dim Z = \dim G - 1$ and $G = Z^0 N$. We deduce $Z = Z^0$ as above. If $\beta = 1$, then $N \leq Z$, so $sN \cap G_s = \{s\}$, which implies $\mathrm{cl}_G(s) = \{s\}$ and hence $Z = G$.) □

Remark 137. *(i), (iii) above carry over to all connected G, as we shall see soon. However, (ii) can fail in general. (For example, take $G = \mathrm{PGL}_2$ in characteristic $\neq 2$ and $s = [\mathrm{diag}(1, -1)]$.)*

Example. D_n is a maximal torus of B_n and $B_n \cong U_n \rtimes D_n$.

Example. If G is connected nilpotent it is clear by Proposition 131 that G_s is the unique maximal torus and the unique Maximal torus.

Lemma 138. *If $\phi : H \to G$ is an injective homomorphism, then $\dim H \leqslant \dim G$.*

Proof. Since $\dim \ker \phi = 0$, $\dim H = \dim \phi(H) \leqslant \dim G$. □

Proposition 139. *Let G be connected and solvable with $H \subset G$ a closed diagonalisable subgroup.*

(i) *H is contained in a Maximal torus.*

(ii) *$\mathcal{Z}_G(H)$ is connected.*

(iii) *$\mathcal{Z}_G(H) = N_G(H)$*

Proof. We shall induct on $\dim G$.

If $H \subset \mathcal{Z}_G$: Let T be a Maximal torus. For $h \in H$, for some $g \in G$,

$$h \in gTg^{-1} \implies h = g^{-1}hg \in T \implies H \subset T$$

Also, $\mathcal{Z}_G(H) = N_G(H) = G$.

If $H \not\subset \mathcal{Z}_G$: let $s \in H - \mathcal{Z}_G$. Then $H \subset Z := \mathcal{Z}_G(s) \neq G$ and so Z is connected by induction. Also by induction, $s \in T$ for some Maximal torus T; hence $T \subset Z$. We have injective morphisms

$$T \to Z/Z_u \to G/G_u \implies \dim T \leqslant \dim(Z/Z_u) \leqslant \dim(G/G_u)$$

But T is maximal, and so all of the dimensions must coincide: T is a Maximal torus of Z. By induction $H \subset gTg^{-1}$ for some $g \in Z$, implying (i). Also, $\mathcal{Z}_G(H) = \mathcal{Z}_Z(H)$ is connected by induction, giving (ii). For (iii), if $n \in N_G(H), h \in H$, then

$$[n, h] \in H \cap [G, G] \subset H \cap G_u = 1 \implies n \in \mathcal{Z}_G(H) \implies N_G(H) \subset \mathcal{Z}_G(H)$$

□

Corollary 140. *Let G be connected and solvable, and let $T \subset G$ be a torus. Then*

$$T \text{ is maximal} \iff T \text{ is Maximal}$$

Proof. If T is Maximal and $T \subset T'$ for some torus T', then $T \to T' \to G/G_u$ are injective morphisms, giving

$$\dim(G/G_u) = \dim T \leqslant \dim T' \leqslant \dim(G/G_u)$$

Hence, $T = T'$ and T is maximal. If T is not Maximal, then $T \subset T'$ for some Maximal T' by the above proposition, so T is not maximal. □

5.4 Cartan subgroups.

Remark 141. *From now on, G denotes a connected algebraic group.*

Theorem 142. *Any two maximal tori in G are conjugate.*

Proof. Let T, T' be maximal. Since both are connected and solvable they are each contained in Borels: $T \subset B$, $T' \subset B'$. There is a $g \in G$ such that $gBg^{-1} = B'$. gTg^{-1} and T' are two maximal tori in B and so, by Proposition 136, for some $b \in B$, $bgTg^{-1}b^{-1} = T'$. □

Corollary 143. *A maximal torus in a Borel subgroup of G is a maximal torus in G.*

Proof. Let B be a Borel subgroup. By the previous proof, any maximal torus of G is conjugate to a maximal torus of B... □

Definition 144. *A* **Cartan subgroup** *of G is $\mathcal{Z}_G(T)^0$, for a maximal torus T. All Cartan subgroups are conjugate. (We will see in Proposition 150 that $\mathcal{Z}_G(T)$ is connected.)*

Examples.
•. $G = \mathrm{GL}_n$, $T = D_n$, $\mathcal{Z}_G(T) = T = D_n$
•. If G is nilpotent, then the unique maximal torus G_s is central, so G is the unique Cartan subgroup.

Proposition 145. *Let $T \subset G$ be a maximal torus. $C := \mathcal{Z}_G(T)^0$ is nilpotent and T is its (unique) maximal torus.*

Proof. $T \subset C$ and so T is a maximal torus of C. Moreover, $T \subset \mathcal{Z}_C$. Now T lies in a Borel subgroup B of C and $T \subset \mathcal{Z}_B$, so by Theorem 136 we have $B = T \times B_u$, so B is nilpotent. By Proposition 130, $C = B$, so C is nilpotent. Finally T is the unique maximal torus of C by Proposition 131. □

Lemma 146. *Let $S \subset G$ be a torus. There exists $s \in S$ such that $\mathcal{Z}_G(S) = \mathcal{Z}_G(s)$.*

Proof. Let $G \hookrightarrow \mathrm{GL}_n$ be a closed immersion. Since S is a collection of commuting, diagonalisable elements, without loss of generality, $S \hookrightarrow D_n$. It is enough to show that $\mathcal{Z}_{\mathrm{GL}_n}(S) = \mathcal{Z}_{\mathrm{GL}_n}(s)$, for some $s \in S$. Let $\chi_i \in X^*(D_n)$ be given by $\mathrm{diag}(x_1, \dots, x_n) \mapsto x_i$. It is easy to show that

$$\mathcal{Z}_G(S) = \{(x_{ij}) \in \mathrm{GL}_n \mid \forall\, i, j \ \ x_{ij} = 0 \text{ if } \chi_i|_S \neq \chi_j|_S\}.$$

The set

$$\bigcap_{\substack{i,j \\ \chi_i|_S \neq \chi_j|_S}} \{s \in S \mid \chi_i(s) \neq \chi_j(s)\}$$

is nonempty and open, and thus is dense; any s from the set will do. □

Lemma 147. *For a closed, connected subgroup $H \subset G$, let $X = \bigcup_{x \in G} xHx^{-1} \subset G$.*

(i) *X contains a nonempty open subset of $\overline{X}$.*

(ii) *H parabolic $\implies$ X closed*

(iii) *If $(N_G(H) : H) < \infty$ and there is $y \in G$ lying in only finitely many conjugates of H, then $\overline{X} = G$.*

Proof.
(i):

$$Y := \{(x, y) \mid x^{-1}yx \in H = \{(x, y) \mid y \in xHx^{-1}\} \subset G \times G$$

is a closed subset. Note that

$$\mathrm{pr}_2(Y) = \{y \in \ \mid y \in xHx^{-1} \text{ for some } x\} = X$$

By Chevalley, X contains a nonempty open subset of $\overline{X}$.

(ii): Let P be parabolic.

$$\begin{array}{ccc} G \times G & \xrightarrow{\pi \times \mathrm{id}} & G/H \times G \\ & \searrow^{\mathrm{pr}_2} & \downarrow{\scriptstyle \mathrm{pr}_2'} \\ & & G \end{array}$$

Note that $\pi \times \mathrm{id}$ is open (Corollary 89) and that

$$(x, y) \in Y \iff \forall\, h \in H \quad (xh, y) \in Y.$$

By the usual argument, $(\pi \times \mathrm{id})(Y)$ is closed. Since G/P is complete,

$$\mathrm{pr}_2'((\pi \times \mathrm{id})(Y)) = \mathrm{pr}_2(Y) = X$$

is closed.

(iii): We have an isomorphism

$$Y \xrightarrow{\sim} G \times H, \quad (x, y) \mapsto (x, x^{-1}yx)$$

and so Y is irreducible (as H, G are connected). Consider the diagram

$$G \overset{\mathrm{pr}_1}{\twoheadleftarrow} Y \xrightarrow{\mathrm{pr}_2} G.$$

$$\begin{aligned} \mathrm{pr}_1^{-1}(x) = \{(x, xhx^{-1}) \mid h \in H\} \cong H \implies & \text{ all fibers of } \mathrm{pr}_1 \text{ have dimension } \dim H \\ \implies & \dim Y = \dim G + \dim H \quad \text{(Theorem 87).} \end{aligned}$$

Moreover,

$$\mathrm{pr}_2^{-1}(y) = \{(x, y) \mid y \in xHx^{-1}\} \cong \{x \mid y \in xHx^{-1}\}$$

Pick $y \in G$ lying in finitely many conjugates of H: $x_1Hx_1^{-1}, \ldots, x_nHx_n^{-1}$. Then

$$\mathrm{pr}_2^{-1}(y) = \bigcup_{i=1}^{n} x_i N_G(H)$$

which is a finite union of H cosets by hypothesis ($(N_G(H) : H) < \infty$). This implies that

$$\begin{aligned} \dim \mathrm{pr}_2^{-1}(y) = \dim H \implies & \mathrm{pr}_2 : Y \to \overline{\mathrm{pr}_2(Y)} \quad \text{is a dominant map with minimal fibre dimension} \leqslant \dim H \\ \implies & \dim Y - \dim \overline{\mathrm{pr}_2(Y)} \leqslant \dim H \\ \implies & \dim \overline{\mathrm{pr}_2(Y)} \geqslant \dim Y - \dim H = \dim G \\ \implies & \overline{\mathrm{pr}_2(Y)} = G \end{aligned}$$

□

Theorem 148.

(i) *Every $g \in G$ is contained in a Borel subgroup.*

(ii) *Every $s \in G_s$ is contained in a maximal torus.*

Proof.
(i): Pick a maximal torus $T \subset G$. Let $C = \mathcal{Z}_G(T)^0$ be the associated Cartan subgroup. Because C is connected and nilpotent (Proposition 145), there is a Borel $B \supset C$.

$$\begin{aligned} T = C_s \text{ (Proposition 145)} &\implies N_G(C) = N_G(T) \quad (\text{“}\supset\text{” is obvious}) \\ &\implies (N_G(C) : C) = (N_G(T) : \mathcal{Z}_G(T)^0) < \infty \quad \text{(Corollary 55)} \end{aligned}$$

By Lemma 146 there is $t \in T$ such that $\mathcal{Z}_G(t)^0 = \mathcal{Z}_G(T)^0 = C$. t is contained in a unique conjugate, i.e.,

$$t \in xCx^{-1} \implies xCx^{-1} = C$$

by the following.

$$\begin{aligned} t \in xCx^{-1} &\implies x^{-1}tx \in C, \quad \text{which is a semisimple element} \\ &\implies x^{-1}tx \in C_s = T \subset \mathcal{Z}_G(C) \\ &\implies C \subset \mathcal{Z}_G(x^{-1}tx)^0 = x^{-1}\mathcal{Z}_G(t)^0 x = x^{-1}Cx \\ &\implies C = x^{-1}Cx \quad \text{(compare dimensions)} \end{aligned}$$

Hence, we can apply Lemma 147 (iii) with $H = C$ to get

$$G = \overline{\bigcup_x xCx^{-1}} \subset \overline{\bigcup xBx^{-1}} = \bigcup xBx^{-1}$$

with the last equality following from Lemma 147 (ii) (this time with $H = B$). Hence, $G = \bigcup xBx^{-1}$, giving (i) of the theorem.

(ii):

$$\begin{aligned} s \in G_s &\implies s \in B, \quad \text{for some Borel } B \text{ by (i)} \\ &\implies s \in T, \quad \text{for some maximal torus } T \text{ of } B \text{ by Theorem 136 (i).} \end{aligned}$$

(A maximal torus in B is a maximal torus in G by Theorem 142.)

Corollary 149. *If $B \subset G$ is a Borel then $\mathcal{Z}_B = \mathcal{Z}_G$.*

Proof. The inclusion $\mathcal{Z}_B \subset \mathcal{Z}_G$ follows Corollary 128. For the reverse inclusion, if $z \in \mathcal{Z}_G$, we have $z \in gBg^{-1}$ for some g by the above Theorem, and so $z = g^{-1}zg \in B$. □

Proposition 150. *Let $S \subset G$ be a torus.*

(i) *$\mathcal{Z}_G(S)$ is connected.*

(ii) *If $B \subset G$ is a Borel containing S, then $\mathcal{Z}_G(S) \cap B$ is a Borel in $\mathcal{Z}_G(S)$, and all Borels of $\mathcal{Z}_G(S)$ arise this way.*

Proof.
(i): Let $g \in \mathcal{Z}_G(S)$ and B a Borel containing g. Define

$$X = \{xB \mid g \in xBx^{-1}\} \subset G/B$$

which is nonempty by Theorem 148. Consider the diagram

$$G/B \xleftarrow{\pi} G \xrightarrow{\alpha} G$$

in which π is the natural surjection and $\alpha : x \mapsto x^{-1}gx$. We have $X = \pi(\alpha^{-1}(B))$. Since $\pi^{-1}(B)$ is a union of fibres of π and is closed, and π is open, we have that X is closed. X is thus complete, being a closed subset of the complete G/B.

S acts on $X \subset G/B$, as for all $s \in S$

$$xBx^{-1} \ni g \implies sxBx^{-1}s^{-1} \ni g \quad (\text{since } g = s^{-1}gs).$$

By the Borel Fixed Point Theorem (120), S as some fixed point $xB \in X$, so

$$SxB = xB \implies Sx \subset xB \implies S \subset xBx^{-1}.$$

Hence, since g also lies in xBx^{-1}, we have

$$g \in \mathcal{Z}_{xBx^{-1}}(S) \subset \mathcal{Z}_G(S)^0$$

where $\mathcal{Z}_{xBx^{-1}}(S)$ is connected by Proposition 139. Thus, $\mathcal{Z}_G(S) \subset \mathcal{Z}_G(S)^0$: equality.

(ii): Let B be a Borel containing S and set $Z = \mathcal{Z}_G(S)$. $Z \cap B = \mathcal{Z}_B(S)$ is connected by Proposition 139 and is also solvable. Therefore, $Z \cap B$ is a Borel of Z if and only if it is parabolic, i.e., if $Z/Z \cap B$ is complete. By the bijective map

$$Z/(Z \cap B) \to ZB/B$$

of homogeneous Z-spaces, we see that suffices to show that

$ZB/B \subset G/B$ is closed $\iff$ $Y := ZB \subset G$ is closed (by the definition of the quotient topology)

Z being irreducible implies that

$$Y = \operatorname{im}(Z \times B \xrightarrow{\text{mult}} G) \text{ is irreducible } \implies \overline{Y} \text{ irreducible.}$$

Let $\pi : B \to B/B_u$ be the natural surjection and define

$$\phi : \overline{Y} \times S \to B/B_u, \quad (y, s) \mapsto \pi(y^{-1}sy).$$

(To make sure that this definition makes sense, i.e., that $y^{-1}sy \in B$, first check it when $y \in Y = ZB$.) For fixed y,

$$\phi_y : S \to B/B_u, \quad s \mapsto \phi(y, s) = \pi(y^{-1}sy)$$

is a homomorphism. Therefore, by rigidity (Theorem 54), for all $y \in Y$, $\phi_e = \phi_y$: for all $s \in S$

$$\pi(y^{-1}sy) = \pi(s).$$

If $T \supset S$ is a maximal torus, by the conjugacy of maximal tori in B, we have

$$uy^{-1}Syu^{-1} = T$$

for some $u \in B_u$. But then, by the above,

$$\pi(uy^{-1}uyu^{-1}) = \pi(y^{-1}sy) = \pi(s) \qquad \text{for all } s \in S$$

while $\pi|_T : T \to B/B_u$ is injective (an isomorphism even) (Jordan decomposition). Therefore,

$$uy^{-1}syu^{-1} = s \implies yu^{-1} \in \mathcal{Z}_G(S) = Z \implies y \in ZB = Y$$

and thus Y is closed: $Z \cap B \subset Z$ is Borel. Moreover, any other Borel of Z is

$$z(Z \cap B)z^{-1} = Z \cap (zBz^{-1}),$$

zBz^{-1} containing S. □

Corollary 151.

(i) *The Cartan subgroups are the $\mathcal{Z}_G(T)$, for maximal tori T.*

(ii) *If a Borel B contains a maximal torus T, then it contains $\mathcal{Z}_G(T)$.*

Proof.
(i) follows immediately from the above. For (ii), we have that $\mathcal{Z}_G(T)$ is a Borel of $\mathcal{Z}_G(T)$. But $\mathcal{Z}_G(T)$ is nilpotent (Proposition 145) and so $\mathcal{Z}_G(T) \cap B = \mathcal{Z}_G(T)$. □

5.5 Conjugacy of parabolic and Borel subgroups.

Theorem 152.

(i) *If $B \subset G$ is Borel, then $N_G(B) = B$.*

(ii) *If $P \subset G$ is parabolic, then $N_G(P) = P$ and P is connected.*

Proof.

(i): Induct on the dimension of G. If G is solvable, then $B = G$ and we are done; suppose otherwise. Let $H = N_G(B)$ and $x \in H$. We want to show that $x \in B$. Pick a maximal torus $T \subset B$. Then $xTx^{-1} \subset B$ is another maximal torus, and so T, xTx^{-1} are B-conjugate. Without loss of generality – changing x modulo B if necessary – suppose that $T = xTx^{-1}$. Consider

$$\phi : T \to T, \quad t \mapsto [x,t] = (xtx^{-1})t^{-1}.$$

Check that ϕ is a homomorphism. (Use that T is commutative.)

<u>Case 1. $\operatorname{im}\phi \neq T$:</u>
Let $S = (\ker\phi)^0$, which is a torus and is nontrivial since $\operatorname{im}\phi \neq T$. x lies in $Z = \mathcal{Z}_G(S)$ and normalises $Z \cap B$ (which is a Borel of Z by Proposition 150). If $Z \neq G$, then $x \in Z \cap B \subset B$ by induction. Otherwise, if $Z = G$, then $S \subset \mathcal{Z}_G$ and $B/S \subset G/S$ is a Borel by Corollary 127; hence,

$$[x] \text{ normalises } B/S \implies [x] \in B/S \text{ by induction} \implies x \in B.$$

<u>Case 2. $\operatorname{im}\phi = T$:</u>
If $\operatorname{im}\phi = T$, then

$$T \subset [x,T] \subset [H,H].$$

By Corollary 104, there is a G-representation V and a line $kv \subset V$ such that $H = \mathrm{Stab}_G(kv)$. Say $hv = \chi(h)v$ for some character $\chi : H \to \mathbf{G}_m$. $\chi(T) = \{e\}$ since $T \subset [H, H]$ and $\chi(B_u) = \{e\}$ by Jordan decomposition. Thus, as $B = TB_u$ (Theorem 136), B fixes v. By the universal property of quotients, we have a morphism

$$G/B \to V, \quad gB \mapsto gv.$$

However, the image of the morphism must be a point, as V is affine, while G/B is complete and connected; hence, G fixes v and $H = G$, i.e., $B \trianglelefteq G$. Therefore, G/B is affine, complete, and connected, and we must have $G = B$. (In particular, $x \in B$.)

(ii): By Theorem 125, $P \supset B$ for some Borel B of G. Suppose $n \in N_G(P)$. Then nBn^{-1}, B are both contained in – and are Borels of – P^0. Therefore, there must be $g \in P^0$ such that

$$nBn^{-1} = gBg^{-1} \implies g^{-1}n \in N_G(B) = B \text{ by (i)} \implies n \in gB \subset P^0.$$

Hence,

$$P \subset N_G(P) \subset P^0 \subset P.$$

□

Proposition 153. *Fix a Borel B. Any parabolic subgroup is conjugate to a unique parabolic containing B.*

Remark 154. *For a fixed B, the parabolics containing B are called* **standard parabolic subgroups**.

Example. If $G = \mathrm{GL}_n$ and $B = B_n$, then the standard parabolic subgroups are the subgroups, for integers $n_i \geqslant 1$ with $n = \sum_i^m n_i$, consisting of matrices

$$\begin{pmatrix} A_{n_1} & * & * & * \\ & A_{n_2} & * & * \\ & & \ddots & * \\ & & & A_{n_m} \end{pmatrix}$$

where $A_{n_i} \in \mathrm{GL}_{n_i}$.

Proof of proposition.
Let P be a parabolic. P contains some Borel gBg^{-1}, so $B \subset g^{-1}Pg$. This takes care of existence. For uniqueness, let $P, Q \supset B$ be two conjugate parabolics; say, $P = gQg^{-1}$.

$$\begin{aligned} gBg^{-1}, B \subset Q \text{ Borels} &\implies g^{-1}Bg = qBq^{-1} \text{ for some } q \in Q \\ &\implies gq \in N_G(B) = B \\ &\implies g \in Bq^{-1} \subset Q \\ &\implies P = Q \end{aligned}$$

□

Proposition 155. *If T is a maximal torus and B is a Borel containing T, then we have a bijection*

$$\begin{aligned} N_G(T)/\mathcal{Z}_G(T) &\xrightarrow{\sim} \{\text{Borels containing T}\} \\ [n] &\mapsto nBn^{-1} \end{aligned}$$

Exercise. If $G = \mathrm{GL}_n$, $B = B_n$, and $T = D_n$, we have that $\mathcal{Z}_G(T) = T$, $N_G(T)$ = permutation matrices, and that $N_G(T)/\mathcal{Z}_G(T) \cong S_n$. When $n = 2$, the two Borels containing T are $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ and $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$.

Proof of proposition.
If $B' \supset T$ is a Borel, then

$$\begin{aligned} B' = gBg^{-1} \text{ for some } g &\implies g^{-1}Tg, T \subset B \text{ are maximal tori} \\ &\implies g^{-1}Tg = bTb^{-1} \text{ for some } b \in B \\ &\implies n := gb \in N_G(T) \\ &\implies B' = gBg^{-1} = nBn^{-1}. \end{aligned}$$

Also,

$$nBn^{-1} = B \iff n \in N_G(B) \cap N_G(T) = B \cap N_G(T) = N_B(T) \overset{139}{=} \mathcal{Z}_B(T) \overset{151}{=} \mathcal{Z}_G(T).$$

□

Remark 156. *Given a Borel $B \subset G$, we have a bijection*

$$\begin{aligned} G/B &\xrightarrow{\sim} \{\text{Borels of } G\} \\ gB &\mapsto gBg^{-1} \end{aligned}$$

The projective variety G/B is called the **flag variety** *of G (independent of B up to isomorphism).*

Example. When $G = \mathrm{GL}_n$, $B = B_n$

$$\begin{aligned} G/B &\xrightarrow{\sim} \{\text{flags } 0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = k^n\} \\ gB &\mapsto g\left(0 \subsetneq \begin{pmatrix} * \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subsetneq \begin{pmatrix} * \\ * \\ 0 \\ \vdots \\ 0 \end{pmatrix} \subsetneq \cdots \subsetneq \begin{pmatrix} * \\ * \\ * \\ \vdots \\ * \end{pmatrix} = k^n\right) \end{aligned}$$

6. Reductive groups.

6.1 Semisimple and reductive groups.

Definitions 157. *The* **radical** *RG of G is the unique maximal connected, closed, solvable, normal subgroup of G. Concretely,*

$$RG = \left(\bigcap_{B \text{ Borel}} B \right)^0$$

(Recall that any two Borels are conjugate.) The **unipotent radical** *of G is the unique maximal connected, closed, unipotent, normal subgroup of G:*

$$R_uG = (RG)_u = \left(\bigcap_{B \text{ Borel}} B_u \right)^0$$

G is **semisimple** *if $RG = 1$ and is* **reductive** *if $R_uG = 1$.*

Remarks 158.
• *G semisimple $\implies$ G reductive*
• *G/RG is semisimple and G/R_uG is reductive. (Exercise!)*
• *If G is connected and solvable, then $G = RG$ and $G/R_uG = G/G_u$ is a torus. Hence a connected, solvable G is reductive $\iff$ G is a torus.*

Example.
• GL_n is reductive. Indeed,

$$R(\mathrm{GL}_n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cap \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} = D_n \implies R_u(\mathrm{GL}_n) = 1$$

Similarly, SL_n is reductive.
• GL_n is not semisimple, as $\{\mathrm{diag}(x, x, \ldots, x) \mid x \in k^\times\} \trianglelefteq \mathrm{GL}_n$. SL_n is semisimple by Proposition 159 (iii) below.

Proposition 159. *G is connected, reductive.*

(i) *$RG = \mathcal{Z}_G^0$, a central torus.*

(ii) *$RG \cap \mathcal{D}G$ is finite.*

(iii) *$\mathcal{D}G$ is semisimple.*

Remark 160. *In fact, $RG \cdot \mathcal{D}G = G$, so $G = \mathcal{D}G$ when G is semisimple. Hence, by (ii) above, $RG \times \mathcal{D}G \xrightarrow{\text{mult.}} G$ is surjective with finite kernel.*

Proof.
(i). $1 = R_uG = (RG)_u \implies RG$ is a torus, by Proposition 133. Hence, by rigidity (Corollary 55) $N_G(RG)^0 = \mathcal{Z}_G(RG)^0$. Moreover, since $RG \trianglelefteq G$

$$G = N_G(RG)^0 = \mathcal{Z}_G(RG)^0 \implies G = \mathcal{Z}_G(RG) \implies RG \subset \mathcal{Z}_G^0$$

The reverse inclusion is clear.

(ii). $S := RG$ is a torus. Embed $G \hookrightarrow \mathrm{GL}(V)$. V decomposes as $V = \bigoplus_{\chi \in X(S)} V_\chi$.

$$S \text{ is central} \implies G \text{ stabilises each } V_\chi \implies G \hookrightarrow \prod_\chi \mathrm{GL}(V_\chi)$$

It follows that $\mathcal{D}G \hookrightarrow \prod_\chi \mathrm{SL}(V_\chi)$ and RG acts by scalars on each V_χ. Since the scalars in SL_n are given by the n-th roots of unity, the result follows.

(iii).

$$\begin{aligned} \mathcal{D}G \trianglelefteq G &\implies R(\mathcal{D}G) \subset RG \\ &\implies R(\mathcal{D}G) \subset RG \cap \mathcal{D}G, \text{ which is finite} \\ &\implies R(\mathcal{D}G) = 1 \end{aligned}$$

□

Definition 161. *For a maximal torus $T \subset G$,*

$$I(T) := \Bigg(\bigcap_{\substack{B \text{ Borel} \\ B \supset T}} B \Bigg)^0$$

which is a connected, closed, solvable subgroup with maximal torus T: $I(T) = I(T)_u \rtimes T$ (see Theorem 136).

Claim:

$$I(T)_u = \Bigg(\bigcap_{B \supset T} B_u \Bigg)^0$$

Proof.
" $\subset$ ": For all Borels $B \supset T$

$$I(T) \subset B \implies I(T)_u \subset B_u \implies I(T)_u \subset \bigcap_{B \supset T} B_u \implies I(T)_u \subset \Bigg(\bigcap_{B \supset T} B_u \Bigg)^0$$

as $I(T)_u$ is connected.
" $\supset$ ": $\left(\bigcap_{B \supset T} B_u \right)^0 \subset I(T)$ and consists of unipotent elements. □

Remark 162.

$$I(T) \supset \Bigg(\bigcap_B B \Bigg)^0 = RG \implies I(T)_u \supset R_uG$$

In fact, the converse is true and equality holds.

Theorem 163 (Chevalley). *$I(T)_u = R_u G$. Hence,*

$$G \text{ reductive} \iff I(T)_u = 1 \iff I(T) = T$$

Corollary 164. *Let G be connected, reductive.*

(i) *$S \subset G$ subtorus $\implies \mathcal{Z}_G(S)$ connected, reductive.*

(ii) *T maximal torus $\implies \mathcal{Z}_G(T) = T$.*

(iii) *$\mathcal{Z}_G$ is the intersection of all maximal tori. (In particular, $\mathcal{Z}_G \subset T$ for all maximal tori T.)*

Proof of corollary.
(i): $\mathcal{Z}_G(S)$ is connected by Proposition 150. Let $T \supset S$ be a maximal torus, so that $T \subset \mathcal{Z}_G(S) =: Z$. Again by Proposition 150

$$\begin{aligned} &\{ \text{ Borels of } Z \text{ containing } T \} = \{ Z \cap B \mid B \supset T \text{ Borel of } G\} \\ \implies\ & I_Z(T) = \Big(\bigcap_{B \supset T} (Z \cap B) \Big)^0 \subset I(T) \overset{163}{=} T \\ \implies\ & I_Z(T) = T \\ \implies\ & Z \text{ is reductive, by the theorem} \end{aligned}$$

(ii): $\mathcal{Z}_G(T)$ is reductive by (i) and solvable (as it is a Cartan subgroup, which is nilpotent by Proposition 145). Hence, $\mathcal{Z}_G(T)$ is a torus: $T = \mathcal{Z}_G(T)$, by maximality, since $T \subset \mathcal{Z}_G(T)$.

(iii): T maximal $\implies T = \mathcal{Z}_G(T) \supset \mathcal{Z}_G$. For the converse, let $H = \bigcap_{T \text{ max.}} T$, which is a closed, normal subgroup of G (normal because all maximal tori are conjugate). Since H is commutative and $H = H_s$, H is diagonalisable, and by Corollary 55

$$G = N_G(H)^0 = \mathcal{Z}_G(H)^0 \implies G = \mathcal{Z}_G(H) \implies H \subset \mathcal{Z}_G$$

□

We will now build up several results in order to prove Theorem 163, following D. Luna's proof from 1999 [1].

Proposition 165. *Suppose V is a $\mathbf{G}_m$-representation. $\mathbf{G}_m$ acts on $\mathbf{P}V$. If $v \in V - \{0\}$, write $[v]$ for its image in $\mathbf{P}V$. Then either, $\mathbf{G}_m \cdot [v] = [v]$, i.e., v is a $\mathbf{G}_m$-eigenvector, or $\overline{\mathbf{G}_m \cdot [v]}$ contains two distinct $\mathbf{G}_m$-fixed points.*

<u>*Precise version of the proposition:*</u> *Write $V = \bigoplus_{n \in \mathbf{Z} = X^*(\mathbf{G}_m)} V_n$, where*

$$V_n = \{ v \in V \mid t \cdot v = t^n v \ \ \forall\, t \in \mathbf{G}_m \ \text{, i.e., ``v has weight n''} \}$$

For $v \in V$, write $v = \sum_{n \in \mathbf{Z}} v_n$ with $v_n \in V_n$. Then

$$[v_r], [v_s] \in \overline{\mathbf{G}_m \cdot [v]}$$

where $r = \min\{n \mid v_n \neq 0\}$ and $s = \max\{n \mid v_n \neq 0\}$. Clearly, $[v_r], [v_s]$ are $\mathbf{G}_m$-fixed. In fact, if $\mathbf{G}_m \cdot [v] \neq [v]$, then

$$\overline{\mathbf{G}_m \cdot [v]} = (\mathbf{G}_m \cdot [v]) \sqcup \{[v_r]\} \sqcup \{[v_s]\}$$

[1] See for example P. Polo's M2 course notes (§21 in Séance 5/12/06) at `www.math.jussieu.fr/~polo/M2`

Proof. Pick a basis $e_0, e_1, \dots, e_n$ of V such that $e_i \in V_{m_i}$. Without loss of generality $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_n$. Write $v = \sum_i \lambda_i e_i$, $\lambda_i \in k$. The orbit map $f : \mathbf{G}_m \to \mathbf{P}V$ is given by mapping t to

$$t \cdot [v] = (t^{m_0}\lambda_0 : t^{m_1}\lambda_1 : \cdots : t^{m_n}\lambda_n) = (0 : \cdots : 0 : \lambda_u : \cdots : t^{m_i - r}\lambda_i : \cdots : t^{s-r}\lambda_v : 0 : \cdots : 0)$$

where $u = \min\{i \mid \lambda_i \neq 0\}$ and $v = \max\{i \mid \lambda_i \neq 0\}$, so that $m_u = r$ and $m_v = s$.

Define $\tilde{f} : \mathbf{P}^1 \to \mathbf{P}V$ by

$$(T_0 : T_1) \mapsto (0 : \cdots : 0 : T_1^{s-r}\lambda_u : \cdots : T_0^{m_i - r}T_1^{s-m_i}\lambda_i : \cdots : T_0^{s-r}\lambda_v : 0 : \cdots : 0)$$

Check that this a morphism and that $\tilde{f}|_{\mathbf{G}_m} = f$. (In fact, $\tilde{f}$ is the unique extension of f, since $\mathbf{P}V$ is separated and $\mathbf{G}_m$ is dense.) We have

$$\tilde{f}(\mathbf{P}^1) = \tilde{f}(\overline{\mathbf{G}_m}) \subset \overline{\tilde{f}(\mathbf{G}_m)} = \overline{\mathbf{G}_m \cdot [v]}$$

and

$$\tilde{f}(0:1) = (0 : \cdots : \lambda_u : \cdots : 0 : \cdots 0) = [v_r] \quad \text{and } \tilde{f}(1:0) = \cdots = [v_s]$$

(In fact, we actually have $\tilde{f}(\mathbf{P}^1) = \overline{\mathbf{G}_m \cdot [v]}$, using the fact that $\mathbf{P}^1$ is complete). □

Informally, above, we have

$$[v_r] = \lim_{t \to 0} t \cdot [v] \in (\mathbf{P}V)^{\mathbf{G}_m}$$
$$[v_s] = \lim_{t \to \infty} t \cdot [v] \in (\mathbf{P}V)^{\mathbf{G}_m}$$

Lemma 166. *Let M be a free abelian group, and $M_1, \dots, M_r \subsetneq M$ subgroups such that each M/M_i is torsion-free. Then*

$$M \neq M_1 \cup \cdots \cup M_r$$

Proof. Since M/M_i is torsion-free, it is free abelian, and

$$0 \to M_i \to M \to M/M_i \to 0$$

splits, giving that M_i is a (proper) direct summand of M. Thus, $M_i \otimes \mathbf{C} \subsetneq M \otimes \mathbf{C}$; hence

$$M \otimes \mathbf{C} \neq \bigcup_{i=1}^{r} M_i \otimes \mathbf{C}$$

as the former is irreducible and the latter are proper closed subsets. □

Lemma 167. *Let T be a torus and V and algebraic representation of T, so that T acts on $\mathbf{P}V$. Then, there is a cocharacter $\lambda : \mathbf{G}_m \to T$ such that $(\mathbf{P}V)^T = (\mathbf{P}V)^{\lambda(\mathbf{G}_m)}$.*

Proof. Let $\chi_1, \dots, \chi_r \in X^*(T)$ be distinct such that $V = \bigoplus_{i=1}^r V_{\chi_i}$ and $V_{\chi_i} \neq 0$ for all i. Then

$$[v] \in (\mathbf{P}V)^T \iff v \in V_{\chi_i} \quad \text{for some } i$$

So it is enough to show that there is a cocharacter λ such that

$$\forall\, i \neq j \quad \chi_i \circ \lambda \neq \chi_j \circ \lambda \iff (\chi_i - \chi_j) \circ \lambda \neq 0$$

Recall from Proposition 33 we have that

$$X^*(T) \times X_*(T) \to X^*(\mathbf{G}_m) \cong \mathbf{Z}, \quad (\chi, \lambda) \mapsto \chi \circ \lambda$$

is a perfect pairing.

Let $M = X_*(T)$, which is free abelian, and for all $i \neq j$

$$M_{ij} := \{\lambda \in X_*(T) \mid \langle \chi_i - \chi_j, \lambda \rangle = 0\} \neq M \quad (\text{ as } \chi_i \neq \chi_j)$$

For $n > 0$, if $n\lambda \in M_{ij}$, then $\lambda \in M_{ij}$, and so M/M_{ij} is torsion-free. By the above lemma, $M \neq \bigcup_{i \neq j} M_{ij}$, so there is a $\lambda \in M$ such that

$$\forall\, i \neq j \quad 0 \neq \langle \chi_i - \chi_j, \lambda \rangle = (\chi_i - \chi_j) \circ \lambda$$

$\square$

Theorem 168 (Konstant-Rosenlicht)**.** *Suppose that G is unipotent and X is an affine G-space. Then all orbits are closed.*

Proof. Let $Y \subset X$ be an orbit.Without loss of generality, we replace X by $\overline{Y}$ (which is affine). Since Y is locally closed and dense, it is open. Let $Z = X - Y$, which is closed. G acts (locally-algebraic) on $k[X]$, preserving $I_X(Z) \subset k[X]$. $I_X(Z) \neq 0$, as $Z \neq X$. By Theorem 40, since G is unipotent, it has a nonzero fixed point, say, f in $I_X(Z)$. f is G-invariant and hence is constant on Y. But then

$$Y \text{ is dense} \implies f \text{ is constant } (\neq 0) \implies k[X] = I_X(Z) \implies Z = \varnothing \implies Y = X \text{ is closed}$$

$\square$

Now, we want to prove Theorem 163. Fix a Borel $B \subset G$ and set $X = G/B$, a homogeneous G-space. Note that

$$X^T = \{gB \mid Tg \subset gB \iff T \subset gBg^{-1}\} \leftrightarrow \{\text{Borel subgroups containing } T\}$$

Furthermore, by Proposition 155, X^T in bijection with $N_G(T)/\mathcal{Z}_G(T)$ and hence is finite. Thus $N_G(T)/\mathcal{Z}_G(T)$ acts simply transitively on X^T. For $p \in X^T$, define

$$X(p) = \{x \in X \mid p \in \overline{Tx}\}$$

Proposition 169 (Luna)**.** *For $p \in X^T$, $X(p)$ is open (in X), affine, and $I(T)$-stable.*

Proof. By Corollary 104 there exists a G-representation V and a line $L \subset V$ such that $B = \mathrm{Stab}_G(L)$ and $\mathrm{Lie}\, B = \mathrm{Stab}_{\mathfrak{g}}(L)$. This gives a map of G-spaces

$$i : X = G/B \to \mathbf{P}V, \quad g \mapsto gL.$$

i and di are injective (Corollary 105); hence, i is a closed immersion (Corollary 105). Without loss of generality, $X \subset \mathbf{P}V$ is a closed G-stable subvariety – and, replacing V by the G-stable $\langle G \cdot L \rangle$, we may also suppose that X is not contained in any $\mathbf{P}V' \subset \mathbf{P}V$ for any subspace $V' \subset V$.

By Lemma 167, there is a cocharacter $\lambda : \mathbf{G}_m \to T$ such that $X^T = X^{\mathbf{G}_m}$, considering X and $\mathbf{P}V$ as $\mathbf{G}_m$-spaces via λ. For $p \in X^T$, write $p = [v_p]$ for some $v_p \in V_{m(p)}$, $m(p) \in \mathbf{Z}$ (weight). Pick $p_0 \in X^T$ such that $m_0 := m(0)$ is minimal. Set $e_0 = v_{p_0}$ and extend e_0 to a basis $e_0, e_1, \ldots, e_n$ of V such that $\lambda(t)e_i = t^{m_i}e_i$. Without loss of generality, $m_1 \leqslant \cdots \leqslant m_n$. Let $e_0^*, \ldots, e_n^* \in V^*$ denote the dual basis.

Claim 1. $m_0 < m_1$:
Suppose that $m_0 > m_1$. There is $[v] \in X$ such that $e_1^*(v) \neq 0$ (otherwise $X \subset \mathbf{P}(\ker e_1^*) \subsetneq \mathbf{P}V$). Then, by Proposition 165,

$$[v_{m_1}] = \lim_{t\to 0} \lambda(t)[v] \in (\mathbf{P}V)^{\mathbf{G}_m} \cap X = X^T$$

(with the inclusion following from the fact that X is complete). This contradicts the minimality of m_0, so we must have $m_0 \leqslant m_1$.

Suppose that $m_0 = m_1$. Define

$$Z = \{z \in k \mid \text{ there is some point of the form } (1 : z : \cdots) \text{ in } X\}$$

If $(1 : z : \cdots) \in X$, then by Proposition 165, as $m_0 = m_1$,

$$(1 : z : \cdots)' = \lim_{t\to 0} \lambda(t)(1 : z : \cdots) \in X^T.$$

Since X^T is finite, so too is Z. Writing $Z = \{z_1, \ldots, z_r\}$, we have

$$X \subset \mathbf{P}(\ker e_0^*) \cup \bigcup_{i=1}^{r} \mathbf{P}(\ker(e_1^* - z_i e_0^*)).$$

Since X is irreducible, it is contained in one of these subspaces, which is a contradiction.

Therefore, $m_0 < m_1$.

Claim 2. $X(\lambda, p_0) := \{x \in X | e_0^*(x) \neq 0\}$ *is open in* X*, affine, and* T*-stable. Also,* $X(\lambda, p_0) = X(p_0)$*, and it is* $I(T)$*-stable:*
$X(\lambda, p_0) = X \cap (e_0^* \neq 0)$ is open in X and affine (as $(e_0^* \neq 0)$ is open and affine in $\mathbf{P}V$). It is T-stable, as e_0^* is an eigenvector for T (as e_0 is an eigenvector for T).

If $x \in X(\lambda, p_0)$, as $m_0 < m_i$ for all $i \neq 0$ (Claim 1),

$$\lim_{t\to 0} \lambda(t)x = [e_0] = p_0.$$

Hence, $p_0 \in \overline{\mathbf{G}_m \cdot x} \subset \overline{Tx}$, so $x \in X(p_0)$. Let $x \in X(p_0)$ and suppose that $e_0^*(x) = 0$. Then

$$p_0 \in \overline{Tx} \subset X - X(\lambda, p_0)$$

with $X - X(\lambda, p_0)$ T-stable and closed. This is a contradiction and so we must have $x \in X(\lambda, p_0)$. Hence, $X(\lambda, p_0) = X(p_0)$.

To show that the set is $I(T)$-stable, we need to show that from the of G on $\mathbf{P}(V^*)$ (which arises from the action on V^*), we have

$$e_0^\perp = \{\ell \in V^* \mid \langle \ell, e_0 \rangle = 0\}$$

First, let us address a third claim.

Claim 3. *(i) Each G-orbit in $\mathbf{P}(V^*)$ intersects the open subset $\mathbf{P}(V^*) - \mathbf{P}(e_0^\perp)$ and (ii) $G \cdot [e_0^*]$ is closed in $\mathbf{P}(V^*)$:* (i): Pick $v \in V^* - \{0\}$. If $G\ell \subset e_0^\perp$, then for all $g \in G$

$$0 = \langle g\ell, e_0 \rangle = \langle \ell, g^{-1}e_0 \rangle.$$

But Ge_0 spans V (otherwise, $X = Ge_0 \subset \mathbf{P}(V') \subsetneq \mathbf{P}V$, which is a contradiction) and so

$$\langle \ell, V \rangle = 0 \implies \ell = 0$$

which is another contradiction. Hence, $G[\ell] \not\subset \mathbf{P}(e_0^\perp)$.

(ii): e_i^* has weight $-m_i$ under the $\mathbf{G}_m$-action and

$$-m_n \leqslant \cdots \leqslant -m_1 < -m_0.$$

Hence by Proposition 165, if $x \in \mathbf{P}(V^*) - \mathbf{P}(e_0^\perp)$ then $[e_0^*] \in \overline{\mathbf{G}_m \cdot x}$. So, for all $x \in \mathbf{P}(V^*)$, by (i),

$$[e_0^*] \in \overline{Gx} \implies G[e_0^*] \subset \overline{Gx}.$$

If Gx is a closed orbit (which exists), we deduce that it is equal to $G[e_0^*]$.

Let us return to Claim 2, that $X(\lambda, p_0)$ is $I(T)$-stable. Recall that $I(T) = \left(\bigcap_{B' \supset T} B' \right)^0$. Define $P = \mathrm{Stab}_G([e_0^*])$. Since $G/P \to G[e_0^*]$ is bijective map of G-spaces and the latter space is complete (Claim 3), it follows that P is parabolic. Hence, there is a parabolic B' of G contained in P. Moreover, since e_0^* is a T-eigenvector, $T \subset P$. There is a maximal torus of B' conjugate to T in P, so without loss of generality suppose that $T \subset B' \subset P$. It follows that $I(T)$ $(\subset B')$ stabilises $[e_0^*]$ and hence also stabilises the set

$$X(\lambda, p_0) = \{x \in X \mid e_0^*(x) \neq 0\},$$

completing Claim 2.

Now, $N_G(T)$ acts transitively on X^T by above. If $p \in X^T$, then $p = np_0$ for some $n \in N_G(T)$; hence $X(p) = nX(p_0)$ is open, affine, and stable under $nI(T)n^{-1} = I(T)$ (equality following from the fact that n permutes the Borels containing T). □

Corollary 170. $\dim X \leqslant 1 + \dim(X - X(p_0))$

Proof. Either $X = X(p_0)$ or otherwise. If equality holds, then X is complete, affine, and connected, and is thus a point. In this case, $\dim X = 0$ and the inequality is true. Suppose that $X \neq X(p_0)(= X(\lambda, p_0))$. Pick $y \in X - X(\lambda, p_0)$. Then $e_0^*(y) = 0$, and $e_i^*(y) \neq 0$ for some $i > 0$. Let

$$U = \{x \in X \mid e_i^*(x) \neq 0\} \subset X,$$

which is nonempty and open. Define the morphism

$$f : U \to \mathbf{A}^1, \quad x \mapsto \frac{e_0^*(x)}{e_i^*(x)}$$

$f^{-1}(0) \subset X - X(\lambda, p_0)$. By Corollary 89,

$$\dim(X - X(\lambda, p_0)) \geqslant \dim U - \dim \overline{f(U)} \geqslant \dim U - 1 = \dim X - 1$$

□

Proposition 171 (Luna). *$I(T)_u$ acts trivially on $X = G/B$.*

Proof. $J := I(T)_u$. If $x \in X$, then $\overline{Tx}$ contains a T-fixed point by the Borel Fixed Point Theorem; hence

$$X = \bigcup_{x \in X^T} X(p).$$

Fix $x \in X$. J being connected, solvable implies that $\overline{Jx}$ contains a J-fixed point y. By the above, we see that $y \in X(p)$ for some $p \in X^T$. If

$$Jx \cap (X - X(p)) \neq \varnothing,$$

with $X - X(p)$ closed and J-stable by Proposition 169, then

$$y \in \overline{Jx} \subset X - X(p)$$

which is a contradiction. Hence, $Jx \subset X(p)$, $X(p)$ being affine by Proposition 169, and J being unipotent implies that $Jx \subset X(p)$ is closed by Kostant-Rosenlicht (168). But

$$\begin{aligned} y \in X(p) \cap \overline{Jx} = Jx \quad (Jx \text{ is closed}) &\implies Jx = Jy = y, \quad \text{as } y \text{ is } J\text{-fixed} \\ &\implies x = y \text{ is } J\text{-fixed} \\ &\implies J \text{ acts trivially on } X. \end{aligned}$$

□

Proof of Theorem 163.
Let $J = I(T)_u$ again. We want to show that $J = R_u G$ and we already know that $J \supset R_u G$. For the reverse inclusion, we have that for all $g \in G$,

$$\begin{aligned} J(gB) = gB \text{ (Theorem 171)} &\implies Jg \subset gB \\ &\implies J \subset gBg^{-1} \\ &\implies J \subset (gBg^{-1})_u, \quad \text{as } J \text{ is unipotent} \\ &\implies J \subset \left(\bigcap_g (gBg^{-1})_u \right)^0 = R_u G, \quad \text{as } J \text{ is connected} \end{aligned}$$

□

6.2 Overview of the rest.

Plan for the rest of the course: Given connected, reductive G (and a maximal torus T) we want to show the following:
• $\mathfrak{g} = \operatorname{Lie} T \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, under the adjoint action of T, where $\Phi \subset X^*(T)$ is finite.
• There is a natural bijection $\Phi \xrightarrow{\sim} \Phi^\vee$, where $\Phi^\vee \subset X_*(T)$ is such that $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is a root datum (to be defined shortly).
• For all $\alpha \in \Phi$, there is a unique closed subgroup $U_\alpha \subset G$, normalised by T, such that $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$.
• $G = \langle T \cup \bigcup_{\alpha\in\Phi} U_\alpha\rangle$.

From now on G denotes a connected, reductive algebraic group. Fix a maximal torus T, so that

$$\mathfrak{g} = \bigoplus_{\lambda\in X^*(T)} \mathfrak{g}_\lambda$$

for the adjoint T-action. We write $X^*(T)$ additively, so

$$\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid \operatorname{Ad}(t)X = X \text{ for all } t \in T\} = \mathfrak{z}_{\mathfrak{g}}(T) \overset{100}{=} \operatorname{Lie} \mathcal{Z}_G(T) \overset{164}{=} \operatorname{Lie} T = \mathfrak{t}$$

Define $\Phi = \Phi(G,T) := \{\alpha \in X^*(T) - \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$, which is finite. The $\alpha \in \Phi$ are the **roots** of G (with respect to T). Hence,

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

Definition 172. *The* **Weyl group** *of* (G,T) *is*

$$W = W(G,T) := N_G(T)/\mathcal{Z}_G(T) \overset{164}{=} N_G(T)/T$$

which is finite by Corollary 55. W acts faithfully on T by conjugation , and hence acts on $X^*(T)$ and $X_*(T)$:

$$w \in W \quad \mapsto \begin{cases} (w^{-1})^* : X^*(T) \to X^*(T) \\ w_* : X_*(T) \to X_*(T) \end{cases}$$

Explicitly,

$$w\mu = \mu(\dot{w}^{-1}(\cdot)\dot{w}), \quad \text{for } \mu \in X^*(T)$$
$$w\lambda = \dot{w}\lambda(\cdot)\dot{w}^{-1}, \quad \text{for } \lambda \in X_*(T)$$

where $\dot{w} \in N_G(T)$ lifts w.

Remarks 173.
• *The natural perfect pairing $X^*(T) \times X_*(T) \to \mathbf{Z}$ is W-invariant: $\langle w\mu, w\lambda\rangle = \langle \mu, \lambda\rangle$.*
• *W preserves $\Phi \subset X^*(T)$ because $N_G(T)$ permutes the eigenspaces $\mathfrak{g}_\alpha$. (Check that $\operatorname{Ad}(\dot{w})\mathfrak{g}_\alpha = \mathfrak{g}_{w\alpha}$.)*

Example. $G = \mathrm{GL}_n$, $T = D_n$.
$\mathfrak{g} = M_n(k)$ and T acts by conjugation.

$$\mathfrak{g} = \begin{pmatrix} * & & & \\ & * & & \\ & & \ddots & \\ & & & * \end{pmatrix} \oplus \bigoplus_{\substack{i,j \\ i\neq j}} \begin{pmatrix} & & * & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

where in the summands on the right $*$ appears in the (i,j)-th entry. On the (i,j)-th *summand*, $\mathrm{diag}(x_1,\dots,x_n)\in T$ acts as multiplication by $x_i x_j^{-1}$. Letting $\epsilon_i \in X^*(T)$ denote $\mathrm{diag}(x_1,\dots,x_n)\mapsto x_i$, we get that $\Phi=\{\epsilon_i-\epsilon_j \mid i\neq j\}$. Also, $W=N_G(T)/T\cong S_n$ acts by permuting the ϵ_i.

Lemma 174. *If $\phi : H \to H'$ is a surjective morphism of algebraic groups and $T\subset H$ is a maximal torus, then $\phi(T)\subset H'$ is a maximal torus.*

Proof. Pick a Borel $B\supset T$, so that $B=B_u\rtimes T$ and $\phi(B)=\phi(B_u)\phi(T)$. $\phi(B)$ is a Borel of H' by Corollary 127. $\phi(T)$ is a torus, as it is connected, commutative, and consists of semisimple elements. $\phi(B_u)\subset\phi(B)_u$ is unipotent (Jordan decomposition). Finally,

$$\begin{aligned}\phi(T)\to\phi(B)/\phi(B)_u \text{ bijective (Jordan decomposition)} &\implies \dim\phi(T)=\dim\phi(B)/\dim(B)_u\\ &\implies \phi(T)\subset\phi(B)\text{ maximal torus}\\ &\implies \phi(T)\subset H'\text{ maximal torus}\end{aligned}$$

□

Lemma 175. *If $S\subset T$ be a subtorus, then*

$$\mathcal{Z}_G(S)\supsetneq T \iff S\subset(\ker\alpha)^0 \text{ for some } \alpha\in\Phi$$

Proof. We always have $\mathcal{Z}_G(S)\supset T$. Note that

$$\mathrm{Lie}\,\mathcal{Z}_G(S)\overset{100}{=}\mathfrak{z}_{\mathfrak{g}}(S)=\{X\in\mathfrak{g}\mid \mathrm{Ad}(s)(X)=X \text{ for all } s\in S\}=\mathfrak{t}\oplus\bigoplus_{\substack{\alpha\in\Phi\\ \alpha|_S=1}}\mathfrak{g}_\alpha$$

$$\begin{aligned}\text{“}\supsetneq\text{”} &\iff \mathrm{Lie}\,\mathcal{Z}_G(S)\supsetneq\mathfrak{t},\ \text{by dimension considerations}\\ &\iff \mathfrak{t}\oplus\bigoplus_{\substack{\alpha\in\Phi\\ \alpha|_S=1}}\mathfrak{g}_\alpha\supsetneq\mathfrak{t}\\ &\iff S\subset\ker\alpha,\ \text{for some } \alpha\in\Phi\end{aligned}$$

□

For $\alpha\in\Phi$, define $T_\alpha:=(\ker\alpha)^0$, which is a torus of dimension $\dim T-1$, as $\mathrm{im}\,\alpha=\mathbf{G}_m$. Define $G_\alpha:=\mathcal{Z}_G(T_\alpha)$, which is connected, reductive by Corollary 164. Note that

$$T_\alpha\subset\mathcal{Z}^0_{G_\alpha}\overset{159}{=}R(G_\alpha)$$

Let π denote the natural surjection $G_\alpha\to G_\alpha/R(G_\alpha)$. By Lemma 174, $\pi(T)$ is a maximal torus of $G_\alpha/R(G_\alpha)$.

$$T_\alpha\subset R(G_\alpha)\implies T/T_\alpha\twoheadrightarrow\pi(T)\implies\dim\pi(T)\leqslant 1$$

If $\dim\pi(T)=0$, then

$$T\subset R(G_\alpha)\subset\mathcal{Z}_{G_\alpha}\implies G_\alpha\subset\mathcal{Z}_G(T)=T$$

which is a contradiction by Lemma 175. Hence, $\dim\pi(T)=1$.

Definitions 176.

$$\text{the } \textbf{rank} \text{ of } G = \text{rk } G := \dim T, \text{ where } T \text{ is a maximal torus}$$
$$\text{the } \textbf{semisimple rank} \text{ of } G = \text{ss-rk } G := \text{rk}(G/RG)$$

Hence, ss-rk $G_\alpha = 1$. Note that since all maximal tori are conjugate, rank is well-defined, and that ss-rk $G \leqslant$ rk G by Lemma 174.

Example. $G = \mathrm{GL}_n$, $\alpha = \epsilon_i - \epsilon_{i+1}$. We have

$$T_\alpha = \{\text{diag}(x_1, \ldots, x_n) \mid x_i = x_{i+1}\}$$

and

$$G_\alpha = D_{i-1} \times \mathrm{GL}_2 \times D_{n-i-1}.$$

$G_\alpha/RG_\alpha \cong \mathrm{PGL}_2$ and $\mathcal{D}G_\alpha \cong \mathrm{SL}_2$.

6.3 Reductive groups of rank 1.

Proposition 177. *Suppose that G is not solvable and* rk $G = 1$. *Pick a maximal torus T and a Borel B containing T. Let $U = B_u$.*

(i) $\#W = 2, \quad \dim G/B = 1,$ *and $G = B \sqcup UnB$, where $n \in N_G(T) - T$.*

(ii) $\dim G = 3$ *and $G = \mathcal{D}G$ is semisimple.*

(iii) $\Phi = \{\alpha, -\alpha\}$ *for some $\alpha \neq 0$, and* $\dim \mathfrak{g}_{\pm\alpha} = 1$.

(iv) $\psi : U \times B \to UnB$, $(u, b) \mapsto unb$, *is an isomorphism of varieties.*

(v) $G \cong \mathrm{SL}_2$ *or* PGL_2

Remark 178. *In either case, $G/B \cong \mathbf{P}^1$. For example,*

$$\mathrm{SL}_2/\begin{pmatrix} * & * \\ & * \end{pmatrix} \xrightarrow{\sim} \mathbf{P}^1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a : c)$$

Proof of proposition.
(i):

$$W \hookrightarrow \mathrm{Aut}(X^*(T)) \cong \mathrm{Aut}(\mathbf{Z}) = \{\pm 1\} \implies \#W \leqslant 2$$

If $W = 1$, then B is the only Borel containing T, and so by Theorem 163

$$B = I(T) = T \implies B \text{ nilpotent} \overset{130}{\implies} G \text{ solvable}$$

which contradicts our hypothesis; hence, $\#W = 2$.

Set $X := G/B$. $\dim X > 0$ since $B \neq G$. By Proposition 155 we have $\#X^T = \#W = 2$. By Corollary 170

$$\dim X \leqslant 1 + \dim(X - X(p_0))$$

Since $X - X(p_0)$ is T-stable and closed (Proposition 169), it can contain at most one T-fixed point (as $\#X^T = 2, p_0 \in X(p_0)$). By Proposition 165, T acts trivially and so $X - X(p_0)$ is finite:

$$\dim X \leqslant 1.$$

Now,

$$\begin{aligned}\#W = 2 &\implies B, nBn^{-1} \text{ are the two Borels containing } T\\ &\implies X^T = \{x, nx\}, \text{where } x := B \in G/B\end{aligned}$$

We want to show that $X = \{x\} \sqcup Unx$, which will imply that $G = B \sqcup UnB$. Note that x is U-fixed, so $\{x\}$ and Unx are disjoint (as $x \neq nx$). Also, Unx is T-stable, as

$$TUnx = UTnx = UnTx = Unx,$$

and $Unx \neq \{nx\}$, as otherwise

$$\{nx\} = Unx = Bnx \implies \{x\} = n^{-1}Bnx \implies n^{-1}Bn \subset \mathrm{Stab}_G(x) = B \implies \text{contradiction}$$

Hence, $\overline{Unx} = X$, by dimension considerations, so $Unx \subset X$ is open, $X - Unx$ is finite (as $\dim X = 1$), and $X - Unx$ is T-stable. T is connected and so

$$U - Unx \subset X^T = \{x, nx\} \implies X - Unx = \{x\}$$

(ii):

$$\begin{aligned}1 &= \dim Unx\\ &= \dim U - \dim(U \cap nUn^{-1}), \text{ as } Unx \text{ is a } U\text{-orbit}\\ &= \dim U, \text{ as } U \cap nUn^{-1} = \mathrm{Stab}_U(nx) \text{ is finite by Theorem 163}\end{aligned}$$

Hence,

$$\begin{aligned}\dim B &= \dim T + \dim U = 1 + 1 = 2\\ \dim G &= \dim B + \dim(G/B) = 2 + 1 = 3\end{aligned}$$

$\mathcal{D}G$ is semisimple by Proposition 159 and is not solvable (as G is not). rk $\mathcal{D}G \leqslant$ rk $G = 1$. If rk $\mathcal{D}G = 0$, then a Borel of $\mathcal{D}G$ is unipotent, which by Proposition 130 implies that $\mathcal{D}G$ is solvable: contradiction. (Or, $T_1 = \{1\}$ is a maximal torus and $T_1 = \mathcal{Z}_{\mathcal{D}G}(T_1) = \mathcal{D}G$: contradiction.) Hence, rk $\mathcal{D}G = 1$, so $\dim \mathcal{D}G = 3$ by the above: $\mathcal{D}G = G$.

(iii): $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$. Since $\dim \mathfrak{g} = 3$ and $\dim \mathfrak{t} = 1$, we have $\#\Phi = 2$. Moreover, Φ is W-stable and $[n] \in W$ acts by -1 on $X^*(T)$, and so $\Phi = \{\alpha, \alpha\}$ for some α: $\dim \mathfrak{g}_{\pm\alpha} = 1$. From $B = U \rtimes T$ we have $\mathrm{Lie}\, B = \mathfrak{t} \oplus \mathrm{Lie}\, U$ and $\mathrm{Lie}\, U = g_\alpha$ or $\mathfrak{g}_{-\alpha}$, as $\mathrm{Lie}\, U$ is a T-stable subspace of $\mathfrak{g}$ of dimension 1. Without loss of generality, $\mathrm{Lie}\, U - \mathfrak{g}_\alpha$. Likewise,

$$nBn^{-1} = nUn^{-1} \rtimes T \implies \mathrm{Lie}\,(nBn^{-1}) = \mathfrak{t} \oplus \mathrm{Lie}\,(nUn^{-1})$$

Since $\mathrm{Lie}\,(nUn^{-1}) = \mathrm{Ad}(n)(\mathrm{Lie}\, U)$ and $[n] \in W$ acts as -1 on $X^*(T)$, $\mathrm{Lie}\,(nUn^{-1}) = \mathfrak{g}_{-\alpha}$.

(iv). This is a surjective map of homogeneous $U \times B$ spaces.

$$\begin{aligned}unb = n &\implies u \in U \cap nBn^{-1} = U \cap nUn^{-1}, \text{ which is finite by Theorem 163}\\ &\implies U \cap nUn^{-1} = 1,\\ &\quad (\text{as } T \text{, being connected, acts trivially by conjugation } \implies U \cap nUn^{-1} \subset \mathcal{Z}_G(T) = T)\\ &\implies \psi \text{ is injective, hence bijective}\end{aligned}$$

$$
\begin{aligned}
d\phi \text{ bijective} &\iff d\Big(U \times B \xrightarrow{(u,b)\,\mapsto\, unbn^{-1}} UnBn^{-1} \Big) \text{ injective} \\
&\iff d(U \times (nBn^{-1}) \xrightarrow{\text{mult.}} UnBn^{-1}) \text{ injective} \\
&\iff 0 = \operatorname{Lie} U \cap \operatorname{Lie}(nBn^{-1}) = \mathfrak{g}_\alpha \cap (\mathfrak{t} \oplus \mathfrak{g}_{-\alpha})
\end{aligned}
$$

(v). See Springer 7.2.4. □

6.4 Reductive groups of semisimple rank 1.

Lemma 179. *If $\phi : H \to K$ is a morphism of algebraic groups, then*

$$d\phi(\operatorname{Ad}(h) \cdot X) = \operatorname{Ad}(\phi(h)) \cdot d\phi X$$

Proof. Exercise. (Easy!) □

Proposition 180. *Suppose that* ss-rk $G = 1$. *Set $\overline{G} := G/RG$ and $\overline{T} :=$ image of T in $\overline{G}$ (T being a maximal torus). Note that $X^*(\overline{T}) \subset X^*(T)$ as $T \twoheadrightarrow \overline{T}$.*

(i) *There is $\alpha \in X^*(\overline{T})$ such that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, and $\dim \mathfrak{g}_{\pm\alpha} = 1$.*

(ii) *$\mathcal{D}G \cong \mathrm{SL}_2$ or PGL_2*

(iii) *$\#W = 2$, so there are precisely two Borels containing T, and, if B is one, then*

$$\operatorname{Lie} B = \mathfrak{t} \oplus \mathfrak{g}_{\pm\alpha} \quad \text{and} \quad \operatorname{Lie} B_u = \mathfrak{g}_{\pm\alpha}$$

(iv) *If T_1 denotes the unique maximal torus of $\mathcal{D}G$ contained in T, then $\exists!\, \alpha^\vee \in X_*(T_1) \subset X_*(T)$ such that $\langle \alpha, \alpha^\vee \rangle = 2$. Moreover, letting $W = \{1, s_\alpha\}$, we have*

$$
\begin{aligned}
s_\alpha \mu &= \mu - \langle \mu, \alpha^\vee \rangle \alpha \quad \text{for all } \mu \in X^*(T) \\
s_\alpha \lambda &= \lambda - \langle \alpha, \lambda \rangle \alpha^\vee \quad \text{for all } \lambda \in X_*(T)
\end{aligned}
$$

Proof.

(i): $\overline{G}$ is semisimple of rank 1.
We have

$$0 \to \operatorname{Lie} RG \to \operatorname{Lie} G \to \operatorname{Lie} \overline{G} \to 0$$

From Lemma 179, restricting actions, we have that the morphisms $T \to \overline{T}$ and $\operatorname{Lie} G \to \operatorname{Lie} \overline{G}$ are compatible with the action of T on $\operatorname{Lie} G$ and $\overline{T}$ on $\operatorname{Lie} \overline{G}$. T acts trivially on $\operatorname{Lie} RG$ (as $RG \subset T$). Thus,

$$\Phi = \Phi(\overline{G}, \overline{T}) = \{\alpha, -\alpha\} \subset X^*(\overline{T}) \subset X^*(T)$$

and $\dim \mathfrak{g}_{\pm\alpha} = 1$.

(ii): $\mathcal{D}G$ is semisimple by Proposition 159. If $T_1 \subset \mathcal{D}G$ is a maximal torus with image $\overline{T}_1$ in $\overline{G}$, then

$$\dim T_1 = \dim \overline{T}_1 + \dim(T_1 \cap RG) \leqslant 1$$

the inequality being due to the fact that $T_1 \cap RG \subset \mathcal{D}G \cap RG$ is finite by Proposition 159. If $\dim T_1 = 0$, then the Borel of $\mathcal{D}G$ is unipotent, implying that $\mathcal{D}G$ is solvable, which gives that G is solvable, a contradiction. Hence, rk $\mathcal{D}G = 1$. By Proposition 177, $\mathcal{D}G \cong \mathrm{SL}_2$ or PGL_2.

(iii): First a lemma.

Lemma 181. *Suppose that $\pi : G \twoheadrightarrow G'$ with $\ker \pi$ connected and solvable. Then $\pi(T)$ is a maximal torus of G' and we have a bijection*

$$\{\text{Borels of } G \text{ containing } T\} \underset{\pi^{-1}}{\overset{\pi}{\rightleftarrows}} \{ \text{ Borels of } G' \text{ containing } \pi(T) \}$$

Moreover, G' is reductive.

Proof of lemma. In the proposed bijection, $\xrightarrow{\pi}$ is well-defined by Corollary 127. For the inverse, note that $G/\pi^{-1}(B') \to G'/B'$ is bijective, which gives that $\pi^{-1}(B')$ is parabolic as well as connected and solvable ($\ker \pi$ and B' are connected and solvable).

$\pi^{-1}(RG')$ is a connected, solvable, normal subgroup of the torus RG. $RG' = \pi(\pi^{-1}(RG'))$ is then a torus and so G' is reductive. □

By the Lemma, $\#W = \#W(\overline{G}, \overline{T}) \overset{177}{=} 2$. Pick a Borel $B \supset T$, so that $\overline{B} \supset \overline{T}$ is a Borel.

$$1 \to RG \to B \to \overline{B} \to 1$$

being exact implies that

$$0 \to \mathrm{Lie}\, RG \to \mathrm{Lie}\, B \to \mathrm{Lie}\, \overline{B} \to 0$$

is also exact. T again acts trivially on $\mathrm{Lie}\, RG$.

$$\mathrm{Lie}\, \overline{B} = \mathrm{Lie}\, T \oplus \mathfrak{g}_{\pm\alpha} \implies \mathrm{Lie}\, B = \mathfrak{t} \oplus \mathfrak{g}_{\pm\alpha}.$$

Also,

$$\mathrm{Lie}\, B = \mathfrak{t} \oplus \mathrm{Lie}\, B_u \implies \mathrm{Lie}\, B_u = \mathfrak{g}_{\pm\alpha}$$

(iv) T_1 exists, as $\mathcal{D}G \trianglelefteq G$ (exercise). It is unique, as $T_1 = (T \cap \mathcal{D}G)^0$. (Another exercise: $T_1 = T \cap \mathcal{D}G$. Use that $\mathcal{D}G$ is reductive.) Let y be a generator of $X_*(T) \cong \mathbf{Z}$. We have the containment

$$\mathrm{Lie}\, \mathcal{D}G \subset \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

with T_1 acting in the former and T on the latter. $\mathcal{D}G$ being reductive implies – by Proposition 177 –

$$\Phi(\mathcal{D}G, T_1) = \{\pm\alpha|_{T_1}\}.$$

<u>$\mathcal{D}G \cong \mathrm{SL}_2$:</u>

$$T_1 = \{\begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \mid x \in k^\times\} \subset \mathrm{SL}_2.$$

By the adjoint action (conjugation), T_1 acts on

$$\operatorname{Lie}\mathrm{SL}_2 = M_2(k)_{\text{trace } 0} \ = k\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \oplus k\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \oplus k\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Its roots are

$$\alpha : \begin{pmatrix}x & \\ & x^{-1}\end{pmatrix} \mapsto x^2, \quad -\alpha : \begin{pmatrix}x & \\ & x^{-1}\end{pmatrix} \mapsto x^{-2}.$$

Moreover, we can take

$$y = x \mapsto \begin{pmatrix}x & \\ & x^{-1}\end{pmatrix}$$

(or its inverse), which gives

$$\langle \alpha, y\rangle = \pm 2.$$

$\underline{\mathcal{D}G \cong \mathrm{PGL}_2 \cong \mathrm{GL}_2/\mathbf{G}_m}$:
T_1 is equal to the image of D_2 in PGL_2. By the adjoint action, T_1 acts on

$$\operatorname{Lie}\mathrm{PGL}_2 = M_2(k)/k = k\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right] \oplus k\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right] \oplus k\left[\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right].$$

Its roots are

$$\alpha : \left[\begin{pmatrix}x_1 & \\ & x_2\end{pmatrix}\right] \mapsto x_1x_2^{-1}, \quad -\alpha : \left[\begin{pmatrix}x_1 & \\ & x_2\end{pmatrix}\right] \mapsto (x_1x_2^{-1})^{-1} = x_1^{-1}x_2.$$

Moreover, we can take

$$y = x \mapsto \left[\begin{pmatrix}x & \\ & 1\end{pmatrix}\right]$$

(or its inverse), which gives

$$\langle \alpha, y\rangle = \pm 1.$$

Therefore, in any case,

$$\alpha^\vee := \frac{2y}{\langle \alpha, y\rangle} \in X_*(T_1)$$

and it is the unique cocharacter such that $\langle \alpha, \alpha^\vee\rangle = 2$.

If $\lambda \in X_*(T)$,

$$s_\alpha\lambda - \lambda : \mathbf{G}_m \to T, \quad x \mapsto [n, \lambda(x)] = n\lambda(x)n^{-1}\lambda(x)^{-1},$$

where $n \in N_G(T)$ is such that $[n] = s_\alpha$. $s_\alpha\lambda - \lambda$ has image in $(T \cap \mathcal{D}G)^0 = T_1$; hence

$$s_\alpha\lambda - \lambda \in X_*(T_1) = \mathbf{Z}y.$$

Say $s_\alpha\lambda - \lambda = \theta(\lambda)y$. We have

$$\begin{aligned}\theta(\lambda)\langle \alpha, y\rangle &= \langle \alpha, s_\alpha\lambda - \lambda\rangle = \langle \alpha, s_\alpha\lambda\rangle - \langle \alpha, \lambda\rangle\\ &= \langle s_\alpha(\alpha), \lambda\rangle - \langle \alpha, \lambda\rangle.\end{aligned}$$

At this point we see that $s_\alpha(\alpha) = -\alpha$. (Otherwise, $s_\alpha(\alpha) = \alpha$, which implies $\theta = 0$, i.e. that s_α acts trivially on $X_*(T)$, which is a contradiction.) So we can continue:

$$= \langle -\alpha, \lambda\rangle - \langle \alpha, \lambda\rangle$$
$$= -2\langle \alpha, \lambda\rangle$$

Therefore,

$$\theta(\lambda) = \frac{-2\langle \alpha, \lambda\rangle}{\langle \alpha, y\rangle}$$

and

$$s_\alpha\lambda = \lambda + \theta(\lambda)y = \lambda - \frac{2\langle \alpha, \lambda\rangle}{\langle \alpha, y\rangle}y = \lambda - \langle \alpha, \lambda\rangle\alpha^\vee.$$

If $\mu \in X^*(T)$, then for all $\lambda \in X_*(T)$

$$\langle s_\alpha\mu, \lambda\rangle = \langle \mu, s_\alpha\lambda\rangle = \langle \mu, \lambda\rangle - \langle \alpha, \lambda\rangle\langle \mu, \alpha^\vee\rangle = \langle \mu - \langle \mu, \alpha^\vee\rangle\alpha, \lambda\rangle$$

and so

$$s_\alpha\mu = \mu - \langle \mu, \alpha^\vee\rangle\alpha.$$

□

Lemma 182.

(i) *Let $S \subset T$ be a subtorus such that* $\dim S = \dim T - 1$. *Then*

$$\ker(\text{res} : X^*(T) \to X^*(S)) = \mathbf{Z}\mu$$

for some $\mu \in X^*(T)$.

(ii) *If $\nu \in X^*(T)$, $m \in \mathbf{Z} - \{0\}$, then* $(\ker \nu)^0 = (\ker m\nu)^0$.

(iii) *If $\nu_1, \nu_2 \in X^*(T) - \{0\}$, then*

$$(\ker \nu_1)^0 = (\ker \nu_2)^0 \iff m\nu_1 = n\nu_2$$

for some $m, n \in \mathbf{Z} - \{0\}$.

Proof.
(i): res is surjective (exercise, cf. the proof of Proposition 47) and

$$X^*(T) \cong \mathbf{Z}^r, \quad X^*(S) \cong \mathbf{Z}^{r-1}.$$

(ii):
"$\subset$": $\nu(t) = 1 \implies \nu(t)^n = 1$.
"$\supset$": $t \in (\ker m\nu)^0 \implies \nu(t)^n = 1$, so $\nu((\ker m\nu)^0)$ is connected and finite, hence trivial.

(iii):
"$\Leftarrow$": Clear from (ii).
"$\Rightarrow$": Define $S = (\ker \nu_1)^0 = (\ker \nu_2)^0 \subset T$, as in (i). Clearly, $\text{res}(\nu_1) = \text{res}(\nu_2) = 0$, so $v_i \in \mathbf{Z}\mu$. The result follows. □

6.5 Root data.

Definitions 183. *A* **root datum** *is a quadruple* $(X, \Phi, X^\vee, \Phi^\vee)$*, where*

(i) $X, X^\vee$ *are free abelian groups of finite rank with a perfect bilinear pairing* $\langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbf{Z}$

(ii) $\Phi \subset X$ *and* $\Phi^\vee \subset X^\vee$ *are finite subsets with a bijection* $\Phi \to \Phi^\vee$, $\alpha \mapsto \alpha^\vee$

(the pairing and the bijection also being part of the root datum) satisfying the following axioms:

(1) $\langle \alpha, \alpha^\vee \rangle = 2$ *for all* $\alpha \in \Phi$

(2) $s_\alpha(\Phi) = \Phi$ *and* $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$ *for all* $\alpha \in \Phi$

where the "reflections" are given by

$$\begin{aligned} s_\alpha &: X \to X & s_{\alpha^\vee} &: X^\vee \to X^\vee \\ x &\mapsto x - \langle x, \alpha^\vee \rangle \alpha : & y &\mapsto y - \langle \alpha, y \rangle \alpha^\vee \end{aligned}$$

A root datum is **reduced** *if* $\mathbf{Q}\alpha \cap \Phi = \{\pm\alpha\}$ *for all* $\alpha \in \Phi$.

Remark 184. *Note that the axioms imply that* $s_\alpha(\alpha) = -\alpha$*, so* $\Phi = -\Phi$*, and* $s_\alpha^2 = 1$ *(so* s_α *is a group isomorphism). Similarly,* $s_{\alpha^\vee}(\alpha^\vee) = -\alpha^\vee$*, so* $\Phi^\vee = -\Phi^\vee$*, and* $s_{\alpha^\vee}^2 = 1$*. Also* $0 \notin \Phi$ *and* $0 \notin \Phi^\vee$*, and* $\langle s_\alpha(\mu), s_{\alpha^\vee}(\lambda) \rangle = \langle \mu, \lambda \rangle$*. (It is less obvious from the axioms, but also true, that* $(-\alpha)^\vee = -\alpha^\vee$ *and hence that* $s_{-\alpha} = s_\alpha$*. For more on root data, see SGA3, Exposé XXI.)*

Recall that $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, $T_\alpha = (\ker \alpha)^0$, $G_\alpha = \mathcal{Z}_G(T_\alpha)$.

Theorem 185.

(i) *For all* $\alpha \in \Phi$*,* G_α *is connected, reductive of semisimple rank 1.*

- $\operatorname{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$
- $\dim \mathfrak{g}_{\pm\alpha} = 1$
- $\mathbf{Q}\alpha \cap \Phi = \{\pm\alpha\}$

(ii) *Let* s_α *be the unique nontrivial element of* $W(G_\alpha, T) \subset W(G, T)$*. Then there exists a unique* $\alpha^\vee \in X_*(T)$ *such that* $\operatorname{im} \alpha^\vee \subset \mathcal{D}G_\alpha$ *and* $\langle \alpha, \alpha^\vee \rangle = 2$*. Moreover,*

$$\begin{aligned} s_\alpha \mu &= \mu - \langle \mu, \alpha^\vee \rangle \alpha, \quad \text{for all } \mu \in X^*(T), \\ s_\alpha \lambda &= \lambda - \langle \alpha, \lambda \rangle \alpha^\vee, \quad \text{for all } \lambda \in X_*(T). \end{aligned}$$

(iii) *Let* $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$*. Then* $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ *is a reduced root datum.*

(iv) $W(G, T) = \langle s_\alpha \mid \alpha \in \Phi \rangle$.

Proof.
(i). We saw above that G_α is connected, reductive of semisimple rank 1.

$$\operatorname{Lie} G_\alpha = \operatorname{Lie} \mathcal{Z}_G(T_\alpha) \overset{100}{=} \mathfrak{z}_\mathfrak{g}(T_\alpha) = \mathfrak{t} \oplus \bigoplus_{\substack{\beta \in \Phi \\ \beta|_{T_\alpha} = 1}} \mathfrak{g}_\beta$$

By Proposition 180,

$$\operatorname{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$$

with $\dim \mathfrak{g}_{\pm\alpha} = 1$. Hence, for all $\beta \in \Phi$,

$$\begin{aligned} \beta|_{T_\alpha} = 1 &\iff \beta \in \{\pm\alpha\} \\ &\iff (\ker\alpha)^0 \subset (\ker\beta)^0 \\ &\iff (\ker\alpha)^0 = (\ker\beta)^0 \quad \text{(dimension considerations)} \\ &\iff \beta \in \mathbf{Q}\alpha \quad \text{(Lemma 182)} \end{aligned}$$

(ii): Follows from Proposition 180

(iii):

$\alpha \mapsto \alpha^\vee$ is bijective ($\iff$ injective):

If $\alpha^\vee = \beta^\vee$, then

$$\begin{aligned} s_\alpha s_\beta(x) &= (x - \langle x, \beta^\vee\rangle\beta) - \langle (x - \langle x, \beta^\vee\rangle\beta), \alpha^\vee\rangle\alpha \\ &= x - \langle x, \alpha^\vee\rangle(\alpha+\beta) + \langle x, \alpha^\vee\rangle\langle\beta, \beta^\vee\rangle\alpha \\ &= x - \langle x, \alpha^\vee\rangle(\alpha+\beta) + 2\langle x, \alpha^\vee\rangle\alpha \\ &= x + \langle x, \alpha^\vee\rangle(\alpha-\beta) \end{aligned}$$

Therefore, if $\langle \alpha - \beta, \alpha^\vee\rangle = 0$, then for some n

$$\begin{aligned} (s_\alpha s_\beta)^n = 1 &\implies \forall\, x, \quad x = (s_\alpha s_\beta)^n(x) = x + n\langle x, \alpha^\vee\rangle(\alpha-\beta) \\ &\implies \forall\, x, \quad 0 = n\langle x, \alpha^\vee\rangle(\alpha-\beta) \\ &\implies 0 = \alpha - \beta \\ &\implies \alpha = \beta \end{aligned}$$

$s_\alpha\Phi = \Phi$:

The action of $s_\alpha \in W$ on $X^*(T)$ (and $X_*(T)$) agrees with the action of s_α (and $s_{\alpha^\vee}$) in the definition of a root datum by (ii). Also, as noted above, $W = N_G(T)/T$ preserves Φ.

$s_{\alpha^\vee}\Phi^\vee = \Phi^\vee$:

For $w = [n] \in W$, $(n \in N_G(T))$, $\beta \in \Phi$

$$w\beta(\cdot) = \beta(n^{-1}(\cdot)n) \implies \ker(w\beta) = n(\ker\beta)n^{-1} \implies T_{w\beta} = nT_\beta n^{-1}, G_{w\beta} = nG_\beta n^{-1}$$

From

$$\operatorname{im}(w(\beta^\vee)) = \operatorname{im}(n\beta^\vee n^{-1}) \subset n\mathcal{D}G_\beta n^{-1} = \mathcal{D}G_{w\beta}$$

and

$$\langle w\beta, w(\beta^\vee)\rangle = \langle \beta, \beta^\vee\rangle = 2$$

by (ii), we have that $(w\beta)^\vee = w(\beta^\vee)$. (iii) follows.

(iv): Induct on $\dim G$. Let $w = [n] \in W$, $n \in N_G(T)$. As in the proof of Theorem 152 consider the homomorphism

$$\phi : T \to T, \quad t \mapsto [t, n] = ntn^{-1}t^{-1}.$$

$\underline{\operatorname{im}\phi \neq T}$:
$S := (\ker\phi)^0 \neq 1$ is a torus and $n \in \mathcal{Z}_G(S)$. (Note that $\mathcal{Z}_G(S)$ is connected, reductive by Corollary 164. Its roots are $\{\alpha \in \Phi \mid \alpha|_S = 1\}$ and $W(\mathcal{Z}_G(S), T) \subset W(G, T)$.) If $\mathcal{Z}_G(S) \neq G$, we are done by induction.

If $\mathcal{Z}_G(S) = G$, then $S \subset \mathcal{Z}_G$. Define $\overline{G} = G/S$, which is reductive by Lemma 181, and $\overline{T} = T/S$, which is a maximal torus of $\overline{G}$. By induction, the (iv) holds for $\overline{G}$.

$$\Phi(G,T) = \Phi(\overline{G}, \overline{T}) \subset X^*(\overline{T}) \subset X^*(T).$$

It is an easy check that we have

$$N_G(T)/T = W(G,T) \xrightarrow{\sim} W(\overline{G}, \overline{T}) = N_{\overline{G}}(\overline{T})/\overline{T}$$

restricting to

$$W(G_\alpha, T) \xrightarrow{\sim} W(\overline{G}_\alpha, \quad s_\alpha \mapsto s_\alpha.$$

Therefore, (iv) follows for $\overline{G}$.

$\underline{\operatorname{im}\phi = T}$:
ϕ being surjective is equivalent to

$$\phi^* : X^*(T) \to X^*(T), \quad \mu \mapsto (w^{-1} - 1)\mu$$

is injective. Hence, $w - 1 : V \to V$ is injective, thus bijective, where $V = X^*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. Fix $\alpha \in \Phi$. Let $x \in V - \{0\}$ be such that $\alpha = (w-1)x$. Pick a W-invariant inner product $(,) : V \times V \to \mathbf{R}$ (averaging a general inner product over W). Then

$$(x,x) = (wx, wx) = (x + \alpha, x + \alpha) = (x,x) + 2(x,\alpha) + (\alpha,\alpha) \implies 2(x,\alpha) = -(\alpha,\alpha).$$

Also, $s_\alpha x = x + c\alpha$ (where $c = -\langle x, \alpha^\vee\rangle \in \mathbf{Z}$) and, as $s_\alpha^2 = 1$,

$$\begin{aligned}(x,\alpha) + c(\alpha,\alpha) = (s_\alpha x, \alpha) = (x, s_\alpha(\alpha)) = -(x,\alpha) &\implies 2(x,\alpha) = -c(\alpha,\alpha)\\ &\implies c = 1\\ &\implies s_\alpha x = x + \alpha = wx\\ &\implies (s_\alpha w)x = x.\end{aligned}$$

Therefore, redefining ϕ with $s_\alpha w$ instead of w, we have that $\operatorname{im}\phi \neq T$, and we are done by the previous case. □

Remarks 186.
• *Let V be the subspace generated by Φ in $X \otimes \mathbf{R}$ (for X in a root datum). Then Φ is a root system in V. (See* §14.7 *in Borel's Linear Algebraic Groups; references are there.) If $V = X \otimes \mathbf{R}$ (which, in fact, is equivalent to G being semisimple), then (X, Φ) uniquely determines $(X, \Phi, X^\vee, \Phi^\vee)$.*
• *The root datum of Theorem 185 does not depend (up to isomorphism) on the choice of T, as any two maximal tori are conjugate.*

Facts:

1. Isomorphism Theorem: Two connected reductive groups are isomorphic $\iff$ their root data are isomorphic.

2. Existence Theorem: Given a reduced root datum, there exists a reductive group that has the root datum.

(See Springer §9, §10.)

Theorem 187.

(i) *For all* $\alpha \in \Phi$ *there is a unique connected closed* T*-stable unipotent subgroup* $U_\alpha \subset G$ *such that* $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$*. There exists an isomorphism* $u_\alpha : \mathbf{G}_a \xrightarrow{\sim} U_\alpha$ *(unique up to scalar) such that*

$$tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x) \quad \text{for all } x \in \mathbf{G}_a, t \in T.$$

(ii) $G = \langle T,\, U_\alpha\ (\alpha \in \Phi)\rangle$ *(i.e., G is the smallest subgroup containing T and all of the U_α)*

(iii) $\mathcal{Z}_G = \bigcap_{\alpha\in\Phi} \ker\alpha$

Proof.
(i): Let B_α denote the Borel subgroup of G_α containing T with $\operatorname{Lie} B_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha$ (Proposition 180, Theorem 185.) Then $U_\alpha := (B_\alpha)_u$ satisfies all assumptions by Proposition 180. Also, $\dim U_\alpha = \dim \mathfrak{g}_\alpha = 1$ and $U_\alpha \cong \mathbf{G}_a$ by Theorem 60. Let $u_\alpha : \mathbf{G}_a \to U_\alpha$ denote any isomorphism; any other differs by a scalar as $\operatorname{Aut} \mathbf{G}_a \cong \mathbf{G}_m$. So $tu_\alpha(x)t^{-1} = u_\alpha(\chi(t)x)$ for some $\chi(t) \in k^\times$. Via u_α, identify $U_\alpha \xrightarrow{t(\cdot)t^{-1}} U_\alpha$ with $\mathbf{G}_a \xrightarrow{\chi(t)} \mathbf{G}_a$. Since the derivative of the former is $\mathfrak{g}_\alpha \xrightarrow{\operatorname{Ad}(t)=\alpha(t)} \mathfrak{g}_\alpha$, we see that the derivative of the latter is $k \xrightarrow{\alpha(t)} k$. However, by Theorem 78, we must have $\alpha(t) = \chi(t)$ – and thus $\alpha = \chi$.

It remain to show that U_α is unique. If U'_α is another connected, closed, T-stable, and unipotent with $\operatorname{Lie} U'_\alpha = \mathfrak{g}_\alpha$, by the same argument as above we get an isomorphism $u'_\alpha : \mathbf{G}_a \to U'_\alpha$ such that $tu'_\alpha(x)t^{-1} = u'_\alpha(\alpha(t)x)$. Hence, $U'_\alpha \subset G_\alpha$ (as $\alpha(T_\alpha) = 1$).

$$\begin{aligned}
T \text{ normalises } U'_\alpha &\implies TU'_\alpha \text{ is closed, connected, and solvable (exercise)}\\
&\implies TU'_\alpha \text{ is contained in a Borel containing } T\\
&\implies TU'_\alpha \subset B_\alpha, \quad \text{as } \operatorname{Lie} U'_\alpha = \mathfrak{g}_\alpha\\
&\implies U'_\alpha = (TU'_\alpha)_u \subset (B_\alpha)_u = U_\alpha\\
&\implies U'_\alpha = U_\alpha \text{ (dimension)}
\end{aligned}$$

(ii): By Corollary 21, $\langle T, U_\alpha\ (\alpha \in \Phi)\rangle$ is connected, closed. Its Lie algebra contains $\mathfrak{t}$ and all of the $\mathfrak{g}_\alpha$, hence coincides with $\mathfrak{g}$. Thus

$$\dim\langle T, U_\alpha\ (\alpha \in \Phi)\rangle = \dim \mathfrak{g} = \dim G \implies \langle T, U_\alpha\ (\alpha \in \Phi)\rangle = G$$

(iii): $\mathcal{Z}_G \subset T$ by Corollary 164 By (i), $t \in T$ commutes with $U_\alpha \iff \alpha(t) = 1$, which implies that $\mathcal{Z}_G \subset \bigcap_\alpha \ker\alpha$. The reverse inclusion follows by (ii). ☐

Appendix. An example: the symplectic group

Set $G = \mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid g^t J g = J\}$, where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$.
Fact. G is connected. (See, for example, Springer 2.2.9(1) or Borel 23.3.[2])
Define

$$\begin{aligned} T = G \cap D_{2n} &= \{\mathrm{diag}(x_1, \ldots, x_{2n}) \mid \mathrm{diag}(x_1, \ldots, x_{2n}) \cdot \mathrm{diag}(x_{n+1}, \ldots, x_{2n}, x_1, \ldots, x_n) = I\} \\ &= \{\mathrm{diag}(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})\} \\ &\cong \mathbf{G}_m^n \end{aligned}$$

Clearly $\mathcal{Z}_G(T) = T$, implying that T is a maximal torus and $\mathrm{rk}\, G = n$. Write ϵ_i, $1 \leqslant i \leqslant n$, for the morphisms

$$T \to \mathbf{G}_m, \quad \mathrm{diag}(x_1, \ldots, x_n^{-1}) \mapsto x_i,$$

which form a basis of $X^*(T)$.

Lemma 188. *If $\rho : G \to \mathrm{GL}(V)$ is a faithful (or just injective) G-representation that is semisimple, then G is reductive.*

Proof.
$U := R_u G$ is a connected, unipotent, normal subgroup of G. Write $V = \bigoplus_{i=1}^r V_i$ with V_i irreducible (V is semisimple). $V_i^U \neq 0$, as U is unipotent (Proposition 40), and $V_i^U \subset V_i$, is G-stable, as $U \trianglelefteq G$: $V_i^U = V_i$. Hence, U acts trivially on V, and is thus trivial, since ρ is injective. □

We will show that the natural faithful representation $G \hookrightarrow \mathrm{GL}_{2n}$ is irreducible and hence G is reductive. Let $V = k^{2n}$ denote the underlying vector space with standard basis $(e_i)_1^{2n}$. We have $V = \bigoplus_{i=1}^{2n} k e_i$ and, for all $t \in T$,

$$te_i = \begin{cases} \epsilon_i(t) e_i, & i \leqslant n \\ \epsilon_{i-n}(t)^{-1} e_i, & i > n \end{cases}$$

Any G-subrepresentation of V is a direct sum of T-eigenspaces; hence, it is enough to show that $N_G(T)$ acts transitively on the ke_i, which is equivalent to it acting transitively on $\{\pm\epsilon_1, \ldots, \pm\epsilon_n\} \subset X^*(T)$.

[2]For another elementary proof, see my post here: `http://mathoverflow.net/questions/98881/connectedness-of-the-linear-algebraic-group-so-n`.

A calculation shows that the elements

$$g_i := \operatorname{diag}(I_{i-1}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), I_{n-2}, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), I_{n-i-1}), \quad (1 \leqslant i < n)$$

lie in G, where $\operatorname{diag}(A_1, A_2, \dots)$ denotes a matrix with square blocks $A_1, A_2, \dots$ along the diagonal in the given order. As well

$$g_n := \begin{pmatrix} \operatorname{diag}(I_{n-1}, 0) & E_{nn} \\ -E_{nn} & \operatorname{diag}(I_{n-1}, 0) \end{pmatrix},$$

lies in G, where $E_{nn} \in M_n(k)$ has a 1 in the (n, n)-entry and 0's elsewhere. Note that the $g_i \in N_G(T)$ for all i and $g_i : \epsilon_i \mapsto \epsilon_{i+1}$, for $1 \leqslant i < n$, and $g_n : \epsilon_n \mapsto -\epsilon_n$ (with $g_i \cdot \epsilon_j = \epsilon_j$ for $i \neq j$). Hence, $N_G(T)$ does act transitively on $\{\pm\epsilon_i\}$, so V is irreducible and G is reductive.

Lie Algebra:

If $\psi : \mathrm{GL}_{2n} \to \mathrm{GL}_{2n}$, $g \mapsto g^t J g$, then $d\psi_1 : M_{2n}(k) \to M_{2n}(k)$, $X \mapsto X^t J + JX$ (as in the proofs of Propositions 79 and 80). Hence,

$$\mathfrak{g} \subset \{X \in M_{2n}(X) \mid X^t J + JX\} =: \mathfrak{g}'.$$

Checking that $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{g}'$ if and only if $B^t = B, C^t = C$, and $D = -A^t$, we see that

$$\dim \mathfrak{g}' = n^2 + 2\binom{n+1}{2} = n(2n+1)$$

Claim: $\dim G \geqslant n(2n+1)$

Define

$$\phi : \mathrm{GL}_{2n} \to \mathbf{A}^{\binom{2n}{2}}, \quad g \mapsto ((g^t J g)_{ij})_{i<j}.$$

We have $\phi^{-1}((J_{ij})_{i<j}) = G$, (because $g^t J g$ is antisymmetric). (This is still okay when $p = 2$.) So,

$$(2n)^2 = \dim \mathrm{GL}_{2n} \overset{87}{=} \dim \overline{\phi(\mathrm{GL}_{2n})} + \text{ minimal fibre dimension} \leqslant \binom{2n}{2} + \dim G$$

and

$$\dim G \geqslant (2n)^2 - \binom{2n}{2} = n(2n+1).$$

Hence,

$$\dim \mathfrak{g} \leqslant n(2n+1) \leqslant \dim G = \dim \mathfrak{g} \implies \dim \mathfrak{g} = n(2n+1)$$

and so

$$\dim G = n(2n+1), \quad \text{and} \quad \mathfrak{g} = \{X \in M_{2n}(k) \mid X^t J + JX = 0\}.$$

Roots:

Write E_{ij} for the $(2n) \times (2n)$ matrix with a 1 in the (i, j)-entry and 0's elsewhere. By the above,

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{i \neq j} k \left(\begin{smallmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{smallmatrix}\right) \right) \oplus \left(\bigoplus_{i \leqslant j} k \left(\begin{smallmatrix} 0 & E_{ij}+E_{ji} \\ 0 & 0 \end{smallmatrix}\right) \right) \oplus \left(\bigoplus_{i \leqslant j} k \left(\begin{smallmatrix} 0 & 0 \\ E_{ij}+E_{ji} & 0 \end{smallmatrix}\right) \right)$$

(with $E_{ij}+E_{ji}$ in the latter factors replaced with E_{ii} if $i=j$ and $p=2$). Correspondingly,

$$\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j\} \cup \{\epsilon_i + \epsilon_j \mid i \leqslant j\} \cup \{-(\epsilon_i + \epsilon_j) \mid i \leqslant j\}$$

(A check: $\#\Phi = n(n-1) + \binom{n+1}{2} + \binom{n+1}{2} = 2n^2 = \dim \mathfrak{g} - \dim \mathfrak{t}$.)

Coroots:

Let $\epsilon_1^*, \ldots, \epsilon_n^*$ denote the dual basis, so

$$\epsilon_i^*(x) = \mathrm{diag}(1, \ldots, x, \ldots, x^{-1}, \ldots, 1) = \mathrm{diag}(I_{i-1}, x, I_{n-1}, x^{-1}, I_{n-i}).$$

We have

$$G_{\epsilon_i - \epsilon_j} = G \cap (D_{2n} + kE_{ij} + kE_{ji} + kE_{n+i,n+j} + kE_{n+j,n+i})$$

and so $G_{\epsilon_i-\epsilon_j}$ is contained in

$$G \cap \{I_{2n} + (a-1)E_{ii} + bE_{ij} + cE_{ji} + (d-1)E_{jj} + (a'-1)E_{n+i,n+i} + b'E_{n+i,n+j} + c'E_{n+j,n+i} + (d'-1)E_{n+j,n+j}\}$$

where $a, b, c, d, a', b', c', d'$ are such that $ad - bc = 1 = a'd' - b'c'$. It follows that

$$(\epsilon_i - \epsilon_j)^\vee = \epsilon_i^* - \epsilon_j^*.$$

Similarly, $(\epsilon_i + \epsilon_j)^\vee = \epsilon_i^* + \epsilon_j^*$ and $(-\epsilon_i - \epsilon_j)^\vee = -\epsilon_i^* - \epsilon_j^*$.

G is semisimple: $RG = \mathcal{Z}_G^0 = \left(\bigcap_\Phi \ker \alpha\right)^0 = 1$.

A Borel subgroup of G: We can explicitly compute a Borel subgroup, for example as explained for the even orthogonal group in Homework 4 (2017). (For this it would be more convenient to choose an antidiagonal form J when we define G!)