A CLASSIFICATION OF IRREDUCIBLE ADMISSIBLE MOD p REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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Abstract. Let F be a locally compact non-archimedean field, p its residue characteristic, and $\mathbf{G}$ a connected reductive group over F. Let C be an algebraically closed field of characteristic p. We give a complete classification of irreducible admissible C-representations of $G = \mathbf{G}(F)$, in terms of supercuspidal C-representations of the Levi subgroups of G, and parabolic induction. Thus we push to their natural conclusion the ideas of the third-named author, who treated the case $\mathbf{G} = \mathrm{GL}_m$, as further expanded by the first-named author, who treated split groups $\mathbf{G}$. As in the split case, we first get a classification in terms of supersingular representations of Levi subgroups, and as a consequence show that supersingularity is the same as supercuspidality.

Contents

I. Introduction

I.1. The study of congruences between classical modular forms has met considerable success in the past decades. When interpreted in the natural framework of automorphic

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forms and representations, such congruences naturally lead to representations over fields of positive characteristic, rather than complex representations. In our local setting, where the base field is a locally compact non-archimedean field F, this means studying representations of $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a connected reductive group over F, on vector spaces over a field C of positive characteristic p, which we assume algebraically closed. As C is fixed throughout, we usually say representation instead of representation on a C-vector space or C-representation.

Our representations satisfy natural requirements: they are smooth, in that every vector has open stabilizer in G (smoothness is always understood for representations of G or its subgroups), and most of the time they are **admissible**: a representation of G on a C-vector space W is admissible if it is smooth and for every open subgroup J in G, the space W^J of vectors fixed under J has finite dimension. The overall goal is to understand irreducible admissible representations of G.

Here we consider only the case where the residue characteristic of F is p.

I.2. In this paper we classify irreducible admissible representations of G in terms of parabolic induction and supercuspidal representations of Levi subgroups of G. Such a classification was obtained for $\mathbf{G} = \mathrm{GL}_2$ in the pioneering work of L. Barthel and R. Livné [BL1, BL2] – see also some recent work [Abd, Che, Ko, KX, Ly2] on situations where, mostly, $\mathbf{G}$ has relative semisimple rank 1.

New ideas towards the general case were set forth by the third-named author [He1, He2], who gave the classification for $\mathbf{G} = \mathrm{GL}_n$ over a p-adic field F; his ideas were further expanded by the first-named author [Abe] to treat the case of a split group $\mathbf{G}$, still over a p-adic field F. T. Ly extended the arguments of [He1, He2] to treat $\mathbf{G} = \mathrm{GL}_{3/D}$ where D is a division algebra over F, allowing F to have characteristic p. Here, using the first steps accomplished in [HV1, HV2] (see I.5, I.6), we treat general $\mathbf{G}$ and F.

I.3. To express our classification, we recall parabolic induction. If P is a parabolic subgroup of G and τ a representation of P on a C-vector space W, we write $\mathrm{Ind}_P^G \tau$ for the natural representation of G, by right translation, on the space $\mathrm{Ind}_P^G W$ of smooth functions $f : G \to W$ such that $f(pg) = \tau(p)f(g)$ for p in P, g in G. The functor Ind_P^G is exact. In fact we use $\mathrm{Ind}_P^G \tau$ only when τ comes via inflation from a representation σ of the Levi quotient of P, and we write $\mathrm{Ind}_P^G \sigma$ instead of $\mathrm{Ind}_P^G \tau$. A representation of G is said to be **supercuspidal** if it is irreducible, admissible, and does not appear as a subquotient of a parabolically induced representation $\mathrm{Ind}_P^G \sigma$, where P is a proper parabolic subgroup of G and σ an irreducible admissible representation of the Levi quotient of P.

First we construct irreducible admissible representations of G. The construction uses the "generalized Steinberg" representations investigated by E. Große-Klönne [GK1] and the third-named author [He2] when $\mathbf{G}$ is split, and by T. Ly [Ly1] in general: for any pair of parabolic subgroups $Q \subset P$ in G, St_Q^P is the natural representation of P in the quotient of $\mathrm{Ind}_Q^P 1$ by the sum of the subspaces $\mathrm{Ind}_{Q'}^P 1$, for parabolic subgroups Q' with $Q \subsetneq Q' \subset P$; the representation St_Q^P factors through the unipotent radical U_P of P and gives the representation $\mathrm{St}_{Q/U_P}^{P/U_P}$ of its reductive quotient, so St_Q^P is irreducible and admissible [GK1, Ly1].

Start with a parabolic subgroup P of G, with Levi quotient M, and a representation σ of M. Then there is a largest parabolic subgroup $P(\sigma)$ of G, containing P, such that

σ inflated to P extends to $P(\sigma)$ (see II.7). That extension is unique, we write it ${}^e\sigma$; it is trivial on the unipotent radical of $P(\sigma)$. It is irreducible and admissible if σ is. We consider triples (P, σ, Q): a triple consists of a parabolic subgroup P of G, a representation σ of the Levi quotient M of P, and a parabolic subgroup Q of G with $P \subset Q \subset P(\sigma)$; we say that the triple is **supercuspidal** if σ is a supercuspidal representation of M. To a triple (P, σ, Q) we associate the representation $I(P, \sigma, Q) = \mathrm{Ind}_{P(\sigma)}^G({}^e\sigma \otimes \mathrm{St}_Q^{P(\sigma)})$.

Theorem 1. *For a supercuspidal triple (P, σ, Q), $I(P, \sigma, Q)$ is irreducible and admissible.*

Theorem 2. *Let (P, σ, Q) and (P', σ', Q') be supercuspidal triples. Then $I(P, \sigma, Q)$ and $I(P', \sigma', Q')$ are isomorphic if and only if there is an element g of G such that $P' = gPg^{-1}$, $Q' = g\,Qg^{-1}$ and σ' is equivalent to $p' \mapsto \sigma(g^{-1}p'g)$.*

Theorem 3. *Any irreducible admissible representation of G is isomorphic to $I(P, \sigma, Q)$ for some supercuspidal triple (P, σ, Q).*

Hopefully the classification expressed by these theorems will be useful in extending the mod p local Langlands correspondence beyond $\mathrm{GL}_2(\mathbb{Q}_p)$.

I.4. Using the classification results above, it is possible to describe the irreducible components of $\mathrm{Ind}_P^G\,\sigma$ where P is a parabolic subgroup of G and σ an irreducible admissible representation of the Levi quotient M of P; in particular we show that $\mathrm{Ind}_P^G\,\sigma$ has finite length and that all its irreducible subquotients are admissible and occur with multiplicity one.

Also we have a notion of **supercuspidal support**: if (P, σ, Q) is a supercuspidal triple, then $\pi = I(P, \sigma, Q)$ occurs as a subquotient of $\mathrm{Ind}_P^G\,\sigma$ and if π occurs as a subquotient of $\mathrm{Ind}_{P'}^G\,\sigma'$ for a supercuspidal representation σ' of (the Levi quotient of) a parabolic subgroup P' of G then (P', σ') is conjugate to (P, σ) in G as in Theorem 2. All that is proved in VI.3. It is the conjugacy class of (P, σ) that we call the supercuspidal support of π.

Remark Even in the case of $\mathrm{GL}_n(F)$ (for which we refer to the introduction of [He2]), the classification and its consequences are rather simpler than for complex representations: intertwining operators do not exist in our context; this "explains" the multiplicity one result above, which does not hold for complex representations [Ze]. By contrast, supercuspidal mod p representations remain a complete mystery, apart from the case of $\mathrm{GL}_2(\mathbb{Q}_p)$ [Br] and groups closely related to it [Abd, Che, Ko, KX].

The existence of a supercuspidal support for complex irreducible representations is a classical result [BZ, 2.9 Theorem]; for mod ℓ representations with $\ell \neq p$ it is unknown (even for finite reductive groups of characteristic p outside the case of general linear groups), except for inner forms of $\mathrm{GL}_n(F)$ where, as above, it is not proved directly but is a consequence of the classification of irreducible representations [MS, Théorème A].

I.5. As in [He2, Abe] our classification is not established directly using supercuspidality. Rather we get a classification in terms of supersingular representations of Levi subgroups of G – the term was first used by Barthel and Livné for $G = \mathrm{GL}_2(F)$ – and deduce Theorems 1 to 3 from it. To define supersingularity, we need to make some choices, and a priori the notion depends on these choices.

So we fix a maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$ and a special point $\mathbf{x}_0$ in the apartment corresponding to $\mathbf{S}$ in the semisimple Bruhat-Tits building of G; we let K be the special parahoric subgroup of G corresponding to $\mathbf{x}_0$. We also fix a minimal parabolic subgroup B of G with Levi subgroup Z, the F-points of the centralizer of $\mathbf{S}$, and we write U for the unipotent radical of B.

Let V be an irreducible representation of K – it has finite dimension. If (π, W) is an admissible representation of G, then $\mathrm{Hom}_K(V, W)$ is a finite-dimensional C-vector space; by Frobenius reciprocity $\mathrm{Hom}_K(V, W)$ is identified with $\mathrm{Hom}_G(\mathrm{ind}_K^G V, W)$, where ind_K^G denotes compact induction, so that $\mathrm{Hom}_K(V, W)$ is a right-module over the intertwining algebra $\mathcal{H}_G(V) = \mathrm{End}_G(\mathrm{ind}_K^G V)$ of V in G. If $\mathrm{Hom}_K(V, W)$ is not zero we say that V is a **weight** of π; in that case the centre[1] $\mathcal{Z}_G(V)$ of $\mathcal{H}_G(V)$ has eigenvectors in $\mathrm{Hom}_K(V, W)$, and we focus on the corresponding characters of $\mathcal{Z}_G(V)$, which we call the (Hecke) **eigenvalues** of $\mathcal{Z}_G(V)$ in π.

For any parabolic subgroup P of G containing B, with Levi component M containing Z and unipotent radical N, the space of coinvariants $V_{N\cap K}$ of $N \cap K$ in V provides an irreducible representation of $M \cap K$ and by [He1, He2, HV2] there is a natural injective algebra homomorphism

$$\mathcal{S}_M^G : \mathcal{H}_G(V) \to \mathcal{H}_M(V_{N\cap K})$$

with explicit image (see III.3). It induces a homomorphism between centres $\mathcal{Z}_G(V) \to \mathcal{Z}_M(V_{N\cap K})$. Both homomorphisms are localizations at a central element. A character $\chi : \mathcal{Z}_G(V) \to C$ is said to be **supersingular** if, in the above situation, it can be extended to a character of $\mathcal{Z}_M(V_{N\cap K})$ only when $P = G$ (see Chapter III, part A) for details). A **supersingular** representation of G is an irreducible admissible representation (π, W) such that for all weights V of π, all eigenvalues of $\mathcal{Z}_G(V)$ in π are supersingular[2].

A triple (P, σ, Q) as in I.3 is a B**-triple** if P contains B; it is said to be **supersingular** if it is a B-triple and σ is a supersingular representation of the Levi quotient of P.

Theorems 1 to 3 are consequences of the following results.

Theorem 4. *For each supersingular triple (P, σ, Q), the representation $I(P, \sigma, Q)$ is irreducible and admissible. If π is an irreducible admissible representation of G, there is a supersingular triple (P, σ, Q) such that π is isomorphic to $I(P, \sigma, Q)$; moreover P and Q are unique and σ is unique up to isomorphism.*

Theorem 5. *Let π be an irreducible admissible representation of G. Then π is supercuspidal if and only if it is supersingular.*

(For $\mathbf{G} = \mathrm{GL}_2$ this was discovered by Barthel and Livné.)

Note that Theorem 5 implies, in particular, that the notion of supersingularity does not depend on the choices of $\mathbf{S}$, K, B necessary for the definition – beware that in general two choices of K will not even be conjugate under the adjoint group of G.

Remarks 1) We also show that, if π is an irreducible admissible representation of G, and for some weight V of π there is an eigenvalue of $\mathcal{Z}_G(V)$ in π which is supersingular, then π is supersingular/supercuspidal.

2) Let (P, σ, Q) be a supersingular (or supercuspidal) B-triple. Then $I(P, \sigma, Q)$ is finite dimensional if and only if $P = B$ and $Q = G$.

[1]Note that $\mathcal{H}_G(V)$ is commutative in many cases, for example when G is split, but not in general [HV1].

[2]That is consistent with the definition in [He2, Abe]; but the reader should be aware that the definition in [HV2] is slightly different, maybe not equivalent.

I.6. As in [He2] and [Abe], a lot of our arguments bear on the relation between parabolic induction Ind_P^G in G and compact induction ind_K^G from K to G.

Let V be an irreducible representation of K, and let P be a parabolic subgroup of G containing B, with Levi component M containing Z, and unipotent radical N. In [HV2], inspired by [He1], [He2], a canonical intertwiner

$$\mathcal{I} : \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G(\mathrm{ind}_{M\cap K}^M V_{N\cap K})$$

was investigated. In fact the morphism $\mathcal{S}_M^G$ of I.5 is such that for f in $\mathrm{ind}_K^G V$ and Φ in $\mathcal{H}_G(V)$ we have

$$\mathcal{I}(\Phi(f)) = \mathcal{S}_M^G(\Phi)(\mathcal{I}(f)),$$

where the action of $\mathcal{S}_M^G(\Phi)$ on $\mathcal{I}(f)$ is via its natural action on $\mathrm{ind}_{M\cap K}^M V_{N\cap K}$. Under a suitable regularity condition of V with respect to P [HV2], cf. III.14 Theorem, $\mathcal{I}$ induces an isomorphism

$$\chi \otimes \mathrm{ind}_K^G V \xrightarrow{\sim} \mathrm{Ind}_P^G(\chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K})$$

for any character χ of $\mathcal{Z}_G(V)$ which extends to $\mathcal{Z}_M(V_{N\cap K})$: such an extension is unique, we still denote it by χ; the first tensor product is over $\mathcal{Z}_G(V)$, the second one over $\mathcal{Z}_M(V_{N\cap K})$. Here we obtain a generalization of that result, which we now proceed to explain.

We consider an irreducible representation V of K, and a character $\chi : \mathcal{Z}_G(V) \to C$. There is a smallest parabolic subgroup P containing B – we write $P = MN$ as above – such that χ extends to a character, still written χ, of $\mathcal{Z}_M(V_{N\cap K})$; there is a natural parabolic subgroup P_e, containing P, such that the representation $\chi \otimes (\mathrm{ind}_{M\cap K}^M V_{N\cap K})$ of M, inflated to P, extends to a representation of P_e – write ${}^e(\chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K})$ for that extension. Using similar notation as in I.3, we write $I_e(P, \chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K}, Q)$ for $\mathrm{Ind}_{P_e}^G({}^e(\chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K}) \otimes \mathrm{St}_Q^{P_e})$ when Q is a parabolic subgroup between P and P_e.

Theorem 6 (Filtration Theorem)**.** *With the previous notation, $\tau = \chi \otimes \mathrm{ind}_K^G V$ has a natural filtration by subrepresentations τ_Q, where Q runs through parabolic subgroups of G with $P \subset Q \subset P_e$ and $\tau_{Q'} \subset \tau_Q$ if $Q' \subset Q$. The quotient $\tau_Q / \sum_{Q' \subsetneq Q} \tau_{Q'}$ is isomorphic to $I_e(P, \chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K}, Q)$.*

This last theorem should be compared to the following (the proof, in Chapter V, explains that comparison). Let $\pi = \mathrm{Ind}_P^G(\chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K})$. It also has a natural filtration by subrepresentations π_Q for Q as above, but this time $\pi_{Q'} \subset \pi_Q$ if $Q' \supset Q$, and the quotient $\pi_Q / \sum_{Q' \supsetneq Q} \pi_{Q'}$ is isomorphic to $I_e(P_e, \chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K}, Q)$. In particular the filtrations on τ and π give rise to the same subquotients, but in reserve order, so to say. (We note that the representation π_Q above corresponds to the representation I_Q in Chapter V.)

A striking example is when V is trivial character of K and χ is the "trivial" character of $\mathcal{Z}_G(V) = \mathcal{H}_G(V)$: in that case $P = B = ZU$, $P_e = G$, and $\chi \otimes \mathrm{ind}_{Z\cap K}^Z V_{U\cap K}$ is the trivial character of Z. In $\pi = \mathrm{Ind}_B^G 1$, the trivial character of G occurs as a subrepresentation and the Steinberg representation St_B^G as a quotient, whereas the reverse is true in $\chi \otimes \mathrm{ind}_K^G 1$.

Theorem 6 is new even for GL_n $(n > 2)$. A weaker version of this theorem is proved in [Abe, Proposition 4.7] when $\mathbf{G}$ is split with simply connected derived subgroup and $P = B$ (and in [BL2] in the further special case when $\mathbf{G} = \mathrm{GL}_2$). On the way, following the ideas of [Abe], we prove the freeness of $R_M \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V$ as R_M-module, where R_M

denotes the "supersingular quotient" of $\mathcal{Z}_M(V_{N\cap K})$. This may be of independent interest. Again this result was established for $\mathbf{G} = \mathrm{GL}_2$ in [BL1], but see also the recent paper [GK2].

I.7. To prove Theorem 4 we follow the same strategy as in [He2, Abe] (see the introduction of [He2] for an outline). If (P, σ, Q) is a supersingular triple, we need to prove that $\pi = I(P, \sigma, Q)$ is irreducible; that is done by showing that for any weight V of π and any eigenvector φ for $\mathcal{Z}_G(V)$ in $\mathrm{Hom}_K(V, \pi)$ with corresponding eigenvalue χ, π is generated as a representation of G by the image of φ. When V is suitably regular, that is seen as a consequence of the isomorphism $\chi \otimes \mathrm{ind}_K^G V \simeq \mathrm{Ind}_P^G(\chi \otimes \mathrm{ind}_{M\cap K}^M V_{N\cap K})$ recalled in I.6 above (see III.14). We reduce to that suitably regular case by using a *change of weight theorem*, which gives explicit sufficient conditions on V, V', and χ for having an isomorphism $\chi \otimes \mathrm{ind}_K^G V \simeq \chi \otimes \mathrm{ind}_K^G V'$. (Here, V' is an irreducible representation of K that is "slightly less regular" than V and such that $(V')_{U\cap K} \simeq V_{U\cap K}$.) We refer the reader to Sections IV.2, III.23 for the precise statement and its use in the proof of Theorem 4.

The main novelty in our methods is our proof of the change of weight theorem. It is also the hardest and most subtle part of our arguments. Previously, for split groups, a version of this theorem was established in [He2, §6] and [Abe, §4] by computing the composition of two intertwining operators and applying the Lusztig–Kato formula. We do not know if this approach can be generalized. Our new proof does not involve Kazhdan–Lusztig polynomials, but rather proceeds by embedding $\mathrm{ind}_K^G V$, $\mathrm{ind}_K^G V'$ into the parabolically induced representation $\mathcal{X} = \mathrm{Ind}_B^G(\mathrm{ind}_{Z\cap K}^Z \psi_V)$ using the intertwiner $\mathcal{I}$ of I.6, where $\psi_V : Z \cap K \to C^\times$ describes the action of $Z \cap K$ on $V_{U\cap K} \simeq (V')_{U\cap K}$. The representation $\mathrm{ind}_K^G V$ contains a canonical (up to scalar) fixed vector under a pro-p Iwahori subgroup $I \subset K$ which generates $\mathrm{ind}_K^G V$ as a representation of G, and similarly for $\mathrm{ind}_K^G V'$. Our proof then studies the action of the pro-p-Iwahori Hecke algebra $\mathrm{End}_G(\mathrm{ind}_I^G 1)$ on $\mathcal{X}^I$ to relate the two compact inductions inside $\mathcal{X}$. We crucially rely on the description of the pro-p-Iwahori Hecke algebra recently given for general G by the fourth-named author in [Vig3], in particular the Bernstein relations in this algebra.

We arrive at a *dichotomy* in IV.1 Theorem and IV.2 Corollary, namely our change of weight results depend on whether or not ψ_V is trivial on a certain subgroup of $Z \cap K$. When G is split, the triviality is always guaranteed, but that is not always so for inner forms of GL_n [Ly3, Lemme 3.10.1] and even for unramified unitary groups in 3 variables. This dichotomy may explain why we did not find an easy generalization of the previous proofs for split G.

I.8. Let π be an irreducible admissible representation of G, $P = MN$ a parabolic subgroup of G, and τ an irreducible admissible representation of M inflated to P. In a sequel to this article we will apply our classification to tackle natural questions as the computation of the N-coinvariants or the P-ordinary part of π, the description of the lattice of subrepresentations of $\mathrm{Ind}_P^G \tau$, the generic irreducibility of the representations $\mathrm{Ind}_P^G \tau\chi$ where χ runs over the unramified characters of M (this question was raised by J.-F. Dat).

I.9. We end this introduction with some comments on the organization of the paper. In Chapter II we fix notation and we examine when a representation of a parabolic subgroup of G, trivial on its unipotent radical, can be extended to a larger parabolic subgroup. For a triple (P, σ, Q) as in I.3, we construct $I(P, \sigma, Q)$ and show that it is admissible if σ is. In Chapter III we give most of the proof of Theorem 4. The irreducibility proof was outlined in I.7. The proof that $\pi = I(P, \sigma, Q)$ determines P, Q, and σ up to isomorphism comes from examining the possible weights and Hecke eigenvalues for π (III.24). Finally, to prove that every irreducible admissible representation π of G has the form $I(P, \sigma, Q)$ we use the filtration theorem (Theorem 6). The proof of the change of weight theorem is given in Chapter IV; this is the technical heart of our paper. In Chapter V we deduce the filtration theorem from the change of weight theorem. We trust that the reader will see easily that there is no loop in our arguments. Finally, Chapter VI gives the proof of Theorems 1, 2, 3, 5 and other consequences of the classification, already stated in I.4. That section can essentially be read independently, taking Theorem 4 for granted.

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II. Extension to a larger parabolic subgroup

II.1. Let us first fix notation, valid throughout the paper. As stated in the introduction, our base field F is locally compact and non-archimedean, of residue characteristic p; its ring of integers is $\mathcal{O}$, its residue field k, and q is the cardinality of k; we write $|\ |$ for the normalized absolute value of F.

A linear algebraic group over F will be written with a boldface letter like $\mathbf{H}$, and its group of F-points will be denoted by the corresponding ordinary letter $H = \mathbf{H}(F)$[3].

We fix our connected reductive F-group $\mathbf{G}$[4], and a maximal F-split torus $\mathbf{S}$ in $\mathbf{G}$; we write $\mathbf{Z}$ for the centralizer of $\mathbf{S}$ in $\mathbf{G}$, $\boldsymbol{\mathcal{N}}$ for its normalizer, and $W_0 = W(\mathbf{G}, \mathbf{S})$ for the Weyl group $\boldsymbol{\mathcal{N}}/\mathbf{Z}$; we recall that $W_0 = \mathcal{N}/Z$ [Bo, 21.2 Theorem]. We also fix a minimal F-parabolic subgroup $\mathbf{B}$ of $\mathbf{G}$ with Levi subgroup $\mathbf{Z}$, and write $\mathbf{U}$ for its unipotent radical. As is customary, we say that P is a parabolic subgroup of G to mean that $P = \mathbf{P}(F)$, where $\mathbf{P}$ is an F-parabolic subgroup of $\mathbf{G}$. If P contains B, we usually write $P = MN$ to mean that M is the Levi component of P containing Z, and N the unipotent radical of P; we then write $P_{\mathrm{op}} = MN_{\mathrm{op}}$ for the parabolic subgroup opposite to P with respect to M; in particular $B_{\mathrm{op}} = ZU_{\mathrm{op}}$.

We let Φ be the set of roots of $\mathbf{S}$ in $\mathbf{G}$, so Φ is a subset of the group $X^*(\mathbf{S})$ of characters of $\mathbf{S}$; we let Φ^+ be the subset of roots of $\mathbf{S}$ in $\mathbf{U}$, called positive roots, and Δ for the set of simple roots of $\mathbf{S}$ in $\mathbf{U}$. If $X_*(\mathbf{S})$ is the group of cocharacters of $\mathbf{S}$ we write $\langle\ ,\ \rangle$ for the natural pairing $X^*(\mathbf{S}) \times X_*(\mathbf{S}) \to \mathbb{Z}$; for α in Φ, the corresponding coroot [SGA3, exposé XXVI, §7] is written $\alpha^\vee$ and for $I \subset \Phi$ we put $I^\vee = \{\alpha^\vee \mid \alpha \in I\}$. We choose a positive definite symmetric bilinear form on $X^*(\mathbf{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, invariant under W_0, which induces a notion of orthogonality between roots; for roots α, β we have $\alpha \perp \beta$ if and only if $\langle \alpha, \beta^\vee \rangle = 0$.

For α in Φ we write $U_\alpha = \mathbf{U}_\alpha(F)$, for the corresponding root subgroup ($\mathbf{U}_\alpha$ is written $\mathbf{U}_{(\alpha)}$ in [Bo, §21]), and $s_\alpha \in W_0$ for the corresponding reflection. For $I \subset \Delta$ we let W_I be the subgroup generated by $\{s_\alpha \mid \alpha \in I\}$, $\mathcal{N}_I$ for the inverse image of W_I in $\mathcal{N}$, P_I for the parabolic subgroup $U\mathcal{N}_I U$ (it contains B), $P_I = M_I N_I$ for its Levi decomposition, M_I containing Z; if I is a singleton $\{\alpha\}$ we rather write $P_\alpha = M_\alpha N_\alpha$. We set $\Delta_P = I$ if $P = P_I$. We note that for I, $J \subset \Delta$, $P_{I\cap J} = P_I \cap P_J$, $M_{I \cap J} = M_I \cap M_J$.

[3] We shall use a similar convention for groups over k.

[4] $\mathbf{G}$ is fixed, but otherwise arbitrary, so the results we establish for $\mathbf{G}$ can be applied to other reductive groups over F.

II.2. As announced in the introduction, we tackle here a preliminary question: if P is a parabolic subgroup of G and σ a representation of P trivial on its unipotent radical N, when can σ be extended to a larger parabolic subgroup Q of G? Dividing by the unipotent radical of Q, which is contained in N, we loose no generality in assuming that $Q = G$. If σ extends to G, then any extension has to be trivial on the normal subgroup $\langle {}^G N \rangle$ of G generated by N, so that σ has to be trivial on $P \cap \langle {}^G N \rangle$. So we need to understand what $\langle {}^G N \rangle$ is. That question, which involves no representation theory, will be dealt with presently.

II.3. Of particular importance in our setting will be the subgroup G' of G generated by U and U_{op}. Beware that the notation, which will be applied to other reductive groups (like the Levi subgroups of $\mathbf{G}$), is unusual, and that G' is not generally the group of points over F of a reductive subgroup of $\mathbf{G}$: this occurs already for $\mathbf{G} = \mathrm{PGL}_2$. Since G is generated by U, U_{op} and Z, see e.g. [BoT, Proposition 6.25], G' is normal in G so is also the subgroup of G generated by the unipotent radicals of the parabolic subgroups of G, and we have $G = ZG'$. Sometimes we have $G' = G$, though.

Proposition *Assume that* $\mathbf{G}$ *is semisimple, simply connected, almost* F*-simple and isotropic. Then* $G' = G$*, and* G *has no non-central proper normal subgroup. Moreover,* Z *is generated by the* $Z \cap M'_\alpha$*,* α *running through* Δ.

Proof The first assertion is due to Platonov [PlR, Theorem 7.6] and the second one then follows from work of Tits [PlR, Theorem 7.1]. The final assertion is due to Prasad and Raghunathan [PrR] – actually their result is valid over any field. □

Remark Let $\mathbf{G}$ be as in the proposition, let $\alpha \in \Delta$ and $\mathbf{G}_\alpha$ the subgroup of $\mathbf{G}$ generated by $\mathbf{U}_\alpha$ and $\mathbf{U}_{-\alpha}$; since $\mathbf{G}_\alpha$ satisfies the hypotheses of the proposition, we have $M'_\alpha = G'_\alpha = G_\alpha$.

II.4. In the following sections (II.5–II.8) our strategy is to reduce statements for G to a much simpler group G^{is} via a homomorphism $G^{\mathrm{is}} \to G$ whose image is G'. The group G^{is} has the property that it is a product of groups of the form considered in II.3 Proposition, and the homomorphism $G^{\mathrm{is}} \to G$ restricts to an isomorphism on unipotent radicals of parabolic subgroups.

Let $\mathbf{G}^{\mathrm{sc}}$ be the simply connected covering of the derived group $\mathbf{G}^{\mathrm{der}}$ of $\mathbf{G}$. Recall that $\mathbf{G}^{\mathrm{sc}}$ is the direct product of its almost F-simple components. We let $\mathcal{B}$ be an indexing set for the **isotropic** almost F-simple components of $\mathbf{G}^{\mathrm{sc}}$ and for $b \in \mathcal{B}$ we write $\tilde{\mathbf{G}}_b$ for the corresponding component. We put $\mathbf{G}^{\mathrm{is}} = \prod_{b \in \mathcal{B}} \tilde{\mathbf{G}}_b$, and denote by ι the natural homomorphism $\mathbf{G}^{\mathrm{is}} \to \mathbf{G}$, factoring through $\mathbf{G}^{\mathrm{is}} \to \mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{der}} \to \mathbf{G}$.

We need to understand the relation between parabolic subgroups of $\mathbf{G}$ and parabolic subgroups of $\mathbf{G}^{\mathrm{is}}$. The following comes from [Bo, §21, §22], going through the factorization of ι.

The connected component of $\iota^{-1}(\mathbf{S})$ is a maximal F-split torus $\tilde{\mathbf{S}}$ of $\mathbf{G}^{\mathrm{is}}$, and $\mathbf{S}$ is the product of $\iota(\tilde{\mathbf{S}})$ and the maximal F-split torus in the centre of $\mathbf{G}$. The centralizer of $\tilde{\mathbf{S}}$ in $\mathbf{G}^{\mathrm{is}}$ is $\tilde{\mathbf{Z}} = \iota^{-1}(\mathbf{Z})$, its normalizer $\tilde{\boldsymbol{\mathcal{N}}} = \iota^{-1}(\boldsymbol{\mathcal{N}})$, and ι induces an isomorphism $W(\mathbf{G}^{\mathrm{is}}, \tilde{\mathbf{S}}) = W(\mathbf{G}, \mathbf{S})$ (see in particular [Bo, 22.6 Theorem]); in particular W_0 has representatives in $\iota(G^{\mathrm{is}})$. As $\mathbf{G}^{\mathrm{is}}$ is a direct product $\prod \tilde{\mathbf{G}}_b$ (over $b \in \mathcal{B}$) we have corresponding natural decompositions $\tilde{\mathbf{S}} = \prod \tilde{\mathbf{S}}_b$, $\tilde{\mathbf{Z}} = \prod \tilde{\mathbf{Z}}_b$, $\tilde{\boldsymbol{\mathcal{N}}} = \prod \tilde{\boldsymbol{\mathcal{N}}}_b$ and $W(\mathbf{G}^{\text{is}}, \mathbf{S}) = \prod W(\tilde{\mathbf{G}}_b, \tilde{\mathbf{S}}_b)$. Note that $\iota(\tilde{\mathbf{G}}_b)$ is normal in $\mathbf{G}$ for each $b \in \mathcal{B}$.

The map $\mathbf{P} \mapsto \tilde{\mathbf{P}} = \iota^{-1}(\mathbf{P})$ is a bijection between F-parabolic subgroups of $\mathbf{G}$ and F-parabolic subgroups of $\mathbf{G}^{\text{is}}$, and ι induces an isomorphism (cf. [Bo, 22.6 Theorem]) of the unipotent radical $\tilde{\mathbf{N}}$ of $\tilde{\mathbf{P}}$ onto the unipotent radical $\mathbf{N}$ of $\mathbf{P}$. Also, $\tilde{\mathbf{M}} = \iota^{-1}(\mathbf{M})$ is the Levi component of $\tilde{\mathbf{P}}$ containing $\tilde{\mathbf{Z}}$. In particular $\tilde{\mathbf{B}} = \iota^{-1}(\mathbf{B})$ is a minimal F-parabolic subgroup of $\mathbf{G}^{\text{is}}$; it is the direct product of minimal parabolic subgroups $\tilde{\mathbf{B}}_b$ of $\tilde{\mathbf{G}}_b$, and its unipotent radical $\tilde{\mathbf{U}}$ is the direct product of the $\tilde{\mathbf{U}}_b$, with $\tilde{\mathbf{U}}_b$ the unipotent radical of $\tilde{\mathbf{B}}_b$. Via ι we get an identification[5] of the roots of $\mathbf{S}$ in $\mathbf{U}$ with the roots of $\tilde{\mathbf{S}}$ in $\tilde{\mathbf{U}}$, so that Δ, in particular, also appears as the set of simple roots of $\tilde{\mathbf{S}}$ in $\tilde{\mathbf{U}}$; as such Δ is a disjoint union of the sets Δ_b, $b \in \mathcal{B}$, where Δ_b is the set of roots of $\tilde{\mathbf{S}}$ (or $\tilde{\mathbf{S}}_b$) in $\tilde{\mathbf{U}}_b$; that partition of Δ is the finest partition into mutually orthogonal subsets. Those subsets are the connected components of the Dynkin diagram of $\mathbf{G}$ (with set of vertices Δ) so we can view $\mathcal{B}$ as the set of such components.

Proposition $G' = \iota(G^{\text{is}})$.

Proof By II.3 Proposition we have $\tilde{G}_b' = \tilde{G}_b$ for each $b \in \mathcal{B}$ so $(G^{\text{is}})' = G^{\text{is}}$; since ι induces an isomorphism of $\tilde{U}$ onto U and $\tilde{U}_{\text{op}}$ onto U_{op}, we get $G' = \iota(G^{\text{is}})$. □

Note that the proposition implies that $Z \cap G' = \iota(\tilde{Z})$.

II.5. Notation For $I \subset \Delta$, set $\mathcal{B}(I) = \{b \in \mathcal{B} \mid I \cap \Delta_b \neq \Delta_b\}$.

Proposition *Let $I \subset \Delta$. Then the normal subgroup $\langle {}^G N_I \rangle$ of G generated by N_I is $\iota(\prod_{b \in \mathcal{B}(I)} \tilde{G}_b)$.*

Proof We have $\tilde{N}_I = \prod_{b \in \mathcal{B}} (\tilde{N}_I \cap \tilde{G}_b)$ and $\tilde{N}_I \cap \tilde{G}_b$ is the unipotent subgroup of $\tilde{G}_b$ corresponding to $I \cap \Delta_b \subset \Delta_b$. For $b \in \mathcal{B} - \mathcal{B}(I)$, $\tilde{N}_I \cap \tilde{G}_b$ is trivial; for b in $\mathcal{B}(I)$, $\tilde{N}_I \cap \tilde{G}_b$ is non-trivial, and provides a non-central subgroup of $\tilde{G}_b$ so by II.3 Proposition the normal subgroup of G^{is} generated by $\tilde{N}_I \cap \tilde{G}_b$ is $\tilde{G}_b$; the proposition follows. □

Corollary

(i) $P_I \cap \langle {}^G N_I \rangle = \iota\big(\prod_{b \in \mathcal{B}(I)} (\tilde{P}_I \cap \tilde{G}_b)\big)$,

(ii) $M_I \cap \langle {}^G N_I \rangle = \iota\big(\prod_{b \in \mathcal{B}(I)} (\tilde{M}_I \cap \tilde{G}_b)\big)$,

(iii) $M_I \langle {}^G N_I \rangle = G$,

(iv) $\langle {}^G N_I \rangle$ *contains* $N_{I,\text{op}}$.

Proof Parts (i) and (ii) are immediate consequences of the previous considerations. Let us prove (iii). From the proposition $\langle {}^G N_I \rangle$ contains $\iota(\tilde{G}_b)$ for $b \in \mathcal{B}(I)$, but for $b \in \mathcal{B} - \mathcal{B}(I)$, M_I contains $\iota(\tilde{G}_b)$, so finally $M_I \langle {}^G N_I \rangle$ contains $\iota(G^{\text{is}}) = G'$. Since M_I contains

[5] More precisely the natural map $\tilde{\mathbf{S}} \to \mathbf{S}$ induces a group homomorphism $X^*(\mathbf{S}) \to X^*(\tilde{\mathbf{S}})$ through which the roots of $\mathbf{S}$ in $\mathbf{U}$ are identified with the roots of $\tilde{\mathbf{S}}$ in $\tilde{\mathbf{U}}$. By [SGA3, Exp. XXVI, 7.4] if α is a root of $\mathbf{S}$ in $\mathbf{U}$ and $\tilde{\alpha}$ the corresponding root of $\tilde{\mathbf{S}}$ in $\tilde{\mathbf{U}}$, then $\tilde{\alpha}^\vee$ goes to $\alpha^\vee$ via the transposed morphism $X_*(\tilde{\mathbf{S}}) \to X_*(\mathbf{S})$. In the sequel we make no distinction between α and $\tilde{\alpha}$, $\alpha^\vee$ and $\tilde{\alpha}^\vee$.

Z and $G = ZG'$, we get (iii). Part (iv) follows from the proposition because $N_{I,\mathrm{op}}$ is $\iota(\prod_{b\in\mathcal{B}(I)}(\tilde{N}_{I,\mathrm{op}}\cap\tilde{G}_b))$. □

Remark For $b\in\mathcal{B}$, $\tilde{M}_I\cap\tilde{G}_b$ can be also described as the product $\prod_c \tilde{M}_{\Delta_c}$ over the connected components c of the Dynkin diagram obtained from that of $\tilde{G}_b$ by deleting vertices outside I. (We note that the product is not direct.)

II.6. There is another useful characterization of $M_I\cap\langle {}^G N_I\rangle$.

Proposition *Let $I\subset\Delta$. Then $M_I\cap\langle {}^G N_I\rangle$ is the normal subgroup of M_I generated by $Z\cap M'_\alpha$, for α running through $\Delta - I$.*

Proof Let $\alpha\in\Delta-I$ and let $b\in\mathcal{B}$ be such that $\alpha\in\Delta_b$, so that $M'_\alpha\subset\iota(\tilde{G}_b)$. As $\alpha\notin I$, b belongs to $\mathcal{B}(I)$ so $\iota(\tilde{G}_b)$ is included in $\langle {}^G N_I\rangle$ by II.5 Proposition, and consequently $Z\cap M'_\alpha\subset M_I\cap\langle {}^G N_I\rangle$. To prove that $M_I\cap\langle {}^G N_I\rangle$ is the normal subgroup of M_I generated by the $Z\cap M'_\alpha$, $\alpha\in\Delta-I$, it is enough, by II.5, to work within $\tilde{G}_b$. So we now assume that $\mathbf{G}=\mathbf{G}^{\mathrm{is}}$ and $\mathbf{G}$ is almost F-simple. If $I=\Delta$, N_I is trivial so there is nothing to prove. So let us assume $I\neq\Delta$, so that $\langle {}^G N_I\rangle = G$ by II.3 since N_I is not trivial. We can apply to $\mathbf{M}_I$ all the considerations applied to $\mathbf{G}$ in the current chapter, so we see that $M_I = Z\prod H_J$ where J runs through connected components of the Dynkin diagram with set of vertices I associated to M_I, and $\mathbf{H}_J$ is the corresponding semisimple simply connected almost F-simple subgroup of $\mathbf{M}_I$. Let J be such a connected component. As the Dynkin diagram attached to G is by assumption connected, there is α in $\Delta-I$ with $\langle J,\alpha^\vee\rangle\neq 0$. Choose α' in J with $\langle\alpha',\alpha^\vee\rangle\neq 0$ and $x\in F^\times$ with $\alpha'(\alpha^\vee(x))^2\neq 1$. We have $\alpha^\vee(x)\in Z\cap M'_\alpha$, $U_{\alpha'}\subset H_J\subset M_I$, and the map from $U_{\alpha'}$ to itself given by $u\mapsto\alpha^\vee(x)u\alpha^\vee(x)^{-1}u^{-1}$ is onto[6]. The normal subgroup of M_I generated by $Z\cap M'_\alpha$ contains $\alpha^\vee(x)$[7] and $u\alpha^\vee(x)^{-1}u^{-1}$ for $u\in U_{\alpha'}$, so it contains $U_{\alpha'}$. By II.3 Proposition it contains H_J and in particular $Z\cap M'_{\alpha''}$ for all $\alpha''\in J$. We conclude that the normal subgroup of M_I generated by the $Z\cap M'_\alpha$, $\alpha\in\Delta-I$, contains $Z\cap M'_\alpha$ for all $\alpha\in\Delta$. By II.3 Proposition it contains Z; since we have seen that it contains each H_J, it is equal to $M_I = Z\prod H_J$. □

II.7. Keeping the same notation, we can now derive consequences for representations.

Proposition *Let $I\subset\Delta$, and let σ be a representation of M_I. Then the following conditions are equivalent:*

(i) σ extends to a representation of G trivial on N_I,

(ii) for each $b\in\mathcal{B}(I)$, σ is trivial on $\iota(\tilde{M}_I\cap\tilde{G}_b)$,

(iii) for each $\alpha\in\Delta-I$, σ is trivial on $Z\cap M'_\alpha$.

When these conditions are satisfied, there exists a unique extension ${}^e\sigma$ of σ to G which is trivial on N_I, and it is smooth, admissible or irreducible if and only if σ is.

Proof As already said in II.2, if σ extends to a representation of G trivial on N_I, the extension is trivial on $\langle {}^G N_I\rangle$ so σ is certainly trivial on $M_I\cap\langle {}^G N_I\rangle$. Consequently, (i) implies (ii) and (iii) by II.5, II.6. Conversely, under assumptions (ii) or (iii), σ is trivial on

[6] If $2\alpha'$ is not a root, then $\alpha^\vee(x)$ acts on $U_{\alpha'}$ (a vector group) via multiplication by $\alpha'(\alpha^\vee(x))$. If $2\alpha'$ is a root, then $\alpha^\vee(x)$ acts on $U_{2\alpha'}$ via $\alpha'(\alpha^\vee(x))^2$ and on $U_{\alpha'}/U_{2\alpha'}$ via $\alpha'(\alpha^\vee(x))$.

[7] It follows from II.4, footnote 5, that $\alpha^\vee(x)$ belongs to M'_α; on the other hand it belongs to $S\subset M_I$.

$M_I \cap \langle {}^G N_I \rangle$ hence extends, trivially on $\langle {}^G N_I \rangle$, to a representation of $M_I \langle {}^G N_I \rangle$, which is G by II.5 Corollary (iii). The extension ${}^e\sigma$ is necessarily unique. Assume that σ extends to a representation ${}^e\sigma$ of G trivial on N_I. Since σ and ${}^e\sigma$ have the same image, σ is irreducible if and only if ${}^e\sigma$ is. As P_I is a topological subgroup of G, σ is smooth if ${}^e\sigma$ is. Conversely, assume that σ is smooth and let x be a vector in the space of σ, J its stabilizer in P_I; by II.5 Corollary (iv), $N_{I,\mathrm{op}}$ acts trivially on ${}^e\sigma$ and the stabilizer of x in G, which contains $N_{I,\mathrm{op}} J$, is open in G, so ${}^e\sigma$ is smooth.

As P_I is a topological subgroup of G, ${}^e\sigma$ is admissible if σ is. Conversely assume ${}^e\sigma$ is admissible; for each open subgroup J of M_I, a vector in σ fixed by J is also fixed by the subgroup generated by J, N_I and $N_{I,\mathrm{op}}$ which is open in G, so σ is admissible. □

Remark 1 By II.5 Remark, condition (ii) illustrates that σ can extend to G (trivially on N_I) only for very strong reasons: for any connected component Δ_b of the Dynkin diagram of G meeting $\Delta - I$, σ has to be trivial on M'_{Δ_c} for any connected component Δ_c of the Dynkin diagram of M_I included in Δ_b. By II.3 Proposition applied to $M^{\mathrm{is}}_{\Delta_c}$ that last condition is also equivalent to σ being trivial on U_β for some, or any, $\beta \in \Delta_c$.

Remark 2 The coefficient field plays no role here. Properties (i), (ii) and (iii) are equivalent for a representation of M_I over a commutative ring. The last assertion of the proposition remains also true for representations over a commutative ring (for admissibility, suppose as usual that the ring is noetherian).

Notation Let $P = MN$ be a parabolic subgroup of G containing B, and let σ be a representation of M. We let $\Delta(\sigma)$ be the set of $\alpha \in \Delta - \Delta_P$ such that σ is trivial on $Z \cap M'_\alpha$. We let $P(\sigma)$ be the parabolic subgroup corresponding to $\Delta(\sigma) \sqcup \Delta_P$.

Corollary 1 *Let $P = MN$ be a parabolic subgroup of G containing B, and let σ be a representation of M. Then the parabolic subgroups of G containing P to which σ extends, trivially on N, are those contained in $P(\sigma)$. In that case the extension is unique and is smooth, admissible or irreducible if σ is.*

The corollary is immediate from the proposition applied to Levi components of parabolic subgroups of G containing P.

Remark 2 Since any parabolic subgroup P of G is conjugate to one containing B, it follows, as stated in the introduction, that if σ is a representation of P trivial on its unipotent radical, there is a maximal parabolic subgroup $P(\sigma)$ of G to which σ can be extended, and the extension is smooth, admissible or irreducible if (and only if) σ is.

Corollary 2 *Keep the assumptions and notation of Corollary 1, and assume further that $\Delta(\sigma)$ is not orthogonal to Δ_M. Then there is a proper parabolic subgroup Q of M, containing $M \cap B$, such that σ is trivial on the unipotent radical of Q; moreover σ is a subrepresentation of $\mathrm{Ind}_Q^M(\sigma_{|Q})$, and $\sigma_{|Q}$ is irreducible or admissible if σ is. In particular, σ cannot be supercuspidal.*

Proof We may assume that $G = P(\sigma)$. Let $\alpha \in \Delta(\sigma)$ not orthogonal to Δ_M, and let $b \in \mathcal{B}$ such that $\alpha \in \Delta_b$. Then $\Delta_b \cap \Delta_M \neq \Delta_b$, so σ is trivial on $\iota(\tilde{M} \cap \tilde{G}_b)$ by the proposition. As α is not orthogonal to Δ_M, $\Delta_b \cap \Delta_M$ is not empty. If Q is the (proper) parabolic subgroup of M corresponding to $\Delta_M - \Delta_b$, then $\iota(\tilde{M} \cap \tilde{G}_b)$ contains the unipotent radical N_Q of Q and σ is trivial on N_Q. Then, obviously, σ is a subrepresentation of $\mathrm{Ind}_Q^M(\sigma_{|Q})$ and by the proposition, applied to M instead of G, if σ is irreducible or admissible, so is its restriction to the Levi component of Q. By the definition of supercuspidality, σ cannot be supercuspidal. □

Remark 3 The last assertion of Corollary 2 explains why the case of interest to us is when Δ_M and $\Delta(\sigma)$ are orthogonal – an analogous result will be obtained when σ is assumed supersingular instead of supercuspidal (III.17 Corollary). As a special case, assume that the (relative) Dynkin diagram of $\mathbf{G}$ is connected, and σ is a supercuspidal representation of M extending to G. Then either $M = G$ or $M = Z$; in the latter case, σ is trivial on $Z \cap G'$ and finite dimensional.

Remark 4 For the record, let us state a few useful facts when Δ is the disjoint union of two subsets I and J, orthogonal to each other. Then M'_I and M'_J are normal subgroups of G, commuting with each other. We have $G' = M'_I M'_J$, $M_I = ZM'_I$, $M_J = ZM'_J$, $M_I \cap M_J = Z$ and in particular $M_I \cap M'_\alpha = Z \cap M'_\alpha$ for $\alpha \in J$. Also, $M'_I \cap M'_J$ is finite and central in G: indeed, decomposing $\tilde{G}$ as $\tilde{G}_I \times \tilde{G}_J$, $M'_I \cap M'_J$ is simply the image under $(g_1, g_2) \mapsto \iota(g_1)$ of $\operatorname{Ker} \iota \subset \tilde{G}_I \times \tilde{G}_J$. The inclusion of M_I in G induces an isomorphism $M_I/(M_I \cap M'_J) \simeq G/M'_J$ (and similarly for M_J).

Remark 5 Let $\alpha \in \Delta$ belong to the component Δ_b. The normal subgroup of G generated by $Z \cap M'_\alpha$ is $\iota(\tilde{G}_b)$ because $Z \cap M'_\alpha$ is not central in M'_α. If σ is a representation of G which is trivial on $Z \cap M'_\alpha$, it is then trivial on $\iota(\tilde{G}_b)$ and the conclusions of Corollary 2 hold (with $M = G$).

II.8. To go further we need the generalized Steinberg representations already recalled in the introduction.

Lemma *Let Q be a parabolic subgroup of G. Then lifting functions on G to functions on G^{is} via ι gives an isomorphism of $\operatorname{Ind}_Q^G 1$ with $\operatorname{Ind}_{\tilde{Q}}^{G^{\mathrm{is}}} 1$. The representation $\operatorname{St}_Q^G \circ \iota$ of G^{is} is isomorphic to $\operatorname{St}_{\tilde{Q}}^{G^{\mathrm{is}}}$; the restriction of St_Q^G to G' is irreducible and admissible.*

Proof We have $ZG' = G$ and Q contains Z, so $G = QG'$. Besides $Q \cap G' = \iota(\tilde{Q})$. It follows that ι induces a bijection of $\tilde{Q}\backslash G^{\mathrm{is}}$ onto $Q\backslash G$; that bijection is continuous hence is a homeomorphism by Arens' theorem [MZ, p. 65]. The first assertion follows and the others are immediate consequences. □

Now let $P = MN$ be a parabolic subgroup of G, let σ be a representation of M, inflated to P. Then by II.7 Corollary 1, σ extends (uniquely) to a representation ${}^e\sigma$ of $P(\sigma)$. For each parabolic subgroup Q with $P \subset Q \subset P(\sigma)$ we can form the representation ${}^e\sigma \otimes \operatorname{St}_Q^{P(\sigma)}$ of $P(\sigma)$.

Proposition *σ is irreducible (*resp. *admissible) if and only if ${}^e\sigma \otimes \operatorname{St}_Q^{P(\sigma)}$ is irreducible (*resp. *admissible).*

From this, we get (see for instance [Vig2, Lemma 4.7]):

Corollary *σ is admissible if and only if $\operatorname{Ind}_{P(\sigma)}^G({}^e\sigma \otimes \operatorname{St}_Q^{P(\sigma)})$ is admissible.*

Proof of the proposition The unipotent radical of $P(\sigma)$ acts trivially on both ${}^e\sigma$ and $\operatorname{St}_Q^{P(\sigma)}$. Therefore we may assume $P(\sigma) = G$.

By the lemma above $\mathrm{St}_Q^G \circ \iota$ is the generalized Steinberg representation $\mathrm{St}_{\tilde{Q}}^{G^{\mathrm{is}}}$. For b in $\mathcal{B} - \mathcal{B}(\Delta_Q)$, $\Delta_Q \cap \Delta_b = \Delta_b$ so that by construction $\mathrm{St}_{\tilde{Q}}^{G^{\mathrm{is}}}$ is trivial on $\tilde{G}_b$; consequently, its restriction to $H = \prod_{b \in \mathcal{B}(\Delta_Q)} \tilde{G}_b$ is irreducible. On the other hand by II.5, ${}^e\sigma$ is trivial on the normal subgroup $\iota(H)$. If σ is irreducible, the irreducibility of ${}^e\sigma \otimes \mathrm{St}_Q^G$ comes then from Clifford theory as in [Abe, Lemma 5.3][8].

Assume that σ is admissible, so ${}^e\sigma$ is admissible too. As above $\iota(H)$ acts trivially on ${}^e\sigma$ and the restriction of St_Q^G to $\iota(H)$ is admissible. If L is an open subgroup of G, the vectors in St_Q^G fixed under $L \cap \iota(H)$ form a finite dimensional vector space X. The vectors fixed by L in ${}^e\sigma \otimes \mathrm{St}_Q^G$ are in ${}^e\sigma \otimes X$. There is an open subgroup L' of L acting trivially on X and $({}^e\sigma \otimes X)^{L'} = {}^e\sigma^{L'} \otimes X$ is finite dimensional. Consequently, ${}^e\sigma \otimes \mathrm{St}_Q^G$ is admissible.

Conversely, if ${}^e\sigma \otimes \mathrm{St}_Q^G$ is irreducible, obviously σ is irreducible. If ${}^e\sigma \otimes \mathrm{St}_Q^G$ is admissible so is σ. Indeed, if J is an open subgroup of G then $({}^e\sigma)^J \otimes (\mathrm{St}_Q^G)^J$ is contained in $({}^e\sigma \otimes \mathrm{St}_Q^G)^J$, so if J is small enough for $(\mathrm{St}_Q^G)^J$ to be non-zero, we deduce that $({}^e\sigma)^J$ is finite-dimensional; thus ${}^e\sigma$ is admissible and so is σ by II.7 Proposition. ☐

Remark *Assume that Δ_M is orthogonal to $\Delta - \Delta_M$. Let σ be a representation of M which extends to G trivially on N, and let Q be a parabolic subgroup of G containing P.*

1) *The representation ${}^e\sigma \otimes \mathrm{St}_Q^G$ of G determines σ and Q.*

2) *Any subquotient π of ${}^e\sigma \otimes \mathrm{St}_Q^G$ is of the form ${}^e\sigma_\pi \otimes \mathrm{St}_Q^G$ for some representation σ_π of M which extends to G trivially on N.*

Proof 1) We put $J = \Delta - \Delta_M$. As Q contains M, St_Q^G is trivial on the normal subgroup M', and restricting to M_J functions on G gives an isomorphism of St_Q^G onto $\mathrm{St}_{Q \cap M_J}^{M_J}$. The restriction of ${}^e\sigma \otimes \mathrm{St}_Q^G$ to M'_J is a direct sum of irreducible representations $\mathrm{St}_Q^G|_{M'_J}$, and that representation determines Q (II.8 Lemma). Seen as a representation of G, $\mathrm{Hom}_{M'_J}(\mathrm{St}_Q^G, {}^e\sigma \otimes \mathrm{St}_Q^G)$ is isomorphic to ${}^e\sigma$ (use for example [Abe, Lemma 5.3]), and ${}^e\sigma$ determines σ.

2) The restriction of π to M'_J is a sum of copies of the irreducible representation $\mathrm{St}_Q^G|_{M'_J}$. By Clifford theory [Abe, Lemma 5.3], π is isomorphic to $\mathrm{Hom}_{M'_J}(\mathrm{St}_Q^G, \pi) \otimes \mathrm{St}_Q^G$. Moreover, $\mathrm{Hom}_{M'_J}(\mathrm{St}_Q^G, \pi)$ is a representation of G trivial on M'_J hence determines a representation σ_π of M via the map $M \twoheadrightarrow G/M'_J$ and ${}^e\sigma_\pi \simeq \mathrm{Hom}_{M'_J}(\mathrm{St}_Q^G, \pi)$ as a representation of G. ☐

III. Supersingularity and classification

III.1. This chapter is devoted to the proof of I.5 Theorem 4, and is rather long. It is divided into parts A) to H). In part A) we give some more detail on supersingularity, and in part B) we describe a parametrization for the irreducible representations of K. The next step in part C) is to determine the weights and eigenvalues of parabolically induced representations. We then proceed to the analysis of the representations $I(P, \sigma, Q)$: we first determine $P(\sigma)$ in part D), and after that we compute the weights and eigenvalues of $I(P, \sigma, Q)$ for a supersingular triple (P, σ, Q) in part E). The subsequent proof of the irreducibility of $I(P, \sigma, Q)$ in part F) uses a change of weight theorem proved in Chapter IV. From the knowledge of weights and eigenvalues, we easily deduce in part G)

[8] To apply that lemma, note that Schur's lemma is valid for the restriction of St_Q^G to $\iota(H)$.

when $I(P_1, \sigma_1, Q_1)$ is isomorphic to $I(P_2, \sigma_2, Q_2)$ for supersingular triples (P_1, σ_1, Q_1) and (P_2, σ_2, Q_2). In part H) we finally prove exhaustion, i.e. that every irreducible admissible representation of G has the form $I(P, \sigma, Q)$ for some supersingular triple (P, σ, Q): that uses a result established only in Chapter V as a further consequence of the change of weight theorem.

Notation The special maximal parahoric subgroup $K \subset G$ is fixed throughout; we write $K(1)$ for its pro-p-radical and H^0 for $H \cap K$, when H is a subgroup of G. Note that Z^0 is the unique parahoric subgroup of Z and that $Z(1) = Z \cap K(1)$ is the unique pro-p Sylow subgroup of Z^0.

A) Supersingularity

III.2. Consider an irreducible representation (ρ, V) of K; it is finite-dimensional and trivial on $K(1)$. The classification of such objects will be recalled in part B).

We view the intertwining algebra $\mathcal{H}_G(V)$ as a Hecke algebra, the convolution algebra of compactly supported functions $\Phi : G \to \mathrm{End}_C(V)$ satisfying

$$\Phi(kgk') = \rho(k)\Phi(g)\rho(k') \quad \text{for } g \text{ in } G,\ k \text{ and } k' \text{ in } K.$$

The convolution operation is given by

$$(\Phi * \Psi)(g) = \sum_{h \in G/K} \Phi(h)\Psi(h^{-1}g) \quad \text{for } \Phi, \Psi \text{ in } \mathcal{H}_G(V). \tag{III.2.1}$$

The action on $\mathrm{ind}_K^G V$ is also given by convolution:

$$(\Phi * f)(g) = \sum_{h \in G/K} \Phi(h)f(h^{-1}g) \quad \text{for } f \in \mathrm{ind}_K^G V,\ \Phi \in \mathcal{H}_G(V). \tag{III.2.2}$$

III.3. We need to recall the structure of $\mathcal{H}_G(V)$ and its centre $\mathcal{Z}_G(V)$, as elucidated in [HV1], building on [He1, He2]; note that $\mathcal{H}_G(V)$ is commutative in the context of [He1, He2, Abe].

Let $P = MN$ be a parabolic subgroup of G containing B. Then the space of coinvariants V_{N^0} of N^0 in V affords an irreducible representation of M^0 (which is the special parahoric subgroup of M corresponding to the special point $\mathbf{x}_0$). For each representation σ of M on a vector space W, Frobenius reciprocity and the equalities $G = KP = PK$, $P^0 = M^0N^0$, give a canonical isomorphism:

$$\mathrm{Hom}_G(\mathrm{ind}_K^G V, \mathrm{Ind}_P^G W) \xrightarrow{\sim} \mathrm{Hom}_M(\mathrm{ind}_{M^0}^M V_{N^0}, W) \tag{III.3.1}$$

The natural algebra homomorphism $\mathcal{S}_M^G : \mathcal{H}_G(V) \longrightarrow \mathcal{H}_M(V_{N^0})$ of I.5 is given concretely by

$$[\mathcal{S}_M^G(\Psi)(m)]\bar{v} = \sum_{n \in N^0 \backslash N} \overline{\Psi(nm)(v)} \text{ for } m \text{ in } M,\ v \text{ in } V, \tag{III.3.2}$$

where a bar indicates the image in V_{N^0} of a vector in V [HV2, Proposition 2.2]. Recall that (III.3.1) is $\mathcal{H}_G(V)$-linear if we let $\mathcal{H}_G(V)$ act on the right-hand side via $\mathcal{S}_M^G$. Recall also that $\mathcal{S}_M^G$ is injective [HV2, Proposition 4.1].

For varying $P = MN$, the homomorphisms $\mathcal{S}_M^G$ satisfy obvious transitivity properties, and $\mathcal{S}_Z^G$ identifies $\mathcal{H}_G(V)$ with a subalgebra of $\mathcal{H}_Z(V_{U^0})$ which we now describe. For a root α in $\Phi = \Phi(\mathbf{G}, \mathbf{S})$, the group homomorphism $|\alpha| : x \mapsto |\alpha(x)|$ from S to $\mathbb{R}_+^\times$ extends uniquely to a group homomorphism $Z \to \mathbb{R}_+^\times$ trivial on Z^0, and we still write $|\alpha|$ for that extension. We write Z^+ for the set of z in Z such that $|\alpha|(z) \leq 1$ for all $\alpha \in \Delta$. Then by [HV2, Proposition 4.2] $\mathcal{H}_G(V)$ is identified via $\mathcal{S}_Z^G$ with the subalgebra of $\mathcal{H}_Z(V_{U^0})$ consisting of elements supported on Z^+. By [HV1, 1.8 Theorem], $\mathcal{Z}_G(V)$ is the subalgebra $\mathcal{H}_G(V) \cap \mathcal{Z}_Z(V_{U^0})$ of $\mathcal{Z}_Z(V_{U^0})$ consisting of elements supported on Z^+.

III.4. The group Z normalizes Z^0 and its pro-p radical $Z(1)$ and the quotient Z/Z^0 is a finitely generated abelian group. The coinvariant space V_{U^0} is in fact a line, and Z^0 acts on it via a character $\psi_V : Z^0 \to C^\times$ trivial on $Z(1)$: see part B), for the difference between the notation ψ_V here and in [HV2]. For $z \in Z$, the coset $Z^0 z$ supports a non-zero function in $\mathcal{H}_Z(V_{U^0})$ if and only if z normalizes ψ_V, and such a function is in $\mathcal{Z}_Z(V_{U^0})$ if and only if $\psi_V(zz'z^{-1}z'^{-1}) = 1$ for all $z' \in Z$ normalizing ψ_V.

Notation We let Z_{ψ_V} be the subgroup of Z defined by this last condition. It contains S and Z^0.

For $z \in Z$ normalizing ψ_V we write $\tau_z \in \mathcal{H}_Z(V_{U^0})$ for the function with support $Z^0 z$ and value $\mathrm{id}_{V_{U^0}}$ at z; we have

$$\tau_z * \tau_{z'} = \tau_{zz'} \text{ for } z,\ z' \text{ in } Z \text{ normalizing } \psi_V.$$

Identifying $\mathcal{H}_G(V)$ and $\mathcal{H}_M(V_{N^0})$ with subalgebras of $\mathcal{H}_Z(V_{U^0})$ via $\mathcal{S}_Z^G$ and $\mathcal{S}_Z^M$, we can now describe $\mathcal{H}_M(V_{N^0})$ as the localization of $\mathcal{H}_G(V)$ at some central element [HV2, Proposition 4.5] (so that $\mathcal{Z}_M(V_{N^0})$ is the localization of $\mathcal{Z}_G(V)$ at the same element).

Proposition *Let $M = M_I$ for some $I \subset \Delta$, and let $s \in S$ satisfy $|\alpha|(s) < 1$ for $\alpha \in \Delta - I$, $|\alpha|(s) = 1$ for $\alpha \in I$. Then $\mathcal{H}_M(V_{N^0})$ is the localization of $\mathcal{H}_G(V)$ at τ_s, and $\mathcal{Z}_M(V_{N^0})$ the localization of $\mathcal{Z}_G(V)$ at τ_s.*

Notation For each $\alpha \in \Delta$, we choose z_α in S such that $|\alpha|(z_\alpha) < 1$ and $|\alpha'|(z_\alpha) = 1$ for $\alpha' \in \Delta - \{\alpha\}$. For a character χ of $\mathcal{Z}_G(V)$, we let $\Delta_0(\chi) = \{\alpha \in \Delta \mid \chi(\tau_{z_\alpha}) = 0\}$.

In the above proposition, we can take $s = \prod_{\alpha \in \Delta - I} z_\alpha$; then τ_s is the product $\tau_s = \prod_{\alpha \in \Delta - I} \tau_{z_\alpha}$ in any order.

Lemma *Let χ be a character of $\mathcal{Z}_G(V)$. Then $I = \Delta_0(\chi)$ is the smallest subset of Δ such that χ extends to a character of $\mathcal{Z}_{M_I}(V_{N_I^0})$. For z in $Z^+ \cap Z_{\psi_V}$ we have $\chi(\tau_z) \neq 0$ if and only if $|\alpha|(z) = 1$ for all $\alpha \in \Delta_0(\chi)$. In particular, $\Delta_0(\chi)$ does not depend on $\{z_\alpha\}$.*

Proof As $\mathcal{Z}_{M_I}(V_{N_I^0})$ is the localization of $\mathcal{Z}_G(V)$ at $\prod_{\alpha \in \Delta - I} \tau_{z_\alpha}$, χ extends to a character of $\mathcal{Z}_{M_I}(V_{N_I^0})$ if and only if $\chi(\tau_{z_\alpha}) \neq 0$ for $\alpha \in \Delta - I$. The first assertion follows. Let $z \in Z^+ \cap Z_{\psi_v}$; if for some $\alpha \in \Delta_0(\chi)$ we have $|\alpha|(z) < 1$, then for some positive integer r, $z^r = z_\alpha t$ with $t \in Z^+ \cap Z_{\psi_V}$, and $\chi(\tau_z)^r = \chi(\tau_{z_\alpha})\chi(t) = 0$, so $\chi(\tau_z) = 0$; if $|\alpha|(z) = 1$ for all $\alpha \in \Delta_0(\chi)$ then with $s = \prod_{\alpha \in \Delta - \Delta_0(\chi)} z_\alpha$ there is a positive integer x such that $s^r = zt$ for some $t \in Z^+ \cap Z_{\psi_V}$ and similarly $\chi(\tau_z) \neq 0$ since $\chi(\tau_s) \neq 0$. □

We write $Z_\Delta^\perp$ for the set of $z \in Z$ with $|\alpha|(z) = 1$ for all $\alpha \in \Delta$. Using the lemma, we can restate the definition of supersingularity (I.5) for a character of $\mathcal{Z}_G(V)$.

Corollary *For a character χ of $\mathcal{Z}_G(V)$, the following conditions are equivalent:*

(i) χ *is supersingular,*
(ii) $\Delta_0(\chi) = \Delta$,
(iii) $\chi(\tau_z) = 0$ *for all* z *in* $Z^+ \cap Z_{\psi_V}$ *not in* $Z^{\perp}_{\Delta}$.

B) Irreducible representations of K

III.5. For a subgroup $H \subset G$ we put $\overline{H} = (H \cap K)/(H \cap K(1))$. As recalled above, irreducible representations of K factor through $\overline{K} = K/K(1)$. Information about $\overline{K}$ comes from [BT1, BT2], see also [Ti]. The group $\overline{K}$ is naturally the group of points (over the residue field k of F), of a connected reductive group, which we write $\mathbf{G}_k$, so that $\overline{G} = \overline{K} = \mathbf{G}_k(k)$[9]. We also have $\overline{S} = \mathbf{S}_k(k)$, where $\mathbf{S}_k$ is a maximal split torus in $\mathbf{G}_k$, with a natural identification of $X^*(\mathbf{S}_k)$ and $X^*(\mathbf{S})$; if $\mathbf{Z}_k$ is the centralizer of $\mathbf{S}_k$ in $\mathbf{G}_k$ then $\overline{Z} = \mathbf{Z}_k(k)$, and similarly for the normalizer $\boldsymbol{\mathcal{N}}_k$ of $\mathbf{S}_k$ in $\mathbf{G}_k$. As K is a special parahoric subgroup, every element of W_0 has a representative in K so that $W_0 = \mathcal{N}^0/Z^0$, and reduction mod $K(1)$ yields an identification of W_0 with $W(\mathbf{G}_k, \mathbf{S}_k) = \overline{\mathcal{N}}/\overline{Z}$.

Similarly $\overline{B} = \mathbf{B}_k(k)$ for a minimal parabolic subgroup $\mathbf{B}_k$ of $\mathbf{G}_k$ with Levi component $\mathbf{Z}_k$ (which is a torus since k is finite) and unipotent radical $\mathbf{U}_k$ such that $\overline{U} = \mathbf{U}_k(k)$.

III.6. The root system of $\mathbf{S}_k$ in $\mathbf{G}_k$ is a sub-root system of the root system of $\mathbf{S}$ in $\mathbf{G}$, using the above-mentioned identification of $X^*(\mathbf{S}_k)$ and $X^*(\mathbf{S})$. We write Φ_k for the set of roots of $\mathbf{S}_k$ in $\mathbf{G}_k$; we have $\Phi_k \subset \Phi$. A reduced root $\alpha \in \Phi$ belongs to Φ_k if 2α is not a root in Φ; if α and 2α are roots in Φ, then α or 2α or both are in Φ_k – all three cases can occur.

So we get a natural bijection $\alpha \mapsto \overline{\alpha}$ from reduced roots in Φ to reduced roots in Φ_k, which sends positive roots to positive roots, and the set Δ of simple roots in Φ to the set Δ_k of simple roots in Φ_k. When $\alpha \in \Phi$ is reduced, we have $\overline{U}_\alpha = \mathbf{U}_{k,\overline{\alpha}}(k)$. Henceforward we **identify** the reduced roots of Φ_k with those of Φ, hence Φ_k with Φ, Δ_k with Δ, via $\alpha \mapsto \overline{\alpha}$. Then for $I \subset \Delta$ the parabolic subgroup $P_I = M_I N_I$ is such that $\overline{P_I} = \mathbf{P}_{I,k}(k)$, $\overline{M_I} = \mathbf{M}_{I,k}(k)$, $\overline{N_I} = \mathbf{N}_{I,k}(k)$.

III.7. Let $\mathbf{B}_{\mathrm{op}}$ be the parabolic subgroup of $\mathbf{G}$ opposite to $\mathbf{B}$[10] (with respect to $\mathbf{Z}$) and $\mathbf{U}_{\mathrm{op}}$ its unipotent radical; then $\overline{B}_{\mathrm{op}} = \mathbf{B}_{k,\mathrm{op}}(k)$ where $\mathbf{B}_{k,\mathrm{op}}$ is the parabolic subgroup of $\mathbf{G}_k$ opposite to $\mathbf{B}_k$. Similarly we have $\overline{U}_{\mathrm{op}} = \mathbf{U}_{k,\mathrm{op}}(k)$ for their unipotent radicals.

From [BoT, Proposition 6.25] we get that $\overline{G}$ is generated by the union of $\overline{Z}$, $\overline{U}$, $\overline{U}_{\mathrm{op}}$. The subgroup $\overline{G}'$ of $\overline{G}$ generated by the union of $\overline{U}$ and $\overline{U}_{\mathrm{op}}$ is normal in $\overline{G}$; it is the image in $\overline{G}$ of $\mathbf{G}_{k,\mathrm{sc}}(k)$ where $\mathbf{G}_{k,\mathrm{sc}}$ is the simply connected covering of the derived group of $\mathbf{G}_k$. Note[11] that G'^0 certainly contains U^0 and $(U_{\mathrm{op}})^0$ so that its image in $\overline{G}$ contains $\overline{G}'$. But it can be larger, so we need to distinguish $\overline{G}'$ and $\overline{G'}$[12]; the discrepancy is actually quite important in Chapter IV.

Lemma *(i) The map* $(U \cap K(1)) \times Z(1) \times (U_{\mathrm{op}} \cap K(1)) \to K(1)$ *given by the product law is bijective, and similarly for any order of the factors.*

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[9] We warn the reader that when $\mathbf{G}$ is semisimple, $\mathbf{G}_k$ is not necessarily semisimple. If $\mathbf{H}_k$ is an algebraic group over k, we put $H_k = \mathbf{H}_k(k)$, so that for many algebraic subgroups $\mathbf{H}$ of $\mathbf{G}$ in the current chapter, we can use indifferently the notations $\overline{H}$ or H_k for $(H \cap K)/(H \cap K(1))$ – we mostly use $\overline{H}$.

[10] When convenient, we put the index op on top.

[11] Recall G' is the subgroup of G generated by U and U_{op}.

[12] To avoid confusion, we sometimes write G'_k rather than $\overline{G}'$.

(ii) K is generated by the union of U^0, Z^0 and $(U_{\mathrm{op}})^0$.

Proof Assertion (i) is due to Bruhat and Tits [BT2, 4.6.8 Corollaire]. Since $\overline{G}$ is generated by the union of $\overline{Z}$, $\overline{U}$ and $\overline{U}_{\mathrm{op}}$, K is generated by the union of Z^0, U^0, $(U_{\mathrm{op}})^0$ and the normal subgroup $K(1)$; then (ii) follows from (i). □

The lemma has a consequence which will be useful later. As in III.4 we write $Z_\Delta^\perp$ for the set of $z \in Z$ such that $|\alpha|(z) = 1$ for all $\alpha \in \Delta$. Equivalently, $Z_\Delta^\perp = \operatorname{Ker} v_Z$ in the notation of [HV1, 3.2]. (We have in fact that $|\alpha|(z) = q^{-\langle \alpha, v_Z(z)\rangle}$ for $\alpha \in \Delta$ and $z \in Z$.)

Corollary *$Z_\Delta^\perp$ is the normalizer of K in Z.*

Proof If $z \in Z$ normalizes K it also normalizes U_α^0 for all $\alpha \in \Phi$. Given the action of z on the filtration of U_α [Ti], that is equivalent to $|\alpha|(z) = 1$ for $\alpha \in \Phi$. Conversely if $|\alpha|(z) = 1$ for α in Δ then $|\alpha|(z) = 1$ for all α in Φ and z normalizes U_α^0 for all $\alpha \in \Phi$; it then normalizes U^0 and $(U_{\mathrm{op}})^0$, so it normalizes K. That proves that $Z_\Delta^\perp$ is the normalizer of K in Z. □

Remark By the Cartan decomposition the normalizer of K in G is $Z_\Delta^\perp K$.

III.8. We can now recall (see [HV1, HV2] and the references therein) the parametrization of the irreducible representations of $\overline{G}$, up to isomorphism.

If (ρ, V) is an irreducible representation of $\overline{G}$, then $V^{\overline{U}}$ is a line, on which $\overline{Z}$ acts via a character, say $\eta : \overline{Z} \to C^\times$. Let $\Delta(\eta)$ be the set of simple roots $\alpha \in \Delta$ such that η is trivial on $\overline{Z} \cap M'_{\alpha,k}$ (where $M_{\alpha,k}$ is the Levi subgroup of $\overline{G}$ corresponding to $\{\alpha\}$), and as in III.7 $M'_{\alpha,k}$ is the subgroup of $M_{\alpha,k}$ generated by (the union of) $\overline{U}_\alpha$ and $\overline{U}_{-\alpha}$. The stabilizer of the line $V^{\overline{U}}$ in $\overline{G}$ is a parabolic subgroup containing $\overline{B}$ corresponding to a subset Δ_V of $\Delta(\eta)$, and V is characterized up to isomorphism by the pair (η, Δ_V); all such pairs do occur. In [HV2], (η, Δ_V) is called the **standard parameter** of V.

III.9. In this paper, we are interested in coinvariants rather than invariants, so we use different parameters. Let V be an irreducible representation of $\overline{G}$ with standard parameter (η, Δ_V).

Lemma *The group $\overline{Z}$ acts on the line $V_{\overline{U}}$ via the character $\eta \circ w_0$ where w_0 is the longest element in W_0. Moreover the stabilizer of the kernel of $V \to V_{\overline{U}}$ is the parabolic subgroup of $\overline{G}$ corresponding to the subset $-w_0\Delta_V$ of Δ.*

Proof By [HV2, Proposition 3.14] the projection $V \to V_{\overline{U}}$ induces a $\overline{Z}$-equivariant isomorphism of $V^{\overline{U}_{\mathrm{op}}}$ onto $V_{\overline{U}}$; the first assertion comes from [HV2, 3.11]. The stabilizer we look at is also the stabilizer of the line $(V^*)^{\overline{U}}$ in the contragredient representation V^* of V; the second assertion follows from by [HV2, 3.12]. □

Definition *The **parameter** of V is the pair $(\psi_V, \Delta(V))$ where $\overline{Z}$ acts on $V_{\overline{U}}$ via ψ_V and the stabilizer in $\overline{G}$ of the kernel of $V \to V_{\overline{U}}$ is $\overline{P}_{\Delta(V)}$.*

Remarks 1) We have $\psi_V = \eta \circ w_0$ and $\Delta(V) = -w_0\Delta_V$.
2) The antistandard parameter of V [HV2, 3.11] is $(\psi_V, -\Delta(V))$.
3) V is determined up to isomorphism by its parameter. One has $\Delta(V) \subset \Delta(\psi_V)$, and all pairs (ψ, I) with $I \subset \Delta(\psi)$ occur as parameters.

III.10. Lemma *Let V be an irreducible representation of K, and let $P = MN$ be a parabolic subgroup of G containing B.*

(i) *$V_{\overline{N}}$ is an irreducible representation of $\overline{M}$ with parameter $(\psi_V, \Delta_M \cap \Delta(V))$.*
(ii) *V is $\overline{P}_{\mathrm{op}}$-regular in the sense of* [HV2, Def. 3.6] *if and only if $\Delta(V) \subset \Delta_M$.*

Here, $\overline{P}_{\mathrm{op}} = \overline{M}\,\overline{N}_{\mathrm{op}}$ is the parabolic subgroup of $\overline{G}$ opposite to $\overline{P}$ (relative to $\overline{M}$).

Proof By [HV2, 3.11] $V^{\overline{N}_{\mathrm{op}}}$ is an irreducible representation of $\overline{M}$ and its antistandard parameters are $(\psi_V, -(\Delta_M \cap \Delta(V)))$. On the other hand, the projection $V \to V_{\overline{N}}$ induces an $\overline{M}$-equivariant isomorphism of $V^{\overline{N}_{\mathrm{op}}}$ onto $V_{\overline{N}}$, so (i) comes from Remark 2) above. By [HV2, Def. 3.6] V is $\overline{P}_{\mathrm{op}}$-regular if and only if $-\Delta(V) \subset -\Delta_M$ i.e. $\Delta(V) \subset \Delta_M$, whence (ii). □

Remarks 1) Since $\overline{P}_{\Delta(V)}$ is the stabilizer of the kernel of the projection $V \to V_{\overline{U}}$, V is one-dimensional if and only if $\Delta(V) = \Delta$. It follows from part (i) of the lemma that $V_{\overline{N}}$ is one-dimensional if and only if $\Delta_M \subset \Delta(V)$. That provides a useful characterization of $\Delta(V)$.

2) In this paper we will not use the terminology of a weight V being $\overline{P}_{\mathrm{op}}$-regular. We will phrase everything in terms of the equivalent condition $\Delta(V) \subset \Delta_M$ of the above lemma.

Examples 1) Consider the case where V is the trivial representation of $\overline{G}$. Then $\psi_V = 1$ and $\Delta(V) = \Delta$. Representations V with parameter $(1, I)$ for $I \subset \Delta$ are particularly important to us (cf. III.18 below).

2) Let η be a character of $\overline{Z}$; then η extends to a character of $\overline{M}_{\Delta(\eta)}$: indeed, that extension is the irreducible representation of $\overline{M}_{\Delta(\eta)}$ with parameter $(\eta, \Delta(\eta))$.

III.11. Consider the simply connected covering $\mathbf{G}_{k,\mathrm{sc}}$ of the derived group $\mathbf{G}_{k,\mathrm{der}}$ of $\mathbf{G}_k$ and write $j : \mathbf{G}_{k,\mathrm{sc}} \to \mathbf{G}_k$ for the natural morphism. Put $G_{k,\mathrm{sc}} = \mathbf{G}_{k,\mathrm{sc}}(k)$. We can repeat exactly the same considerations as in II.4 in this context of finite reductive groups, and we use the analogous notation – note however that since k is finite, every almost k-simple component of $\mathbf{G}_{k,\mathrm{sc}}$ is isotropic. In particular j induces an isomorphism between $\tilde{\mathbf{U}}_k$ and $\mathbf{U}_k$, and Δ_k also appears as the set of simple roots of $\tilde{\mathbf{S}}_k$ in $\tilde{\mathbf{U}}_k$.

From III.7, recall that

$$G'_k = j(G_{k,\mathrm{sc}}).$$

Proposition *Let (ρ, V) be an irreducible representation of G_k with parameter $(\psi_V, \Delta(V))$. Then $(\rho \circ j, V)$ is an irreducible representation of $G_{k,\mathrm{sc}}$ with parameter $(\psi_V \circ j_{|\tilde{Z}_k}, \Delta(V))$.*

Here, $\tilde{Z}_k = \tilde{\mathbf{Z}}_k(k)$, where $\tilde{\mathbf{Z}}_k$ is the centralizer of $\tilde{\mathbf{S}}_k$ in $\mathbf{G}_{k,\mathrm{sc}}$; we use similarly abbreviated notation below. By the fact above and the inclusion $G'_k = \overline{G}' \subset \overline{G'}$ (III.7), we get:

Corollary *The restriction of ρ to G'_k, and a fortiori to $\overline{G'}$, is irreducible.*

Proof of the proposition Since $V_{\tilde{U}_k}$, equal to $V_{\overline{U}}$, is one-dimensional, the cosocle of $\rho \circ j$ is irreducible. Similarly $V^{\tilde{U}_{k,\mathrm{op}}}$ equal to $V^{\overline{U}_{\mathrm{op}}}$ is one dimensional, so the socle of $\rho \circ j$ is irreducible too. As the projection of $V^{\overline{U}_{\mathrm{op}}}$ to $V_{\overline{U}}$ is non-zero, the map from the socle of $\rho \circ j$ to its cosocle is non-zero, and $\rho \circ j$ is indeed irreducible. Clearly $\tilde{Z}_k$ acts on $V_{\tilde{U}_k} = V_{\overline{U}}$

by $z \mapsto \psi_V \circ j(z)$, and $\tilde{P}_{\Delta(V),k} = j^{-1}(\overline{P}_{\Delta(V)})$ stabilizes the kernel of $V \to V_{\tilde{U}_k}$. But for $I \subset \Delta$, we have $\overline{P_I} = \overline{Z} j(\tilde{P}_{I,k})$, so if $\tilde{P}_{I,k}$ stabilizes that kernel, $I \subset \Delta(V)$. □

C) Weights of parabolically induced representations

III.12. Let $P = MN$ be a parabolic subgroup of G containing B, and (τ, W) a representation of M. We investigate the weights of $\mathrm{Ind}_P^G W$ and the corresponding Hecke eigenvalues. From now on, we identify the irreducible representations of K and those of $\overline{G} = K/K(1)$.

In this part C) we let (ρ, V) be an irreducible representation of K, with parameter $(\psi_V, \Delta(V))$. Recall that if (π, X) is a representation of G, for example $X = \mathrm{Ind}_P^G W$, then $\mathrm{Hom}_K(V, X)$ is a right $\mathcal{H}_G(V)$-module via Frobenius reciprocity. The formula for the action is

$$(\varphi\Phi)(v) = \sum_{g \in G/K} g\varphi(\Phi(g^{-1})v) \quad \text{for } v \in V,\ \varphi \in \mathrm{Hom}_K(V, X), \tag{III.12.1}$$

and $\Phi \in \mathcal{H}_G(V)$.

Proposition *(i) The natural isomorphism*

$$\mathrm{Hom}_K(V, \mathrm{Ind}_P^G W) \xrightarrow[\sim]{\text{can}} \mathrm{Hom}_{M^0}(V_{N^0}, W)$$

is $\mathcal{H}_G(V)$-linear, where $\mathcal{H}_G(V)$ acts on the right-hand side via $\mathcal{S}_M^G$.

(ii) V is a weight for $\mathrm{Ind}_P^G W$ if and only if V_{N^0} is a weight for W.

(iii) The map $\mathcal{S}_M^G$ identifies the eigenvalues of V in $\mathrm{Ind}_P^G W$ and the eigenvalues of V_{N^0} in W.

Proof (i) comes from III.3 and (ii) is an immediate consequence. We have seen that $\mathcal{Z}_M(V_{N^0})$ is the localization of $\mathcal{Z}_G(V)$ at some element τ_s. Clearly τ_s acts invertibly on $\mathrm{Hom}_{M^0}(V_{N^0}, W)$; as the canonical isomorphism is $\mathcal{H}_G(V)$-linear, τ_s also acts invertibly on $\mathrm{Hom}_K(V, \mathrm{Ind}_P^G W)$, which gives (iii). □

A useful consequence of (III.12.1) is the following lemma. Recall that for $z \in Z^+ \cap Z_{\psi_V}$, $\mathcal{Z}_G(V)$ contains a unique element T_z such that $\mathrm{Supp}\, T_z = KzK$ and $T_z(z) \in \mathrm{End}_C(V)$ induces the identity on $V^{\overline{U}_{\mathrm{op}}}$ [HV1, 7.3, 2.9].

Lemma *Let (π, X) be a representation of G and $\varphi \in \mathrm{Hom}_K(V, X)$. Let $z \in Z_{\psi_V}$. Assume $z \in Z_\Delta^\perp$, i.e. that z normalizes K. Then $\mathcal{S}_Z^G(T_z) = \tau_z$ and $(\varphi\tau_z)(v) = z^{-1}\varphi(v)$ for v in $V^{\overline{U}_{\mathrm{op}}}$. If φ is an eigenvector for $\mathcal{Z}_G(V)$ with eigenvalue χ, then z^{-1} acts on $\varphi(V^{\overline{U}_{\mathrm{op}}})$ by $\chi(\tau_z)$.*

Proof By assumption $zK = Kz$, and the endomorphism $T_z(z)$ satisfies $\rho(k)T_z(z) = T_z(z)\rho(z^{-1}kz)$ for $k \in K$ [HV1, 7.3]. As z normalizes U^0 and $(U_{\mathrm{op}})^0$, $T_z(z)$ induces endomorphisms of $V_{\overline{U}}$ and $V^{\overline{U}_{\mathrm{op}}}$; since the natural map $V^{\overline{U}_{\mathrm{op}}} \to V_{\overline{U}}$ is an isomorphism, $T_z(z)$ induces the identity on $V_{\overline{U}}$. From (III.3.2) we get $\mathcal{S}_Z^G(T_z) = \tau_z$, and (III.12.1) gives

$$(\varphi T_z)(v) = z^{-1}\varphi(T_z(z)v) \quad \text{for } v \in V,$$

hence the result. □

III.13. Let $\varphi \in \mathrm{Hom}_K(V, \mathrm{Ind}_P^G W)$ and $\varphi_M \in \mathrm{Hom}_{M^0}(V_{N^0}, W)$ correspond via (III.3.1). Then φ gives rise to a G-morphism, again written φ, from $\mathrm{ind}_K^G V$ to $\mathrm{Ind}_P^G W$, and similarly we get an M-morphism $\varphi_M : \mathrm{ind}_{M^0}^M V_{N^0} \to W$.

Consider the following diagram, where horizontal maps are canonical isomorphisms

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathrm{ind}_K^G V,\ \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0})) & \xrightarrow[\sim]{\text{can}} & \mathrm{Hom}_M(\mathrm{ind}_{M^0}^M V_{N^0}, \mathrm{ind}_{M^0}^M V_{N^0}) \\ \Big\downarrow{\scriptstyle \mathrm{Ind}_P^G \varphi_M} & & \Big\downarrow{\scriptstyle \varphi_M} \\ \mathrm{Hom}_G(\mathrm{ind}_K^G V,\ \mathrm{Ind}_P^G W) & \xrightarrow[\sim]{\text{can}} & \mathrm{Hom}_M(\mathrm{ind}_{M^0}^M V_{N^0}, W) \end{array}$$

By naturality, the vertical maps obtained by composing with $\mathrm{Ind}_P^G \varphi_M$ and φ_M, as indicated, make the diagram commutative. The identity map of $\mathrm{ind}_{M^0}^M V_{N^0}$ yields the canonical intertwiner

$$\mathcal{I} : \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0}) \tag{III.13.1}$$

mentioned in I.6. We get:

Lemma $\mathrm{Ind}_P^G \varphi_M \circ \mathcal{I} = \varphi$.

By [HV2, Proposition 4.1], $\mathcal{I}$ is injective. As $\mathcal{I}$ is $\mathcal{H}_G(V)$-linear, it factors as follows:

$$\mathrm{ind}_K^G V \longrightarrow \mathcal{Z}_M(V_{N^0}) \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V \xrightarrow{u} \mathcal{H}_M(V_{N^0}) \otimes_{\mathcal{H}_G(V)} \mathrm{ind}_K^G V \xrightarrow{\Theta} \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0}),$$

for some canonical map Θ. Since $\mathcal{H}_M(V_{N^0})$ is the localization of $\mathcal{H}_G(V)$ at some central element, and $\mathcal{Z}_M(V_{N^0})$ is the localization of $\mathcal{Z}_G(V)$ at the same element, the map u is an isomorphism.

III.14. The main result of [HV2] is, taking into account III.10 Lemma (ii):

Theorem *Let $(\psi_V, \Delta(V))$ be the parameter of V. If $\Delta(V) \subset \Delta_M$ then the map*

$$\mathcal{H}_M(V_{N^0}) \otimes_{\mathcal{H}_G(V)} \mathrm{ind}_K^G V \xrightarrow{\Theta} \mathrm{Ind}_P^G(\mathrm{ind}_{M^0}^M V_{N^0})$$

of III.13 *is an isomorphism.*

We derive some consequences.

Corollary 1 *Let $\varphi \in \mathrm{Hom}_K(V, \mathrm{Ind}_P^G W)$ be an eigenvector for $\mathcal{Z}_G(V)$. If $\Delta(V) \subset \Delta_M$ and if $\varphi_M(V_{N^0})$ generates W as a representation of M, then $\varphi(V)$ generates $\mathrm{Ind}_P^G W$ as a representation of G.*

Proof By the theorem, Θ is surjective. By hypothesis $\varphi_M : \mathrm{ind}_{M^0}^M V_{N^0} \to W$ is surjective, so by III.13 Lemma the map induced by φ

$$\mathcal{Z}_M(V_{N^0}) \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V \longrightarrow \mathrm{Ind}_P^G W$$

is surjective. But $\mathcal{Z}_G(V)$ acts on φ via a character which extends to $\mathcal{Z}_M(V_{N^0})$ (III.12 Proposition (ii)) so we conclude that $\varphi(\mathrm{ind}_K^G V) = \mathrm{Ind}_P^G W$, hence the result. □

Corollary 2 *Assume that (τ, W) is irreducible and admissible. Then $\mathrm{Ind}_P^G W$ is irreducible if and only if every non-zero subrepresentation of it contains a weight V with $\Delta(V) \subset \Delta_M$.*

Proof Since W has some weight, by III.12 Proposition (i) and III.10 Lemma (i), $\mathrm{Ind}_P^G W$ has a weight V with $\Delta(V) \subset \Delta_M$. Conversely if a subrepresentation X of $\mathrm{Ind}_P^G W$ contains a weight V with $\Delta(V) \subset \Delta_M$, there is an eigenvector $\varphi \in \mathrm{Hom}_K(V, X)$ for $\mathcal{Z}_G(V)$. As τ is irreducible, $\varphi_M(V_{N^0})$ generates W and by the proposition $X = \mathrm{Ind}_P^G W$. $\square$

D) Determination of $P(\sigma)$ for supersingular σ

III.15. We want to apply the preceding corollary to prove the irreducibility of $I(P, \sigma, Q)$ for a supersingular triple (P, σ, Q). That can only be done in stages. First we determine $P(\sigma)$ in terms of weights and eigenvalues of σ. In other words, we determine the set $\Delta(\sigma)$ of $\alpha \in \Delta - \Delta_M$ such that σ is trivial on $Z \cap M'_\alpha$ (II.7).

As the generality will be useful in Chapter V, we consider the situation where $P = MN$ is a parabolic subgroup of G containing B, and (σ, W) is a representation of M satisfying the following hypothesis:

(H) There is an irreducible representation (ρ, V) of M^0 and some φ in $\mathrm{Hom}_{M^0}(V, W)$ such that σ is generated by $\varphi(V)$ as a representation of M.

Hypothesis (H) is certainly true if σ is irreducible and admissible, and then we can take φ to be an eigenvector for $\mathcal{Z}_M(V)$, and the corresponding eigenvalue is supersingular if σ is. As before, write $(\psi_V, \Delta(V))$ for the parameter of V.

Lemma *Assume Hypothesis (H). Let $\alpha \in \Delta$. If σ is trivial on $Z \cap M'_\alpha$, then ψ_V is trivial on $Z^0 \cap M'_\alpha$.*

Proof If σ is trivial on $Z \cap M'_\alpha$, then certainly $Z \cap M'_\alpha$ acts trivially on $\varphi(V)$. As $\varphi \in \mathrm{Hom}_{M^0}(V, W)$ is injective, $Z^0 \cap M'_\alpha$ acts trivially on V hence on V_{U^0} and ψ_V is trivial on $Z^0 \cap M'_\alpha$. $\square$

III.16. Proposition *Let $\alpha \in \Delta$.*
(i) If ψ_V is trivial on $Z^0 \cap M'_\alpha$ then $Z \cap M'_\alpha \subset Z_{\psi_V}$.
(ii) $|\alpha|\ (Z \cap M'_\alpha)$ is isomorphic to $\mathbb{Z}$.
(iii) Let $z \in Z \cap M'_\alpha$. Then $|\alpha|(z) = 1$ if and only if $z \in Z^0 \cap M'_\alpha$.

Notation By the proposition the group $(Z \cap M'_\alpha)/(Z^0 \cap M'_\alpha)$ is isomorphic to $\mathbb{Z}$. By (ii) there is an element a_α in $Z \cap M'_\alpha$ with $|\alpha|(a_\alpha) > 1$, such that $|\alpha|(a_\alpha)$ generates $|\alpha|(Z \cap M'_\alpha)$; by (iii) the element a_α is well-defined modulo $Z^0 \cap M'_\alpha$. Note that if α is orthogonal to Δ_M then $a_\alpha \in Z^\perp_{\Delta_M}$ (see proof of III.7 Corollary) and τ_{a_α} is a unit of $\mathcal{Z}_M(V)$. If ψ_V is trivial on $Z^0 \cap M'_\alpha$ the element τ_{a_α} of $\mathcal{Z}_Z(V_{U \cap M^0})$ does not depend on the choice of a_α, so we write it τ_α.

Proof of the proposition Assume that ψ_V is trivial on $Z^0 \cap M'_\alpha$, and take $z \in Z \cap M'_\alpha$; then, for $z' \in Z$ (in particular for $z' \in Z^0$), $zz'z^{-1}z'^{-1}$ belongs to $Z^0 \cap M'_\alpha$ (because Z/Z^0 is abelian and $Z \cap M'_\alpha$ is normal in Z), so we get $\psi_V(zz'z^{-1}z'^{-1}) = 1$. That shows that z normalizes ψ_V and belongs to Z_{ψ_V}, hence (i).

Let us introduce the isotropic part $\tilde{\mathbf{M}}_\alpha = \mathbf{M}_\alpha^{\mathrm{is}}$ of the simply connected covering of the derived group of $\mathbf{M}_\alpha$, its minimal Levi subgroup $\tilde{\mathbf{Z}}_\alpha$ lifting $\mathbf{Z}$, and the maximal split torus $\tilde{\mathbf{S}}_\alpha$ of $\tilde{\mathbf{Z}}_\alpha$. Write j for the canonical map $\tilde{\mathbf{M}}_\alpha \to \mathbf{M}_\alpha$. We have $M'_\alpha = j(\tilde{M}_\alpha)$ and $j^{-1}(Z) = \tilde{Z}_\alpha$, so $Z \cap M'_\alpha = j(\tilde{Z}_\alpha)$.

Let $v_Z : Z \to X_*(\mathbf{S}) \otimes \mathbb{Q}$ be the homomorphism such that $\chi(v_Z(z)) = \mathrm{val}_F(\chi(z))$ for all $z \in S$ and all F-characters χ of $\mathbf{S}$, where val_F is the valuation of F with image $\mathbb{Z}$;

its kernel is the maximal compact subgroup of Z. Let $w_G : G \to X^*(Z(\widehat{\mathbf{G}})^{I_F})^{\sigma_F}$ be the Kottwitz homomorphism of G [Kot, §7.7], where $\widehat{\mathbf{G}}$ denotes the dual group, I_F the inertia subgroup of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$, and σ_F a Frobenius element of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. The parahoric subgroup Z^0 of Z is equal to $\operatorname{Ker} w_Z$. By [HV1, 6.2], $\operatorname{Ker} w_Z = \operatorname{Ker} v_Z \cap \operatorname{Ker} w_G$.

We have the analogous map $v_{\tilde{Z}_\alpha}$ and a commutative diagram

$$\begin{array}{ccc} \tilde{Z}_\alpha & \xrightarrow{v_{\tilde{Z}_\alpha}} & X_*(\tilde{\mathbf{S}}_\alpha) \otimes \mathbb{Q} \\ \downarrow & & \downarrow \\ Z & \xrightarrow{v_Z} & X_*(\mathbf{S}) \otimes \mathbb{Q} \end{array}$$

where the vertical maps are induced by j.

As $\tilde{\mathbf{M}}_\alpha$ is semisimple and simply connected, $w_{\tilde{M}_\alpha}$ is trivial and by functoriality of the Kottwitz homomorphism w_G is trivial on $M'_\alpha = j(\tilde{M}_\alpha)$; in particular $Z^0 \cap M'_\alpha = \operatorname{Ker} v_Z \cap M'_\alpha$. The vertical map on the right of the above diagram is injective so $j^{-1}(Z^0 \cap M'_\alpha) = \operatorname{Ker} v_{\tilde{Z}_\alpha}$. Thus $(Z \cap M'_\alpha)/(Z^0 \cap M'_\alpha)$ is isomorphic to $\tilde{Z}_\alpha / \operatorname{Ker} v_{\tilde{Z}_\alpha}$, i.e. to the image of $v_{\tilde{Z}_\alpha}$. Since $\tilde{\mathbf{S}}_\alpha$ has dimension 1, that image is isomorphic to $\mathbb{Z}$. Now for $z \in \tilde{Z}_\alpha$ we have $|\alpha|(j(z)) = q^{-\langle \alpha, v_Z(j(z))\rangle} = q^{-\langle \alpha, v_{\tilde{Z}_\alpha}(z)\rangle}$ and (ii), (iii) follow. $\square$

Remark From the above proof it is clear that $v_Z(a_\alpha)$ is a (negative) rational multiple of $\alpha^\vee$. See also IV.11 Example 3.

III.17. Let us derive consequences of III.16.

Proposition *Assume Hypothesis (H)* (III.15). *Let $\alpha \in \Delta$ be orthogonal to Δ_M. Then the following conditions are equivalent*:

(i) σ is trivial on $Z \cap M'_\alpha$,

(ii) ψ_V is trivial on $Z^0 \cap M'_\alpha$ and $(\varphi\tau_\alpha)(v) = \varphi(v)$ for $v \in V^{U_{\mathrm{op}} \cap M^0}$.

Proof Apply first III.12 Lemma to get

$$(*) \qquad (\varphi\tau_\alpha)(v) = a_\alpha^{-1}\varphi(v)$$

for $v \in V^{U_{\mathrm{op}} \cap M^0}$. Now assume (i). By III.15 Lemma, ψ_V is trivial on $Z^0 \cap M'_\alpha$; then, since α is orthogonal to Δ_M, a_α belongs to $Z^\perp_{\Delta_M}$ and $(*)$ implies (ii). Conversely assume (ii). Applying III.16 Proposition and $(*)$ again we get that $Z \cap M'_\alpha$ acts trivially on the line $\varphi(V^{U_{\mathrm{op}} \cap M^0})$. But as α is orthogonal to Δ_M, M normalizes M'_α and hence also $Z \cap M'_\alpha$; consequently, the set of fixed points of $Z \cap M'_\alpha$ in W is invariant under M. As it contains $\varphi(V^{U_{\mathrm{op}} \cap M^0})$ it contains $\varphi(V)$ since $V^{U_{\mathrm{op}} \cap M^0}$ generates V over M^0, and by hypothesis (H), $Z \cap M'_\alpha$ acts trivially on W. $\square$

Corollary *Assume Hypothesis (H) and that moreover φ is a $\mathcal{Z}_M(V)$-eigenvector with supersingular eigenvalue χ. Then $\Delta(\sigma)$ as in* (II.7) *is the set of $\alpha \in \Delta$, orthogonal to Δ_M, such that ψ_V is trivial on $Z^0 \cap M'_\alpha$ and $\chi(\tau_\alpha) = 1$.*

Proof Assume $\alpha \in \Delta(\sigma)$ is not orthogonal to Δ_M. By II.7 Corollary 2 and Remark 5, there is a proper parabolic subgroup $Q = M_Q N_Q$ of M (containing $M \cap B$) such that σ is trivial on N_Q and is a subrepresentation of $\mathrm{Ind}_Q^M(\sigma_{|M_Q})$. By III.12 Proposition (iii), no eigenvalue of σ can be supersingular. Consequently, any α in $\Delta(\sigma)$ is orthogonal to Δ_M and the result follows from the proposition. $\square$

In particular, we have determined $P(\sigma)$ for a supersingular representation σ of M.

E) Weights and eigenvalues of $I(P,\sigma,Q)$

III.18. In this section, for a supersingular triple (P,σ,Q) (I.5), we determine the weights and eigenvalues of $I(P,\sigma,Q)$. A slightly more general situation is useful in part G) though.

Proposition *Consider a B-triple (P,σ,Q) as in* I.5 *with $P = MN$, and assume that $\Delta(\sigma)$ is orthogonal to Δ_M. Let V be an irreducible representation of K, with parameter $(\psi_V, \Delta(V))$.*

1) *The following conditions are equivalent:*

(i) V is a weight of $I(P,\sigma,Q)$,

(ii) V_{N^0} is a weight of σ and $\Delta(V)\cap\Delta(\sigma) = \Delta_Q\cap\Delta(\sigma)$.

2) *If V is a weight of $I(P,\sigma,Q)$, then the eigenvalues of $\mathcal{Z}_G(V)$ in $I(P,\sigma,Q)$ are in bijection with those of $\mathcal{Z}_M(V_{N^0})$ in σ via $\mathcal{S}_M^G$.*

The proof of 1) is in III.19–III.21 below, that of 2) in III.22, which actually gives more precise information.

Remark 1 Consider the case where $P = B$ and σ is the trivial representation of B. Then $P(\sigma) = G$ and $I(B,\sigma,Q) = \mathrm{St}_Q^G$. From [Ly1, §8] we get that St_Q^G has a unique weight V_Q^G, with multiplicity one, and parameter $(1,\Delta_Q)$. That weight also occurs with multiplicity one in $\mathrm{Ind}_Q^G 1$ and the natural map $\mathrm{Hom}_K(V_Q^G, \mathrm{Ind}_Q^G 1) \to \mathrm{Hom}_K(V_Q^G, \mathrm{St}_Q^G)$ is an isomorphism; similarly V_Q^G occurs with multiplicity one in $\mathrm{Ind}_B^G 1$ and the natural map $\mathrm{Hom}_K(V_Q^G, \mathrm{Ind}_Q^G 1) \to \mathrm{Hom}_K(V_Q^G, \mathrm{Ind}_B^G 1)$ is an isomorphism. Those isomorphisms are $\mathcal{H}_G(V_Q^G)$-equivariant, and the algebra $\mathcal{H}_G(V_Q^G)$, isomorphic to the monoid algebra $C[Z^+/Z^0]$, acts via the augmentation character sending τ_z to 1 for $z\in Z^+$. That special case will be used in the proof of part 2) of the proposition.

The proposition may be applied to a supersingular triple, by III.17 Corollary.

Corollary *Assume (P,σ,Q) is a supersingular triple; if V is a weight of $I(P,\sigma,Q)$ then for any eigenvalue χ of $\mathcal{Z}_G(V)$ in $I(P,\sigma,Q)$, we have $\Delta_0(\chi) = \Delta_M$.*

Proof By part 2) of the proposition, χ extends to a character of $\mathcal{Z}_M(V_{N^0})$ so $\Delta_0(\chi)\subset\Delta_M$. On the other hand the extended character is an eigenvalue of σ which is supersingular so $\Delta_M\subset\Delta_0(\chi)$. □

Remark 2 In the context of the corollary, if $P\neq G$, then no eigenvalue of $I(P,\sigma,Q)$ is supersingular.

III.19. By III.12 Proposition, we immediately reduce the proof of part 1) of the proposition to the case where $P(\sigma) = G$. In the course of the proof we shall glean more information on the weights and eigenvalues.

We put $\Delta_1 = \Delta_M$ and $\Delta_2 = \Delta(\sigma)$, so that Δ is the union of two orthogonal subsets Δ_1 and Δ_2. As in II.4 we introduce the group $\tilde{\mathbf{G}} = \mathbf{G}^{\mathrm{is}}$. It appears as the product of two factors $\tilde{\mathbf{G}}_1$ and $\tilde{\mathbf{G}}_2$ attached to Δ_1, Δ_2. Note that $\tilde{G}$ and G have the same semisimple building and their actions on it are compatible. Let $\tilde{K}$ be the parahoric subgroup of $\tilde{G}$ attached to the point $\mathbf{x}_0$. It decomposes as $\tilde{K}_1\times\tilde{K}_2$ where for $i = 1,2$, $\tilde{K}_i = \tilde{K}\cap\tilde{G}_i$ is a parahoric subgroup of $\tilde{G}_i$. Write ι for the natural map $\tilde{G}\to G$. For $i = 1,2$, let M_i be the Levi subgroup M_{Δ_i} of G. Then $M'_i = \iota(\tilde{G}_i)$ and $M_i = ZM'_i$. By II.7 Remark 4, M'_1 and M'_2 commute with each other, Z normalizes each of them and $G = ZM'_1M'_2$.

Proposition *(i) $\tilde{K} = \iota^{-1}(K)$, $\tilde{Z}^0 = \iota^{-1}(Z^0)$ and $\iota(\tilde{K}_i) = K \cap M'_i$ for $i = 1, 2$.*

(ii) Let $\alpha \in \Phi$; then ι induces a group isomorphism of $\tilde{U}^0_\alpha = \tilde{U}_\alpha \cap \tilde{K}$ onto *$U^0_\alpha = U_\alpha \cap K$.*

Here, $\tilde{U}_\alpha$ denotes the root subgroup of $\tilde{G}$ attached to $\alpha \in \Phi$.

Proof By functoriality of the Kottwitz homomorphism, since $\tilde{G}$ is semisimple simply connected, $w_G \circ \iota$ is trivial; on the other hand an element $x \in \tilde{G}$ fixes the point $\mathbf{x}_0$ if and only if $\iota(x)$ fixes $\mathbf{x}_0$. So we have $\tilde{K} = \iota^{-1}(K)$ and intersecting with $\tilde{Z} = \iota^{-1}(Z)$ we get $\tilde{Z}^0 = \iota^{-1}(Z^0)$. If $x \in \tilde{K}_i$ then $\iota(x) \in K \cap \iota(\tilde{G}_i) = K \cap M'_i$. Conversely if $x \in \tilde{G}_i$ and $\iota(x) \in K$ then $x \in \tilde{K} \cap \tilde{G}_i = \tilde{K}_i$. This proves (i).

(ii) Let $\alpha \in \Phi$. As $\iota(\tilde{K}) \subset K$ we have $\iota(\tilde{U}^0_\alpha) \subset U^0_\alpha$. Conversely for $x \in \tilde{U}_\alpha$, $\iota(x) \in U^0_\alpha$ implies $x \in \tilde{U}_\alpha \cap \iota^{-1}(K) = \tilde{U}^0_\alpha$ by (i). □

Corollary *We have $K = Z^0\iota(\tilde{K})$. For $i = 1, 2$, $M^0_i = Z^0\iota(\tilde{K}_i)$.*

Proof This comes from (ii) of the proposition, given III.7 Lemma. □

Remark By II.7 Remark 4, $M'_1 \cap M'_2$ is finite and central in G. As it is contained in $\operatorname{Ker} w_G$, it follows that Z^0 contains $M'_1 \cap M'_2$, which is equal to $\iota(\tilde{K}_1) \cap \iota(\tilde{K}_2)$.

III.20. Let now (ρ, V) be an irreducible representation of K. We want to write V as a tensor product adapted to the orthogonal decomposition $\Delta = \Delta_1 \sqcup \Delta_2$.

Write $(\tilde{\rho}, \tilde{V})$ for the representation of $\tilde{K}$ obtained from ρ via $\iota : \tilde{K} \to K$. By III.19 Proposition (ii) $\overline{\iota(\tilde{K})}$ contains $\overline{G}'$, so by III.11 Corollary $\tilde{\rho}$ is irreducible. Since $\tilde{K} = \tilde{K}_1 \times \tilde{K}_2$, $\tilde{V}$ decomposes as a tensor product $\tilde{V}_1 \otimes \tilde{V}_2$ where for $i = 1, 2$, $\tilde{V}_i$ is an irreducible representation of $\tilde{K}_i$ which is trivial on $\tilde{K}_{3-i}$.

To decompose V as a tensor product $V_1 \otimes V_2$ of irreducible representations of K, where V_1 restricts to $\tilde{V}_1$ via ι, and V_2 to $\tilde{V}_2$, we have to take some care, as K is not the direct product $M^0_1 \times M^0_2$.

Proposition *(i) For $i = 1, 2$, let V_i be an irreducible representation of K trivial on $K \cap M'_{3-i}$. Then $V_1 \otimes V_2$ is irreducible with parameter $(\psi_{V_1}\psi_{V_2}, \Delta(V_1) \cap \Delta(V_2))$. Moreover, $\Delta(V_i)$ contains Δ_{3-i}.*

(ii) Let V be an irreducible representation of K. If V_2 is an irreducible representation of K trivial on $K \cap M'_1$ with $\operatorname{Hom}_{K \cap M'_2}(V_2, V) \neq 0$, then $V_1 = \operatorname{Hom}_{K \cap M'_2}(V_2, V)$ is an irreducible representation of K trivial on $K \cap M'_2$ and $V \simeq V_1 \otimes V_2$.

(iii) Let V be an irreducible representation of K. Then $V \simeq V_1 \otimes V_2$ with V_i as in (i) if and only if V is trivial on $M'_1 \cap M'_2$.

We will not need part (iii), we only included it for completeness.

Proof (i) Let $\tilde{V}_i$ be the pullback of V_i to $\tilde{K}$ via ι. Then $\tilde{V}_i$ is trivial on $\tilde{K}_{3-i}$, so $\tilde{V}_1 \otimes \tilde{V}_2$ is an irreducible representation of $\tilde{K}$. Hence $V := V_1 \otimes V_2$ is an irreducible representation of K. If $Q = M_Q N_Q$ is a parabolic subgroup containing B, then

$$V_{N^0_Q} \simeq (V_1)_{N^0_Q} \otimes (V_2)_{N^0_Q}, \text{ as } N^0_Q = (N^0_Q \cap M'_1) \times (N^0_Q \cap M'_2).$$

Hence by III.10, $\Delta_Q \subset \Delta(V)$ if and only if $\Delta_Q \subset \Delta(V_i)$ for $i = 1, 2$, so $\Delta(V) = \Delta(V_1) \cap \Delta(V_2)$. Taking $Q = B$, we deduce $\psi_V = \psi_{V_1}\psi_{V_2}$. As $K \cap M'_{3-i}$ is trivial on V_i, we get $\Delta_{3-i} \subset \Delta(V_i)$.

(ii) This follows from Clifford theory [Abe, Lemma 5.3].

(iii) The "if" direction is obvious. Assume that V is trivial on $M'_1 \cap M'_2$. Let W be an irreducible representation of $K \cap M'_2$ such that $\mathrm{Hom}_{K\cap M'_2}(W, V) \neq 0$. Via ι, W is an irreducible representation of $\tilde{K}_2$, which we consider as a representation $\tilde{W}$ of $\tilde{K}$ trivial on $\tilde{K}_1$. As V, hence W, is trivial $\iota(\tilde{K}_1) \cap \iota(\tilde{K}_2)$ by assumption, it follows that $\tilde{W}$ is trivial on $\mathrm{Ker}\,\iota$, so we have extended W to an irreducible representation of $K \cap G'$, which is trivial on $K \cap M'_1$. We may view W as an irreducible representation of $\overline{G'}$ and we choose an irreducible representation V_2 of $\overline{G}$ such that W occurs in $V_2|_{\overline{G'}}$. By III.11 Corollary $W \simeq V_2|_{\overline{G'}}$ and hence $\mathrm{Hom}_{K\cap M'_2}(V_2, V) \neq 0$. By part (ii), $V \simeq V_1 \otimes V_2$ with V_i as in (i). □

III.21. Let (P, σ, Q) be a B-triple with $P(\sigma) = G$. We are now finally ready to determine the weights of ${}^e\sigma \otimes \mathrm{St}_Q^G$. We keep the notation of III.19. Recall that by construction ${}^e\sigma$ is trivial on M'_2 and St_Q^G is trivial on M'_1.

Let us fix a weight V of $I(P, \sigma, Q) = {}^e\sigma \otimes \mathrm{St}_Q^G$. We decompose the pullback $\tilde{V}$ of V to a representation of $\tilde{K} = \tilde{K}_1 \times \tilde{K}_2$, via ι, as $\tilde{V} \simeq \tilde{V}_1 \otimes \tilde{V}_2$. Therefore $\mathrm{Hom}_K(V, {}^e\sigma \otimes \mathrm{St}_Q^G)$ injects into

$$\mathrm{Hom}_{\tilde{K}}(\tilde{V}, {}^e\sigma \otimes \mathrm{St}_{\tilde{Q}}^{\tilde{G}}) \simeq \mathrm{Hom}_{\tilde{K}}(\tilde{V}_1, {}^e\sigma) \otimes \mathrm{Hom}_{\tilde{K}}(\tilde{V}_2, \mathrm{St}_{\tilde{Q}}^{\tilde{G}}),$$

where we used that $\tilde{K}_1$ acts trivially on $\tilde{V}_2, \mathrm{St}_{\tilde{Q}}^{\tilde{G}}$ and $\tilde{K}_1$ acts trivially on $\tilde{V}_1, {}^e\sigma$. As $\mathrm{St}_{\tilde{Q}}^{\tilde{G}}$ has a unique weight (III.18), $\tilde{V}_2$ is the pullback via ι of the unique weight V_2 of St_Q^G. By lifting via $\iota : \tilde{K}_2 \to \iota(\tilde{K}_2) = K \cap M'_2$, we deduce $\mathrm{Hom}_{K\cap M'_2}(V_2, V) = \mathrm{Hom}_{\tilde{K}_2}(\tilde{V}_2, \tilde{V}) \neq 0$. By III.20 Proposition (ii), $V \simeq V_1 \otimes V_2$ for some irreducible representation V_1 of K trivial on $K \cap M'_2$. We also see by III.20 Proposition (i) and III.18 Remark 1 that $\Delta(V) \cap \Delta_2 = \Delta(V_2) \cap \Delta_2 = \Delta_Q \cap \Delta_2$. The natural injection $\mathrm{Hom}_K(V_2, \mathrm{St}_Q^G) \hookrightarrow \mathrm{Hom}_{K\cap M'_2}(V_2, \mathrm{St}_Q^G)$ is an isomorphism of 1-dimensional vector spaces, because the right-hand side is isomorphic to $\mathrm{Hom}_{\tilde{K}}(\tilde{V}_2, \mathrm{St}_{\tilde{Q}}^{\tilde{G}})$ via ι. Thus the following lemma, in our situation, implies that V_1 is a weight of ${}^e\sigma$, so V_{N^0} is a weight of σ. This proves that (i) implies (ii) in III.18 Proposition 1).

Lemma *Let σ_1 be a representation of G trivial on M'_2, σ_2 a representation of G trivial on M'_1. Let V_1 be an irreducible representation of K trivial on $K \cap M'_2$, V_2 an irreducible representation of K trivial on $K \cap M'_1$. Assume that the inclusion $\mathrm{Hom}_K(V_2, \sigma_2) \to \mathrm{Hom}_{K\cap M'_2}(V_2, \sigma_2)$ is an isomorphism. Then the natural inclusion of $\mathrm{Hom}_K(V_1, \sigma_1) \otimes \mathrm{Hom}_K(V_2, \sigma_2)$ into $\mathrm{Hom}_K(V_1 \otimes V_2, \sigma_1 \otimes \sigma_2)$ is an isomorphism.*

Proof Look first at points fixed by $K \cap M'_2$ in $\mathrm{Hom}(V_1 \otimes V_2, \sigma_1 \otimes \sigma_2)$. As $K \cap M'_2$ acts trivially in V_1 and σ_1, it is simply $\mathrm{Hom}(V_1, \sigma_1) \otimes \mathrm{Hom}_{K\cap M'_2}(V_2, \sigma_2)$, so by the assumption it is also $\mathrm{Hom}(V_1, \sigma_1) \otimes \mathrm{Hom}_K(V_2, \sigma_2)$. Now K acts trivially on $\mathrm{Hom}_K(V_2, \sigma_2)$, so taking fixed points under K indeed gives $\mathrm{Hom}_K(V_1, \sigma_1) \otimes \mathrm{Hom}_K(V_2, \sigma_2)$. □

We now prove that (ii) implies (i) in III.18 Proposition 1). Let V be an irreducible representation of K satisfying (ii). From III.12 Proposition (i), V is a weight of $\mathrm{Ind}_P^G \sigma \simeq {}^e\sigma \otimes \mathrm{Ind}_P^G 1$. Therefore, V is a weight of $I(P, \sigma, Q')$ for some parabolic $Q' \supset P$. As we have already proved that (i) implies (ii) in III.18 Proposition 1), we deduce that $\Delta_{Q'} \cap \Delta_2 = \Delta_Q \cap \Delta_2$, so $Q' = Q$. □

III.22. It remains to prove part 2) of III.18 Proposition. We in fact establish something more precise, which gives what we need by III.12 Proposition. Also, by that proposition we may assume $P(\sigma) = G$.

Lemma 1 *Let (ρ, V) be a weight of $I(P, \sigma, Q)$ where $P(\sigma) = G$.*

(i) The quotient map $\mathrm{Ind}_Q^G 1 \to \mathrm{St}_Q^G$ induces an $\mathcal{H}_G(V)$-isomorphism

$$\mathrm{Hom}_K(V, \mathrm{Ind}_Q^G {}^e\sigma) \longrightarrow \mathrm{Hom}_K(V, I(P, \sigma, Q)).$$

(ii) The inclusion $\mathrm{Ind}_Q^G 1 \to \mathrm{Ind}_P^G 1$ induces an $\mathcal{H}_G(V)$-isomorphism

$$\mathrm{Hom}_K(V, \mathrm{Ind}_Q^G {}^e\sigma) \longrightarrow \mathrm{Hom}_K(V, \mathrm{Ind}_P^G \sigma).$$

Proof It is clear that the maps in (i), (ii) are $\mathcal{H}_G(V)$-equivariant. As in III.21 write V as $V_1 \otimes V_2$ where V_2 is the unique weight of St_Q^G (it has parameter $(1, \Delta_Q)$). By III.21 Lemma (the hypothesis is verified by pulling back via ι, as in III.21), we get isomorphisms

$$\begin{aligned} \mathrm{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \mathrm{St}_Q^G) &\simeq \mathrm{Hom}_K(V_1, {}^e\sigma) \otimes \mathrm{Hom}_K(V_2, \mathrm{St}_Q^G), \\ \mathrm{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \mathrm{Ind}_Q^G 1) &\simeq \mathrm{Hom}_K(V_1, {}^e\sigma) \otimes \mathrm{Hom}_K(V_2, \mathrm{Ind}_Q^G 1), \\ \mathrm{Hom}_K(V_1 \otimes V_2, {}^e\sigma \otimes \mathrm{Ind}_P^G 1) &\simeq \mathrm{Hom}_K(V_1, {}^e\sigma) \otimes \mathrm{Hom}_K(V_2, \mathrm{Ind}_P^G 1). \end{aligned}$$

The maps $\mathrm{Ind}_Q^G 1 \to \mathrm{St}_Q^G$ and $\mathrm{Ind}_Q^G 1 \to \mathrm{Ind}_P^G 1$ induce on each side vertical maps which give commutative diagrams. As the vertical maps on the right-hand side are isomorphisms by III.18 Remark 1, so are the vertical maps on the left-hand side, and (i), (ii) are implied by the following well-known lemma. □

Lemma 2 *Let H' be a closed subgroup of a locally profinite group H and $\mathrm{ind}_{H'}^H$ the smooth compact induction functor. Let V be a smooth representation of H' and W a smooth representation of H. Then there is an isomorphism Φ of representations of H, $W \otimes \mathrm{ind}_{H'}^H V \xrightarrow{\sim} \mathrm{ind}_{H'}^H (W \otimes V)$, given by the formula*

$$\Phi(w \otimes f) : h \longmapsto hw \otimes f(h) \qquad \text{for } w \in W,\ f \in \mathrm{ind}_{H'}^H V.$$

F) Irreducibility of $I(P, \sigma, Q)$

III.23. Proposition *Let (P, σ, Q) be a supersingular triple. Then $I(P, \sigma, Q)$ is irreducible.*

Proof It is enough to prove that if V is an irreducible representation of K and $\varphi \in \mathrm{Hom}_K(V, I(P, \sigma, Q))$ is a $\mathcal{Z}_G(V)$-eigenvector with eigenvalue χ, then the subrepresentation X of $I(P, \sigma, Q)$ generated by $\varphi(V)$ is $I(P, \sigma, Q)$. So we fix such a situation and write $(\psi_V, \Delta(V))$ for the parameter of V. We prove the result by induction on the cardinality of $\Delta(V)$.

By III.14 Corollary 1 we have $X = I(P, \sigma, Q)$ if $\Delta(V) \subset \Delta_{P(\sigma)}$, so we assume that this is not the case. We pick α in $\Delta(V)$ but not in $\Delta_{P(\sigma)}$, and let V' be an irreducible representation of K with parameters $(\psi_V, \Delta(V) - \{\alpha\})$. Note that V'_{U^0} and V_{U^0} are isomorphic, so that χ defines a character of $\mathcal{Z}_G(V')$ via the Satake isomorphism, which we also denote by χ.

Via φ, X is a quotient of $\chi \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V$. By III.18 Corollary $\Delta_0(\chi) = \Delta_M$, hence $\alpha \notin \Delta_0(\chi)$. By the change of weight theorem (IV.2 Corollary), $\chi \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V$ and $\chi \otimes_{\mathcal{Z}_G(V')} \mathrm{ind}_K^G V'$ are isomorphic unless α is orthogonal to $\Delta_0(\chi)$, ψ_V is trivial on $Z^0 \cap M'_\alpha$ and $\chi(\tau_\alpha) = 1$ (see III.16 for the notation τ_α). By induction then, we are reduced to the case where α is orthogonal to $\Delta_0(\chi)$, ψ_V is trivial on $Z^0 \cap M'_\alpha$ and $\chi(\tau_\alpha) = 1$. As $\Delta_0(\chi) = \Delta_M$, the conditions imply (III.17 Corollary) that α belongs to $\Delta(\sigma) \subset \Delta_{P(\sigma)}$ contrary to assumption. □

G) Injectivity of the parametrization

III.24. Let (P_1, σ_1, Q_1) and (P_2, σ_2, Q_2) be supersingular triples such that

$$I(P_1, \sigma_1, Q_1) \simeq I(P_2, \sigma_2, Q_2).$$

Let V be a weight of $I(P_1, \sigma_1, Q_1)$, with parameter $(\psi_V, \Delta(V))$, and χ an eigenvalue of $\mathcal{Z}_G(V)$ in $I(P_1, \sigma_1, Q_1)$. We have seen $\Delta_0(\chi) = \Delta_{P_1}$ (III.19 Corollary) so we deduce $\Delta_{P_1} = \Delta_{P_2}$ and $P_1 = P_2$. Write $P_i = M_i N_i$ as usual. By III.18 Proposition, $V_{N_i^0}$ is a weight of σ_i with supersingular eigenvalue χ (via $\mathcal{S}_{M_i}^G$). Then III.17 Corollary implies that $P(\sigma_1) = P(\sigma_2)$. Taking the ordinary part functor [Eme, Vig2] with respect to $P(\sigma_1)$, we deduce that ${}^e\sigma_1 \otimes \mathrm{St}_{Q_1}^{P(\sigma_1)}$ and ${}^e\sigma_2 \otimes \mathrm{St}_{Q_2}^{P(\sigma_2)}$ are isomorphic as representations of $P(\sigma_1) = P(\sigma_2)$. From II.8 Remark, we get $Q_1 = Q_2$ and $\sigma_1 \simeq \sigma_2$. This completes the proof of the uniqueness in I.5 Theorem 4.

We insert here a consequence of the irreducibility of $I(P, \sigma, Q)$ and of the injectivity of the parametrization, which we shall use in part H) and generalize in Chapter VI.

Proposition *Let $P = MN$ be a parabolic subgroup of G containing B, and σ a supersingular representation of M, inflated to P. Then the irreducible components of $\mathrm{Ind}_P^G \sigma$ are the $I(P, \sigma, Q)$, Q a parabolic subgroup of G with $P \subset Q \subset P(\sigma)$; each occurs with multiplicity 1. In particular $\mathrm{Ind}_P^G \sigma$ has finite length.*

Proof The representation $\mathrm{Ind}_P^{P(\sigma)} \sigma$ is isomorphic to ${}^e\sigma \otimes \mathrm{Ind}_P^{P(\sigma)} 1$ (III.22 Lemma 2), which has a filtration with subquotients ${}^e\sigma \otimes \mathrm{St}_Q^{P(\sigma)}$, one for each parabolic subgroup Q with $P \subset Q \subset P(\sigma)$. The proposition then follows from III.23 Proposition by parabolic induction from $P(\sigma)$ to G. □

H) Surjectivity of the parametrization

III.25. Let (π, W) be an irreducible admissible representation of G. To prove that π has the form $I(P, \sigma, Q)$ for a supersingular triple (P, σ, Q), we use induction on the semisimple rank of G.

If $\Delta_0(\chi) = \Delta$ for all weights V of π and corresponding eigenvalues χ, then π is supersingular and $\pi \simeq I(G, \pi, G)$. So we fix a weight V for π with $\mathcal{Z}_G(V)$-eigenvalue χ such that $\Delta_0(\chi) \neq \Delta$. By construction π is a quotient of $\chi \otimes_{\mathcal{Z}_G(V)} \mathrm{ind}_K^G V$.

Let $P = MN$ be the parabolic subgroup such that $\Delta_P = \Delta_0(\chi)$. Consider $\sigma = \chi \otimes \mathrm{ind}_{M^0}^M V_{N^0}$. By the filtration theorem (I.6 Theorem 6, proved in Chapter V), $\chi \otimes \mathrm{ind}_K^G V$ has a filtration with subquotients $I_e(P, \sigma, Q) = \mathrm{Ind}_{P_e}^G({}^e\sigma \otimes \mathrm{St}_Q^{P_e})$ where $P \subset Q \subset P_e$. So π is a quotient of some $I_e(P, \sigma, Q)$. If $P_e \neq G$, then by [HV2, Proposition 7.9] (note that σ has a central character by III.12 Lemma) there is an irreducible admissible representation ρ of the Levi quotient of P_e such that π is a quotient of $\mathrm{Ind}_{P_e}^G \rho$. By the induction hypothesis and III.24 Proposition, ρ is an irreducible constituent of $\operatorname{Ind}_{P_1}^{P_e} \rho_1$ where P_1 is a parabolic subgroup of P_e containing B, and ρ_1 is a supersingular representation of the Levi quotient of P_1. Then π is an irreducible constituent of $\operatorname{Ind}_{P_1}^{G} \rho_1$, so by III.24 Proposition it is isomorphic to $I(P_1, \rho_1, Q')$ for some Q'.

If $P_e = G$, π is a quotient of some ${}^e\sigma \otimes \mathrm{St}_Q^G$. By II.8 Proposition and Remark, π is isomorphic to ${}^e\sigma_\pi \otimes \mathrm{St}_Q^G$ for some irreducible admissible representation σ_π of M. The eigenvalues of σ_π are those of π by III.18 Proposition, and since $\Delta_M = \Delta_0(\chi)$, σ_π has a supersingular eigenvalue. As $\Delta_M \neq \Delta$, the induction hypothesis implies that σ_π is supersingular, cf. III.18 Remark 2, and $\pi \simeq I(P, \sigma_\pi, Q)$. $\square$

III.26. It is worth commenting on the admissibility assumptions in our results. The reader may notice that, since admissibility plays no rôle in Chapters IV and V, our results would still be true if instead of irreducible admissible representations, we considered irreducible representations (σ, W) such that for some weight (ρ, V) of σ, $\operatorname{Hom}_K(V, W)$ contains an eigenvector for $\mathcal{Z}_G(V)$. But the classification thus obtained would depend on the choice of K, $\mathbf{S}$, $\mathbf{B}$, whereas we shall see in Chapter VI that with the admissibility assumption it does not depend on those choices. Of course one may hope that the condition above actually implies admissibility or even, as is the case for complex representations, that any irreducible representation of G is admissible. Note that because of our admissibility condition we do not assert that G has any supersingular representation. When $G = \mathrm{GL}_n(F)$ and F has characteristic 0, we will show in forthcoming work that supersingular representations of G exist.

IV. Change of weight

IV.1. The main goal of this chapter is to establish our change of weight theorem (IV.2 Corollary) used in III.23. Before commenting on the method of proof, let us state precisely what we prove here. We fix an irreducible representation ρ of K on a space V, with parameter $(\psi_V, \Delta(V))$ as defined in III.9. We consider the "universal" representation $\operatorname{ind}_K^G V$, which we see as a sub-representation of $\operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z(V_{U^0}))$ via the injective canonical intertwiner (III.13.1).

We assume that $\Delta(V)$ is non-empty, and we choose $\alpha \in \Delta(V)$ and let (ρ', V') be the irreducible representation of K with parameter $(\psi_V, \Delta(V) - \{\alpha\})$. Similarly we consider the universal representation $\operatorname{ind}_K^G V'$ as a subrepresentation of $\operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z V'_{U^0})$.

To compare the two universal representations, we fix non-zero vectors v in V and v' in V' which are invariant under U_{op}^0. The image of v in V_{U^0} is then a basis of V_{U^0}, and similarly for v'. Using those images as basis vectors, we obtain embeddings of $\operatorname{ind}_K^G V$ and $\operatorname{ind}_K^G V'$ into the same representation $\operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z \psi_V)$. Moreover the Satake isomorphism induces an algebra homomorphism $\mathcal{H}_G(V) \to \mathcal{H}_Z(\psi_V)$; the algebra $\mathcal{H}_Z(\psi_V)$ acts on $\operatorname{ind}_{Z^0}^Z \psi_V$, hence on $\operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z \psi_V)$, and the embedding $\operatorname{ind}_K^G(V) \to \operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z \psi_V)$ is $\mathcal{H}_G(V)$-equivariant. We have similar properties for V'. Note that $\mathcal{H}_G(V)$ and $\mathcal{H}_G(V')$ have the same image in $\mathcal{H}_Z(\psi_V)$, so we identify them with that common image, which we write $\mathcal{H}_G$, and similarly we write $\mathcal{Z}_G$ for their common centre.

For z in Z normalizing ψ_V, we have the function τ_z in $\mathcal{H}_Z(\psi_V)$ with support $Z^0 z$ and value 1_C at z. Recall from III.16 the notation $a_\alpha \in Z \cap M'_\alpha$ and $\tau_\alpha = \tau_{a_\alpha} \in \mathcal{Z}_Z(\psi_V)$, when ψ_V is trivial on $Z^0 \cap M'_\alpha$.

Theorem *Let $z \in Z^+$. Assume that z normalizes ψ_V and that $|\alpha|(z) < 1$. We have:*
(i) $\tau_z(\mathrm{ind}_K^G V) \subset \mathrm{ind}_K^G V'$.
(ii) If ψ_V is not trivial on $Z^0 \cap M'_\alpha$, then $\tau_z(\mathrm{ind}_K^G V') \subset \mathrm{ind}_K^G V$.
(iii) If ψ_V is trivial on $Z^0 \cap M'_\alpha$, then $\tau_z(1-\tau_\alpha)(\mathrm{ind}_K^G V') \subset \mathrm{ind}_K^G V$.

Remark In (iii) $\tau_z(1-\tau_\alpha) = \tau_z - \tau_{za_\alpha}$ belongs to $\mathcal{Z}_G(V)$ if $z \in Z_{\psi_V}$ and za_α belongs to Z^+; moreover, if $|\alpha|(z)$ is small enough, za_α belongs to Z^+.

IV.2. We obtain our change of weight theorem:

Corollary *Let χ be a character of $\mathcal{Z}_G$ and assume that $\alpha \notin \Delta_0(\chi)$. Then $\chi \otimes_{\mathcal{Z}_G} \mathrm{ind}_K^G V$ and $\chi \otimes_{\mathcal{Z}_G} \mathrm{ind}_K^G V'$ are isomorphic unless α is orthogonal to $\Delta_0(\chi)$, ψ_V is trivial on $Z^0 \cap M'_\alpha$ and $\chi(\tau_\alpha) = 1$.*

We remark that $\chi(\tau_\alpha)$ is well defined if α is orthogonal to $\Delta_0(\chi)$ (III.4, III.16 Notation).

Proof Choose z as in the theorem, with $\chi(\tau_z) \neq 0$. For example, we can take for z the element z_α of III.4, since $\alpha \notin \Delta_0(\chi)$: then $\chi(\tau_{z_\alpha}) \neq 0$. Multiplying by τ_z in $\mathrm{Ind}_B^G(\mathrm{ind}_{Z^0}^Z \psi_V)$ is $\mathcal{Z}_G$-linear, so, when ψ_V is not trivial on $Z^0 \cap M'_\alpha$, by (i) and (ii) of the theorem, τ_z induces G-equivariant maps from $\mathrm{ind}_K^G V$ to $\mathrm{ind}_K^G V'$ and back. The composites in both directions are given by the action of τ_z^2. Tensoring with χ, we see that the representations $\chi \otimes_{\mathcal{Z}_G} \mathrm{ind}_K^G V$ and $\chi \otimes_{\mathcal{Z}_G} \mathrm{ind}_K^G V'$ are isomorphic, because $\chi(\tau_z^2) \neq 0$. That gives the desired result when ψ_V is non-trivial on $Z^0 \cap M'_\alpha$.

Assume then that ψ_V is trivial on $Z^0 \cap M'_\alpha$. Replacing z by a positive power, we may assume $za_\alpha \in Z^+$. If α is not orthogonal to $\Delta_0(\chi)$ then there is β in $\Delta_0(\chi)$ with $|\beta|(za_\alpha) < 1$ and then $\chi(\tau_{za_\alpha}) = 0$, so the same reasoning applies, using (iii) instead of (ii). It similarly applies if α is orthogonal to $\Delta_0(\chi)$ and $\chi(\tau_\alpha) \neq 1$. $\square$

IV.3. Let us now comment on the proof of IV.1 Theorem. We abbreviate $\psi = \psi_V$, $J = \Delta(V)$, $J' = J - \{\alpha\}$, and $\mathcal{X} = \mathrm{Ind}_B^G(\mathrm{ind}_{Z^0}^Z \psi)$. Let I be the pro-p Iwahori subgroup of G which is the inverse image in K of U_k^{op}.[13]

We first remark that $\mathrm{ind}_K^G V$ is generated, as a representation of G, by a single element, the function with support K and value v at 1_G. We write f for its image in $\mathcal{X}$; it is described explicitly in IV.4 below. Similarly we have a function f' in $\mathcal{X}$, corresponding to v', which generates the subrepresentation $\mathrm{ind}_K^G V'$. We use work of the fourth-named author [Vig3] which determines the structure of the Hecke algebra $\mathcal{H} = \mathcal{H}(G, I)$, the intertwining algebra in G of the trivial character of I. The space $\mathcal{X}^I$ is a right module over $\mathcal{H}$, and for $x \in \mathcal{X}^I$ and T in $\mathcal{H}$, xT belongs to the G-subspace generated by x. By construction, f and f' belong to $\mathcal{X}^I$ and to prove the theorem we show that: for (i) $\tau_z f \in f'\mathcal{H}$; for (ii) $\tau_z f' \in \mathcal{Z}_G f + f\mathcal{H}$; for (iii) $\tau_z(1-\tau_\alpha) f' \in \mathcal{Z}_G f + f\mathcal{H}$. That is not an easy matter and takes up the rest of this chapter.

IV.4. Let us first identify the element $f \in \mathcal{X}^I$; the obvious analogue will hold for f'.

As $G = BK$ it is enough to specify f at $g \in K$. Going through the construction of the embedding $\mathrm{ind}_K^G V \to \mathrm{Ind}_B^G(\mathrm{ind}_{Z^0}^Z \psi)$ we get that for g in K, $f(g)$ is the function in $\mathrm{ind}_{Z^0}^Z \psi$ with support Z^0 and value $\varepsilon(g)$ at 1, where $\overline{gv} = \varepsilon(g)\overline{v}$ in V_{U^0}, bars indicating the images under $V \to V_{U^0}$.

[13] Beware of the notation: here, for convenience, we write I for a pro-p "lower" Iwahori subgroup.

The value $\varepsilon(g)$ depends only on the image $\overline{g}$ of g in $K/K(1)$, we write accordingly $\varepsilon(\overline{g})$. By [HV2, Corollary 3.19] we have $\varepsilon(\overline{g}) \neq 0$ if and only if $\overline{g}$ belongs to $B_k P_{J,k} B_k^{\mathrm{op}}$ (recall from III.9 Definition that $P_{J,k}$ is the stabilizer in G_k of the kernel of the quotient map $V \to V_{U^0}$); that last set is also $P_{J,k} U_k^{\mathrm{op}}$. We can be more precise; we obviously have $\varepsilon(\overline{g}x) = \varepsilon(\overline{g})$ for $x \in U_k^{\mathrm{op}}$, so it is enough to describe $\varepsilon_{|P_{J,k}}$. Since $P_{J,k}$ is the stabilizer in G_k of the kernel of $V \to V_{U^0}$, the restriction $\varepsilon_{|P_{J,k}}$ is a character $P_{J,k} \to C^\times$; as such it is trivial on unipotent elements. On Z_k it is given by the action of Z_k on $V^{U_k^{\mathrm{op}}}$ or V_{U_k}, so it is equal to ψ there. In other words, on $P_{J,k}$ the character ε is simply the (unique) extension of ψ to $P_{J,k}$.

IV.5. To relate f and f' we shall express both of them in terms of Hecke operators in the subalgebra $\mathcal{H}(K,I)$ of $\mathcal{H}(G,I)$ acting on a single function f_0 in $\mathcal{X}^I$.

We first describe the double coset spaces $I\backslash G/I$ and $B\backslash G/I$. Recall that the Weyl group W_0 of G can be seen as $\mathcal{N}^0/Z^0$ or $\mathcal{N}_k/Z_k$. As $G = BK$ the inclusion of K in G induces a bijection $B^0\backslash K/I \simeq B\backslash G/I$; as moreover I contains the normal subgroup $K(1)$ of K, reduction mod $K(1)$ induces a bijection $B^0\backslash K/I \simeq B_k\backslash G_k/U_k^{\mathrm{op}}$ and the Bruhat decomposition in G_k gives a bijection $\mathcal{N}_k/Z_k \simeq B_k\backslash G_k/U_k^{\mathrm{op}}$. All in all, we see that the map $\mathcal{N}^0 \to B\backslash G/I$ $g \mapsto BgI$ induces a bijection $W_0 = \mathcal{N}^0/Z^0 \simeq B\backslash G/I$.

On the other hand, the map $\mathcal{N} \to I\backslash G/I$ induces a bijection $\mathcal{N}/(Z \cap K(1)) \simeq I\backslash G/I$ and, by restriction, a bijection $\mathcal{N}^0/(Z \cap K(1)) \simeq I\backslash K/I$. Under reduction modulo $K(1)$ we get the bijection $\mathcal{N}_k \simeq U_k^{\mathrm{op}}\backslash G_k/U_k^{\mathrm{op}}$ given by the Bruhat decomposition.

Notation Recall that $Z(1) = Z \cap K(1)$ is the unique pro-p Sylow subgroup of Z^0 and that it is normal in Z. We write ${}_1W$ for the group $\mathcal{N}/Z(1)$ and ${}_1W_0$ for the group $\mathcal{N}^0/Z(1)$ (naturally isomorphic to $\mathcal{N}_k$), W for the group $\mathcal{N}/Z^0$. We have obvious exact sequences of groups

$$1 \longrightarrow Z_k \longrightarrow {}_1W_0 \longrightarrow W_0 \longrightarrow 1,$$
$$1 \longrightarrow Z_k \longrightarrow {}_1W \longrightarrow W \longrightarrow 1.$$

Moreover W is the semi-direct product of $\Lambda = Z/Z^0$ with W_0 viewed as $\mathcal{N}^0/Z^0$. We also put ${}_1\Lambda = Z/Z(1)$ and ${}_1\Lambda^+ = Z^+/Z(1)$.

For g in G we write $T(g)$ for the double coset IgI viewed as an element of $\mathcal{H}(G,I)$. On an element φ in $\mathcal{X}^I$ it acts via

$$(\varphi T(g))(h) = \sum_{x \in I/(I \cap g^{-1}Ig)} \varphi(hxg^{-1}) \text{ for } h \in G. \tag{IV.5.1}$$

When $g \in \mathcal{N}$, $T(g)$ depends only on the class w of g modulo $Z(1)$, and we write $T(w)$ for $T(g)$. In a similar manner, reduction modulo $K(1)$ gives an isomorphism of $\mathcal{H}(K,I)$ onto $\mathcal{H}(G_k, U_k^{\mathrm{op}})$; accordingly for $g \in K$, $T(g)$ depends only on the reduction $\overline{g}$ of g in G_k and we write also $T(\overline{g})$. In fact we shall also have use of the Hecke algebras with integer coefficients $\mathcal{H}_{\mathbb{Z}}(G,I)$ and $\mathcal{H}_{\mathbb{Z}}(K,I)$ (isomorphic to $\mathcal{H}_{\mathbb{Z}}(G_k, U_k^{\mathrm{op}})$) and we use the same notations $T(g), T(w), T(\overline{g})$.

IV.6. Basic generators and relations for $\mathcal{H}_{\mathbb{Z}}(G,I)$ and $\mathcal{H}_{\mathbb{Z}}(K,I)$ are given in [Vig3]. By tensoring with C they give generators and relations for $\mathcal{H}(G,I)$ and $\mathcal{H}(K,I)$. We now state the results we use, referring to [Vig3] for details. We need a bit more notation, though.

For $\beta \in \Delta$, we let s_β be the corresponding reflection in W_0. We put $\Sigma_0 = \{s_\beta \mid \beta \in \Delta\}$. The pro-$p$ Iwahori subgroup I is attached to an alcove $\mathfrak{a}$ in the (semisimple) Bruhat-Tits building of G, with vertex the special point $\mathbf{x}_0$, and we let Σ be the set of reflections across the walls of $\mathfrak{a}$, so that Σ_0 appears as the subset of reflections across walls passing through $\mathbf{x}_0$. Then Σ generates an affine Weyl group W^a canonically identified with the subgroup $(\mathcal{N} \cap \operatorname{Ker} w_G)/Z^0$ of W; also W is the semi-direct product of its normal subgroup W^a and the subgroup Ω stabilizing the alcove $\mathfrak{a}$. We let ℓ be the length function of the Coxeter system (W^a, Σ) and we extend it to W, trivially on Ω, i.e. so that $\ell(w\omega) = \ell(w)$ for $w \in W^a$, $\omega \in \Omega$; on W_0 it restricts to the length function of the Coxeter system (W_0, Σ_0). Inflating through ${}_1W \to W$ we get a length function on ${}_1W$ and ${}_1W_0$, still written ℓ. The operators $T(w)$ in $\mathcal{H}_{\mathbb{Z}}(G, I)$ for $w \in {}_1W$ satisfy the "braid relations"

$$T(w)T(w') = T(ww') \text{ when } \ell(ww') = \ell(w) + \ell(w'). \qquad \text{(IV.6.1)}$$

There are other relations, the "quadratic relations" [Vig3, Proposition 4.3]. Essentially there is one such relation for each $s \in \Sigma$. It comes directly from the finite field case, treated in [CE, 6.8]. For $s \in \Sigma_0$, $s = s_\beta$ for some $\beta \in \Delta$, we may describe the relation as follows: let n_s be a lift of s_β in $\mathcal{N}_k \cap M'_{\beta,k}$ and define $Z_{k,s} = Z_k \cap M'_{\beta,k}$ (so that n_s^2 belongs to $Z_{k,s}$); then the quadratic relation for $T(n_s)$ is

$$T(n_s)(T(n_s) - c_{n_s}) = q_s T(n_s^2), \qquad \text{(IV.6.2)}$$

where $q_s > 1$ is a power of p and

$$c_{n_s} = \sum_{t \in Z_{k,s}} c_{n_s}(t) T(t)$$

for positive integers $c_{n_s}(t) = c_{n_s}(-t)$, constant on each coset of $\{xs(x)^{-1} \mid x \in Z_k\}$, of sum $q_s - 1$. Moreover, we have $c_{n_s} \equiv c_s \mod p$, where

$$c_s := (q_s - 1)|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} T(t). \qquad \text{(IV.6.3)}$$

We have $T(n_s)c_{n_s} = c_{n_s} T(n_s)$.

Remark In the C-algebra $\mathcal{H}(G, I)$, q_s equals 0 and c_{n_s} equals $-|Z_{k,s}|^{-1} \sum_{t \in Z_{k,s}} T(t)$, so the relations simplify somewhat. We always embed the group algebra of Z_k over C into $\mathcal{H}(G, I)$ by sending t to $T(t)$; for $s = s_\beta$ as above we have $\psi(c_{n_s}) = -1$ if ψ is trivial on $Z_{k,s}$ (i.e. β belongs to the set $\Delta(\psi)$ of III.8, which contains J), and $\psi(c_{n_s}) = 0$ otherwise.

Proposition *There is a unique extension of the map $s \mapsto n_s$ from Σ_0 to $\mathcal{N}_k$ to a map $w \mapsto n_w$ from W_0 to $\mathcal{N}_k$ such that $n_{ww'} = n_w n_{w'}$ for w, w' in W_0 such that $\ell(ww') = \ell(w) + \ell(w')$.*

Proof (Another proof is in [Vig3, Proposition 3.4].) Uniqueness is obvious, as we must have $n_w = n_{s_1} \cdots n_{s_r}$ for each reduced decomposition $w = s_s \cdots s_r$ of w in W_0 with the s_i in Σ_0. Existence will be consequence of [Bk, §1, n^o 5, Proposition 5] once we prove:

$(*)$ For s, s' distinct in Σ_0, and m the order of ss', then $(n_s n_{s'})^\ell = (n_{s'} n_s)^\ell$ if $m = 2\ell$ and $(n_s n_{s'})^\ell n_s = (n_{s'} n_s)^\ell n_{s'}$ if $m = 2\ell + 1$.

To prove $(*)$ we may assume that $\mathbf{G}_k$ is semisimple simply connected of relative rank 2, with W_0 generated by s and s', corresponding to the two simple roots β and β'. But then the result follows from [BT1, 6.1.8] applied to the valued root datum associated to $(\mathbf{G}_k, \mathbf{S}_k, \mathbf{B}_k)$: indeed, we can always put reduced roots of Φ in a "circular order" as required by [BT1, 6.1.8], with β first and β' in the m-th position, in which case formula (9) of [BT1, 6.1.8] gives exactly the required equality $(*)$ above. □

Henceforward we use the extension $w \mapsto n_w$, and we put $\nu_w = n_{w^{-1}}^{-1}$ for $w \in W_0$; in particular if w, w' in W_0 satisfy $\ell(ww') = \ell(w) + \ell(w')$, then $\nu_{ww'} = \nu_w \nu_{w'}$.

IV.7. We are now ready to define f_0 (as promised in IV.5) and study the action of $\mathcal{H}(K, I)$ on it. We let w_0 be the longest element in W_0.

Definition *The function f_0 in $\mathcal{X}^I$ has support $B\nu_{w_0}I$ and its value at ν_{w_0} is the function e_ψ in $\operatorname{ind}_{Z^0}^{Z} \psi$ with support Z^0 and equal to ψ on Z^0.*

Note the abuse of notation: we should choose a representative $\tilde{\nu}_{w_0}$ of ν_{w_0} in $\mathcal{N}^0$ but neither the coset $B\tilde{\nu}_{w_0}I$ nor the value at $\tilde{\nu}_{w_0}$ depend on that choice. We shall allow similar abuse of notation below. Note also that f_0 depends on the choice of ν_{w_0} (but the support of f_0 is independent of this choice).

Notation For $z \in Z_k$ and $w \in W_0$ we put $w \cdot z = n_w z n_w^{-1}$ (it is simply the natural action of $w \in W_0 = \mathcal{N}_k/Z_k$ on Z_k); more generally we shall use a dot to denote a conjugation action, which will be clear from the context.

Lemma *For $z \in Z_k$ we have $z^{-1} f_0 = \psi(w_0 \cdot z^{-1}) f_0 = f_0 T(z) = \tau_{w_0 \cdot z} f_0$.*

The last equality in the lemma will be generalized below (IV.10).

Proof Since Z^0 normalizes I, the first equality in the lemma comes from an immediate computation, whereas the equality $f_0 T(z) = z^{-1} f_0$ comes from (IV.5.1). The equality $\tau_z f_0 = \psi(z^{-1}) f_0$ is equally easy. □

Proposition *Let $w \in W_0$. Then $f_0 T(n_w)$ has support $B\nu_{w_0 w} I$ and value e_ψ at $\nu_{w_0 w}$.*

Proof As $f_0 T(n_w)$ is I-invariant, it is enough to compute its value at $\nu_{w'}$ for w' in W_0. By definition $(f_0 T(n_w))(g) = \sum f_0(g h n_w^{-1})$ for $g \in G$, where the sum runs over h in $I/(n_w^{-1} I n_w \cap I)$. Assume that for such an h, f_0 is not 0 at $\nu_{w'} h n_w^{-1}$. Then looking modulo $K(1)$, we get that $\nu_{w'} U_k^{\mathrm{op}} n_w^{-1} \cap B_k \nu_{w_0} U_k^{\mathrm{op}}$ is non-empty, and, multiplying on the right by $\nu_{w_0}^{-1}$, that $\nu_{w'} U_k^{\mathrm{op}} n_w^{-1} \nu_{w_0}^{-1} \cap B_k \neq \emptyset$ and hence $B_k \nu_{w'} U_k^{\mathrm{op}} \cap B_k \nu_{w_0} n_w U_k^{\mathrm{op}} \neq \emptyset$; by the Bruhat decomposition in G_k, that implies $w' = w_0 w$. Assume that $w' = w_0 w$; then h belongs to $\nu_{w'}^{-1} B^0 \nu_{w_0} I n_w$. However note that $\ell(w_0 w) + \ell(w^{-1}) = \ell(w_0)$ (because w_0 is the longest element in W_0), so that $\nu_{w'} \nu_{w^{-1}} = \nu_{w_0}$; we deduce that the image of h in G_k belongs to $n_w^{-1} B_k^{\mathrm{op}} n_w \cap U_k^{\mathrm{op}} = n_w^{-1} U_k^{\mathrm{op}} n_w \cap U_k^{\mathrm{op}}$. But that shows that h belongs to $n_w^{-1} I n_w \cap I$ and consequently $(f_0 T(n_w))(\nu_{w_0 w}) = f_0(\nu_{w_0 w} n_w^{-1}) = f_0(\nu_{w_0}) = e_\psi$. □

Corollary $f = \sum_{w \in w_0 W_J} f_0 T(n_w)$.

Proof By the description in IV.4, for w in W_0, $f(\nu_w)$ is equal to e_ψ if w belongs to W_J and is 0 otherwise: we only have to remark that $P_{J,k} U_k^{\mathrm{op}} = B_k W_J U_k^{\mathrm{op}}$, and since $\psi(Z_k \cap M'_{\beta,k}) = 1$ for $\beta \in J$, the character ε of IV.4 is trivial on ν_w for $w \in W_J$. □

IV.8. We need to determine the action of c_{n_s} on $f_0T(n_w)$ for $s = s_\beta$, $\beta \in J$. We recall that $J \subset \Delta(\psi)$.

Proposition *Let $\beta \in \Delta(\psi)$, $s = s_\beta$ and $z \in Z_k \cap M'_{\Delta(\psi),k}$. For $w \in w_0 W_{\Delta(\psi)}$, we have*

$$f_0\, T(n_w)T(z) = f_0\, T(n_w) \text{ and}$$
$$f_0\, T(n_w)c_{n_s} = -f_0\, T(n_w).$$

In particular $f c_{n_s} = -f$.

Proof By III.10 Example 2, ψ is trivial on $Z_k \cap M'_{\Delta(\psi),k}$. By IV.7 Lemma then, we get $f_0T(z) = f_0$ for $z \in Z_k \cap w_0 M'_{\Delta(\psi),k} w_0^{-1}$. The braid relation gives $T(n_w)T(t) = T(w \cdot t)T(n_w)$ for $t \in Z_k$, $w \in W_0$. For $z \in Z_k \cap M'_{\Delta(\psi),k}$ we have $w \cdot z \in Z_k \cap M'_{\Delta(\psi),k}$ for $w \in W_{\Delta(\psi)}$, hence $(w_0 w) \cdot z \in Z_k \cap w_0 M'_{\Delta(\psi),k} w_0^{-1}$, and consequently $f_0T(n_{w_0w})T(z) = f_0T(n_{w_0w})$. That gives the first assertion.

The second one comes from the expression of c_{n_s} in (IV.6.3), noting that q_s gives 0 in C; the last assertion follows from IV.7 Corollary. □

IV.9. Notation Let w_J be the longest element in $W_J \subset W_0$ and put $w^J = w_0 w_J$ (note that w_J and w_0 have order 2). We put $f_J = f_0T(n_{w^J})$.

Lemma 1 *For $w \in W_J$ we have* (i) $\ell(w^J w) = \ell(w^J) + \ell(w)$, (ii) $T(n_{w^J w}) = T(n_{w^J})T(n_w)$, and (iii) $f_0T(n_{w^J w}) = f_J T(n_w)$.

Proof We have $\ell(w^J w) = \ell(w_0 w_J w) = \ell(w_0) - \ell(w_J w)$; if $w \in W_J$ we also have $\ell(w_J w) = \ell(w_J) - \ell(w)$ so we get $\ell(w^J w) = \ell(w^J) + \ell(w)$; by the braid relation $T(n_{w^J w}) = T(n_{w^J})T(n_w)$, and the last assertion follows. □

By Lemma 1, and IV.7 Corollary, IV.8 Proposition, we have $f = \sum_{w \in W_J} f_J T(n_w)$ and for $w \in W_J$

$$f_J T(n_w) c_{n_s} = -f_J T(n_w). \tag{IV.9.1}$$

For $s \in \Sigma_0$ we put $T^*(n_s) = T(n_s) - c_{n_s}$, so that in $\mathcal{H}_{\mathbb{Z}}(K, I)$ we get

$$T(n_s)T^*(n_s) = T^*(n_s)T(n_s) = q_s T(n_s^2) \quad (= 0 \text{ in } \mathcal{H}(K, I)).$$

That definition can be extended to defining $T^*(n_w)$ for $w \in W_0$, so that $T^*(n_{ww'}) = T^*(n_w)T^*(n_{w'})$ if $\ell(ww') = \ell(w) + \ell(w')$ [Vig3, Proposition 4.13]. We now use the Bruhat order $\leq$ on the Coxeter group W_J (see for example [Deo]).

Proposition *For $w \in W_J$ we have*

$$f_J\big(\sum_{v \leq w} T(n_v)\big) = f_J T^*(n_w) \text{ and in particular}$$
$$f = f_J T^*(n_{w_J}) = f_0 T(n_{w^J}) T^*(n_{w_J}).$$

A similar proposition can be found in [Oll2, Lemma 5.1].

Proof We use induction on $\ell(w)$. The result is true for $w = 1$. If $\ell(w) = \ell \geq 1$, we write $w = w's$ with $\ell(w') = \ell - 1, \ell(s) = 1$. As $\ell(w) = \ell(w') + \ell(s)$ we have $T^*(n_w) =$

$T^*(n_{w'})T^*(n_s) = T^*(n_{w'})(T(n_s) - c_{n_s})$. By induction $f_J T^*(n_{w'}) = \sum_{v \leq w'} f_J T(n_v)$. Remembering that for v in W_J we have $T(n_{w^J})T(n_v) = T(n_{w^J v})$ and by (IV.9.1) $f_J T(n_v) c_{n_s} = -f_J T(n_v)$. So finally we obtain

$$f_J T^*(n_w) = f_J T^*(n_{w'})(T(n_s) + 1).$$

By induction $f_J T^*(n_{w'}) = \sum_{v \leq w'} f_J T(n_v)$, so we want to compute $A = \sum_{v \leq w'} f_J T(n_v) T(n_s)$.

Divide the set of $v \leq w'$ in the disjoint union $X \sqcup Y \sqcup Ys$ where

$$\begin{aligned} Y &= \{v \in W_J,\ v < vs \leq w'\}, \\ Ys &= \{v \in W_J,\ vs < v \leq w'\}, \\ X &= \{v \in W_J,\ v \leq w' \text{ and } vs \not\leq w'\}. \end{aligned}$$

In A, the subsum over $Y \sqcup Ys$ is

$$\sum_{v \in Y} f_J(T(n_{vs}) + T(n_v))T(n_s).$$

But for $v \in Y$, we have $v < vs$ so $T(n_{vs}) = T(n_v)T(n_s)$ and $f_J(T(n_{vs}) + T(n_v))T(n_s) = f_J T(n_v)(T(n_s)+1)T(n_s)$. By (IV.9.1) that equals $f_J T(n_v) T^*(n_s) T(n_s)$ which is 0 because $T^*(n_s)T(n_s) = 0$ in $\mathcal{H}$. So $A = \sum_{v \in X} f_J T(n_v) T(n_s)$. Since for $v \in X$, we have $v < vs$ we get $A = \sum_{v \in X} f_J T(n_{vs})$.

The proof will be complete once we get:

Lemma 2 $Xs = \{v \in W_J,\ v \leq w \text{ and } v \not\leq w'\}$.

Proof We use properties of the Bruhat order [Deo, Theorem 1.1 (II) (ii)]. Let a, b in W_J with $a \leq b$. Then:

(1) If $a < as$ then $a \leq bs$; (2) if $b > bs$ then $as \leq b$.

Let $v \in X$, i.e. $v \leq w'$, $vs \not\leq w'$. Then by (2) applied to $a = v, b = w$, we get $vs \leq w$. Conversely let $v \in W_J$ verify $v \leq w$ and $v \not\leq w'$; if $v < vs$ then $v \leq w'$ by (1) applied to $a = v$, $b = w$, which is a contradiction; so $vs < v \leq w$, which gives $vs \leq w'$ by (1) applied to $a = vs$ and $b = w$. That proves the lemma. □

IV.10. We now turn to the promised generalization of IV.7 Lemma which will be used in IV.15.

Proposition *Let $z \in Z$ with $z^{-1} \in Z^+$. Assume that $\nu_{w_0} \cdot z$ normalizes ψ. Then $f_0 T(z) = \tau_{\nu_{w_0} \cdot z} f_0$.*

Remark If z^{-1} belongs to Z^+, $\nu_{w_0} \cdot z$ also belongs to Z^+, and conversely.

Proof As both terms are I-invariant, we only need to check that they are equal at ν_w for $w \in W_0$. Now $(f_0 T(z))(g) = \sum f_0(ghz^{-1})$ for $g \in G$, where the sum runs over $h \in I/(z^{-1}Iz \cap I)$. But I has an Iwahori decomposition and the assumption that z^{-1} belongs to Z^+ gives $z^{-1}(I \cap U)z \subset I \cap U$, $z^{-1}(I \cap U_{\mathrm{op}})z \supset I \cap U_{\mathrm{op}}$, thus the inclusion of $I \cap U$ into I induces of bijection of $(I \cap U)/(z^{-1}Iz \cap U)$ onto $I/(z^{-1}Iz \cap I)$, and it is enough to let h run through $(I \cap U)/(z^{-1}Iz \cap U)$. For such an h, $\nu_w h z^{-1}$ belongs to $B\nu_{w_0} I$ only if $w = w_0$: indeed, $\nu_w h z^{-1} \in B n_w U$ and $B \nu_{w_0} I \subset B n_{w_0} U$, so the Bruhat decomposition in G implies $w = w_0$. Consequently, both terms of the desired equality vanish at ν_w for $w \neq w_0$.

Consider now $(f_0T(z))(\nu_{w_0})$. Let $h \in I \cap U$ with $\nu_{w_0}hz^{-1} = b\nu_{w_0}j$ for some b in B, j in I; again by the Iwahori decomposition of I, we may assume that j belongs to $I \cap U$ and then the equality $h = (\nu_{w_0}^{-1}b\nu_{w_0})z(z^{-1}jz)$, where $\nu_{w_0}^{-1}b\nu_{w_0}z \in B^{\mathrm{op}}$ and $z^{-1}jz \in U$, shows that h is equal to $z^{-1}jz$ and belongs to $z^{-1}Iz \cap U$; consequently, $(f_0T(z))(\nu_{w_0}) = f_0(\nu_{w_0}z^{-1}) = f_0((\nu_{w_0} \cdot z^{-1})\nu_{w_0}) = (\nu_{w_0} \cdot z^{-1})f_0(\nu_{w_0})$. That is equal to $(\nu_{w_0} \cdot z^{-1})e_\psi$, which sends z' to $e_\psi(z'(\nu_{w_0} \cdot z^{-1}))$. On the other hand if $\nu_{w_0} \cdot z$ normalizes ψ, we have $(\tau_{\nu_{w_0}\cdot z}f)(\nu_{w_0}) = \tau_{\nu_{w_0}\cdot z}e_\psi$, sending z' to $e_\psi((\nu_{w_0} \cdot z^{-1})z')$. That gives the result since $\nu_{w_0} \cdot z$ normalizes ψ. $\square$

IV.11. To go further, we need more notation. We have the vector space $V_{\mathrm{ad}} = X_*(\mathbf{S}_{\mathrm{ad}}) \otimes \mathbb{R}$, where $\mathbf{S}_{\mathrm{ad}}$ is the torus image of $\mathbf{S}$ in the adjoint group $\mathbf{G}_{\mathrm{ad}}$ of $\mathbf{G}$, the dominant Weyl chamber $\mathcal{D}^+ = \{v \in V_{\mathrm{ad}},\ \beta(v) > 0 \text{ for } \beta \in \Delta\}$, and the antidominant Weyl chamber $\mathcal{D}^- = -\mathcal{D}^+ = w_0\mathcal{D}^+$. We recall the natural map $\nu : Z \to V_{\mathrm{ad}}$ used in [Vig3, 3.3]: the action of $z \in Z$ on V_{ad} is via translation by $\nu(z)$. We remark that ν is the composite of $-v_Z : Z \to X_*(\mathbf{S}) \otimes \mathbb{R}$ with $X_*(\mathbf{S}) \otimes \mathbb{R} \to V_{\mathrm{ad}}$. By [Vig3], Z^+ is the set of $z \in Z$ such that $\nu(z)$ belongs to the closure of $\mathcal{D}^-$ (i.e. $\beta \circ \nu(z) \leq 0$ for $\beta \in \Delta$). The map ν factors through ${}_1\Lambda$ and Λ, and we still write ν for the corresponding maps.

Note however that in citing [Vig3, Ch. 5], some care is needed:

Firstly, the roots in [Vig3, Ch. 5] are in the reduced root system Φ_a on V_{ad} attached to the collection of affine root hyperplanes in V_{ad} (it is denoted by Σ in [Vig3, Ch. 5]). It is **not** in general the root system Φ attached to $(\mathbf{G}_{\mathrm{ad}}, \mathbf{S}_{\mathrm{ad}})$. Let us describe what is happening. The space $V_{\mathrm{ad}} = X_*(\mathbf{S}_{\mathrm{ad}}) \otimes \mathbb{R}$ is naturally a quotient of $X_*(\mathbf{S}) \otimes \mathbb{R}$, and its dual $X^*(\mathbf{S}_{\mathrm{ad}}) \otimes \mathbb{R}$ appears as the subspace of $X^*(\mathbf{S}) \otimes \mathbb{R}$ generated by the roots in Φ, which are then the same for $(\mathbf{G}, \mathbf{S})$ and $(\mathbf{G}_{\mathrm{ad}}, \mathbf{S}_{\mathrm{ad}})$. The coroot in V_{ad} attached to a given root β in Φ is the image of $\beta^\vee \in X_*(\mathbf{S})$, we also write it $\beta^\vee$. The root system Φ_a on V_{ad} can be described from Φ as follows. For each $\beta \in \Phi$, there is a positive integer e_β such that Φ_a is the set of $\beta_a := e_\beta\beta$ for $\beta \in \Phi$; in particular, $e_{2\beta} = e_\beta/2$ if $2\beta \in \Phi$. The root systems Φ_a and Φ share the same Weyl group W_0, and consequently the same Weyl chambers. The choice of Weyl chamber defining Φ^+ also defines Φ_a^+ and $\beta \mapsto \beta_a$ gives a bijection of Δ onto the set Δ_a of simple roots in Φ_a. Note also that $(\beta_a \circ \nu)(\Lambda) \subset \mathbb{Z}$ and that the coroot in V_{ad} associated to $\beta_a \in \Phi_a$ is $\beta_a^\vee = e_\beta^{-1}\beta^\vee$.

Examples 1) If $\mathbf{G}$ is split, then $\Phi_a = \Phi$, $e_\beta = 1$ for $\beta \in \Phi$.

2) For $G = \mathrm{GL}_r(D)$, where D is a central division algebra over F, of finite degree d^2, then $e_\beta = d$ for all $\beta \in \Phi$.

3) Assume that $\mathbf{G}$ is semisimple simply connected of relative rank 1. Then there is only one positive root β and $\beta_a \circ \nu(\Lambda) = 2\mathbb{Z}$ [Vig3, 5.14]. Going back to the situation of III.16 with no condition on the reductive group $\mathbf{G}$ we deduce that $\nu(a_\beta) = \beta_a^\vee$, since $\langle \beta_a, \beta_a^\vee \rangle = 2$. In particular $v_Z(a_\beta) = -e_\beta^{-1}\beta^\vee$.

Secondly, the choice of Iwahori subgroup corresponds to a choice of alcove with vertex $\mathbf{x}_0$, and positivity conditions are with respect to that choice. As we work with the "lower" pro-p Iwahori subgroup I, the alcove with vertex $\mathbf{x}_0$ which corresponds to I is the one contained in $\mathcal{D}^-$, so positive roots in [Vig3, Ch. 5] correspond to negative roots here. In citing [Vig3, Ch. 5] therefore we either have to exchange positive and negative roots or replace ν with $-\nu$; we choose the first solution. For example $\Sigma^+, \mathcal{D}^+$ in [Vig3, Ch. 5] correspond to $\Phi_a^-, \mathcal{D}^-$ here.

IV.12. Other bases of $\mathcal{H}_{\mathbb{Z}}(G,I)$ are constructed in [Vig3, Ch. 5] using (spherical) orientations. They generalize the Bernstein-Lusztig basis of an affine Hecke algebra. We need not know what such an object is, only that it is determined by a Weyl chamber in $V_{\rm ad}$; the action of W_0 on Weyl chambers determines an action on orientations; but as in [Vig3, Ch. 5], we let W_0 (and hence ${}_1W$ via ${}_1W \to W_0$) act **on the right** on orientations by $(o,w) \mapsto o \cdot w$, so that if an orientation o corresponds to the Weyl chamber $\mathcal{D}$, then $o\cdot w$ corresponds to $w^{-1}(\mathcal{D})$.

Let o be an orientation. By [Vig3, Corollary 5.26] it gives a basis $(E_o(w))_{w\in {}_1W}$ for $\mathcal{H}_{\mathbb{Z}}(G,I)$. In $\mathcal{H}_{\mathbb{Z}}(G,I)$ some computations are easier because it is a "characteristic zero" algebra. The above basis of $\mathcal{H}_{\mathbb{Z}}(G,I)$ specializes to a basis $(E_o(w))_{w\in {}_1W}$ of $\mathcal{H}$ over C: we use the same notation, making the context precise when necessary.

To $w \in {}_1W$ is attached an element q_w in $\mathbb{Z}$, such that $q_{n_s} = q_s$ for $s \in \Sigma_0$ and $q_w = 1$ if $\ell(w) = 0$. The main relations in $\mathcal{H}_{\mathbb{Z}}(G,I)$ satisfied by the $E_o(w)$ are the following relations: for w, w' in ${}_1W$,

$$\text{(IV.12.1)} \qquad E_o(w)E_{o\cdot w}(w') = q_{w,w'}E_o(ww') \text{ with } q_{w,w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2}.$$

Beware that in general $o\cdot w \neq o$, although it is the case when $w \in {}_1\Lambda$. Note that $q_{w,w'} = 1$ if and only if $\ell(ww') = \ell(w) + \ell(w')$, and $q_{w,w'}$ gives 0 in C otherwise [Vig3, Remark 4.18 and Lemma 4.19].

In particular, if $\mathcal{A}_o$ is the subspace of $\mathcal{H}$ with basis $(E_o(\lambda))$ for $\lambda \in {}_1\Lambda$, the multiplication in $\mathcal{A}_o$ is straightforward:

$$\text{(IV.12.2)} \qquad E_o(\lambda)E_o(\lambda') = \begin{cases} E_o(\lambda\lambda') & \text{if } \ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda'), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\mathcal{A}_o$ is a subalgebra of $\mathcal{H}$. In fact the condition $\ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda')$ means that $\nu(\lambda)$ and $\nu(\lambda')$ belong to the same closed Weyl chamber in $V_{\rm ad}$ [Vig3, 5.12].

If o is an orientation, we let Λ_o be the set of $\lambda \in \Lambda$ such that $\nu(\lambda)$ belongs to the closure of the corresponding Weyl chamber; we similarly define ${}_1\Lambda_o$. For λ in ${}_1\Lambda_o$, we have $E_o(\lambda) = T(\lambda)$ [Vig3, Example 5.30].

We shall need the orientation o_I attached to a subset I of Δ: by definition it is the orientation corresponding to the Weyl chamber $w_I(\mathcal{D}^-)$. Hence o_Δ corresponds to $\mathcal{D}^+$, $o_\emptyset$ corresponds to $\mathcal{D}^-$, $o_I = o_\Delta \cdot w^I$, $\Lambda_{o_I} = w_I \cdot \Lambda^+$ (hence ${}_1\Lambda_{o_I} = \nu_{w_I} \cdot {}_1\Lambda^+$). For $w \in W_I$ we then have $E_{o_I}(n_w) = T(n_w)$ [Vig3, Example 5.32]. (Note that $w_I(\mathcal{D}^-)$ here equals $w_I(\mathcal{D}^+)$ in [Vig3], which corresponds to $o_{w_I(\Delta)}$ in [Vig3].)

IV.13. We need some length formulas ([Vig3, Corollaries 5.10 and 5.11]). We have to be careful to remember that Σ^+ in [Vig3] corresponds to Φ_a^-. For $\lambda \in \Lambda$, $w \in W_0$, we have

$$\text{(IV.13.1)} \qquad \ell(w\cdot\lambda) = \sum_{\beta\in\Phi_a^+} |\beta\circ\nu(\lambda)| = \ell(\lambda),$$

$$\text{(IV.13.2)} \qquad \ell(w\lambda) = \sum_{\beta\in\Phi_a^+\cap w^{-1}(\Phi_a^+)} |\beta\circ\nu(\lambda)| + \sum_{\beta\in\Phi_a^+\cap w^{-1}(\Phi_a^-)} |\beta\circ\nu(\lambda) - 1|,$$

$$\text{(IV.13.3)} \qquad \ell(\lambda w) = \sum_{\beta\in\Phi_a^+\cap w(\Phi_a^+)} |\beta\circ\nu(\lambda)| + \sum_{\beta\in\Phi_a^+\cap w(\Phi_a^-)} |\beta\circ\nu(\lambda) + 1|.$$

Note that for $\beta \in \Delta$ and $w = s_\beta = w^{-1}$, s_β permutes $\Phi_a^+ - \{\beta_a\}$ and sends β_a to $-\beta_a$ so $\Phi_a^+ \cap s_\beta(\Phi_a^-) = \{\beta_a\}$.

Lemma *Let $I \subset \Delta$. Then, for $\lambda \in \Lambda_{o_I}$, $\ell(w^I\lambda) = \ell(w^I) + \ell(\lambda)$.*

Proof By (IV.13.2) we need to check that $\beta \circ \nu(\lambda) \leq 0$ for $\beta \in \Phi^+ \cap (w^I)^{-1}(\Phi^-)$; but $\lambda \in \Lambda_{o_I}$ means that $\beta \circ \nu(\lambda) \geq 0$ for $\beta \in w_I(\Phi^-) = (w^I)^{-1}(\Phi^+)$. □

IV.14. An important result in this chapter is the following.

Theorem *Let $w \in W_J$. Then for $\lambda \in {}_1\Lambda$,*

$$f_J T^*(n_w) E_{o_J}(\lambda) = \begin{cases} \tau((\nu_{w_J} n_w)\cdot\lambda) f_J T^*(n_w) & \text{if } (\nu_{w_J} n_w)\cdot\lambda \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } (\nu_{w_J} n_w)\cdot\lambda \notin {}_1\Lambda^+. \end{cases}$$

The proof of the theorem is in IV.15–IV.18. Taking $w = w_J$ we get by IV.9 Proposition:

Corollary *For $\lambda \in {}_1\Lambda$,*

$$f E_{o_J}(\lambda) = \begin{cases} \tau(\lambda) f & \text{if } \lambda \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } \lambda \notin {}_1\Lambda^+. \end{cases}$$

Remarks

1) We have used the notation $\tau(\mu)$ for $\mu \in {}_1\Lambda^+$ to mean τ_z for $z \in Z^+$ with image $\mu \in {}_1\Lambda^+$. The shift of indices is only for typographical convenience.
2) As ψ extends to a character of $M_{J,k}$ by IV.4, each n_w for $w \in W_J$ normalizes ψ, and it follows that λ normalizes ψ if and only if so does $(\nu_{w_J} n_w)\cdot\lambda$.
3) The subspace of $\mathcal{A}_{o_J}$ generated by the $E_{o_J}(\lambda)$ for λ in ${}_1\Lambda$ normalizing ψ is a subalgebra $\mathcal{A}_{o_J}(\psi)$ of $\mathcal{A}_{o_J}$ The map $\mathcal{A}_{o_J}(\psi) \to \mathcal{H}_Z(\psi)$ sending $E_{o_J}(\lambda)$ to $\tau((\nu_{w_J} n_w)\cdot\lambda)$ if $(\nu_{w_J} n_w)\cdot\lambda \in {}_1\Lambda^+$ and to 0 otherwise is an algebra homomorphism $\theta_{\nu_{w_J} n_w}$, and for $T \in \mathcal{A}_{o_J}(\psi)$ we have
$$f_J T^*(n_w) T = \theta_{\nu_{w_J} n_w}(T) f_J T^*(n_w).$$
4) The theorem says nothing when $(\nu_{w_J} n_w)\cdot\lambda \in {}_1\Lambda^+$ and does not normalize ψ. We do not use this case.

IV.15. We treat first the case where $w = 1$. Recalling that $f_J = f_0 T(n_{w^J})$, we want to compute $f_0 T(n_{w^J}) E_{o_J}(\lambda)$. By IV.12, we have $T(n_{w^J}) = E_{o_\Delta}(n_{w^J})$, so we look at $E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda)$.

Assume first that $\nu_{w_J}\cdot\lambda$ belongs to ${}_1\Lambda^+$, i.e. that λ belongs to ${}_1\Lambda_{o_J}$, and that $\nu_{w_J}\cdot\lambda$ normalizes ψ. Then IV.13 Lemma gives $\ell(n_{w^J}) + \ell(\lambda) = \ell(n_{w^J}\lambda)$, hence $E_{o_\Delta}(n_{w^J}\lambda) = E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda)$. Since $\ell(n_{w^J}\cdot\lambda) = \ell(\lambda)$ by (IV.13.1), we also obtain $\ell(n_{w^J}\cdot\lambda) + \ell(n_{w^J}) = \ell(n_{w^J}\lambda)$ hence $E_{o_\Delta}(n_{w^J}\lambda) = E_{o_\Delta}(n_{w^J}\cdot\lambda) E_{o_\Delta}(n_{w^J})$, and finally

$$E_{o_\Delta}(n_{w^J}\cdot\lambda) E_{o_\Delta}(n_{w^J}) = E_{o_\Delta}(n_{w^J}) E_{o_J}(\lambda) = T(n_{w^J}) E_{o_J}(\lambda).$$

We can apply IV.10 Proposition to $n_{w^J}\cdot\lambda$. Indeed, $\nu_{w_0}\cdot(n_{w^J}\cdot\lambda) = (\nu_{w_0} n_{w^J})\cdot\lambda$ and $\nu_{w_0} n_{w^J} = \nu_{w_J}$. Since by IV.12 $E_{o_\Delta}(n_{w^J}\cdot\lambda) = T(n_{w^J}\cdot\lambda)$, that gives $f_0 E_{o_\Delta}(n_{w^J}\cdot\lambda) = \tau(\nu_{w_J}\cdot\lambda) f_0$, so $\tau(\nu_{w_J}\cdot\lambda) f_J = f_0 E_{o_\Delta}(n_{w^J}\cdot\lambda) T(n_{w^J}) = f_J E_{o_J}(\lambda)$, which is the desired formula when $\nu_{w_J}\cdot\lambda$ belongs to ${}_1\Lambda^+$.

Fix a regular such λ and let $\lambda' \in {}_1\Lambda - {}_1\Lambda_{o_J}$. Then $E_{o_J}(\lambda) E_{o_J}(\lambda') = 0$ by (IV.12.2), and $f_J E_{o_J}(\lambda) E_{o_J}(\lambda') = 0$, implying $\tau(\nu_{w_J}\cdot\lambda) f_J E_{o_J}(\lambda') = 0$. Since $\tau(\nu_{w_J}\cdot\lambda)$ is invertible in

$\mathcal{H}_Z(\psi)$, we get $f_J E_{o_J}(\lambda') = 0$, which is the formula we want for λ'. The theorem is proved for $w = 1$.

IV.16. We prove the theorem by induction on $\ell(w)$ (see [Oll1, Section 5] for GL_n). Let $\ell(w) = \ell \geq 1$, and write $w = w's$ with $\ell(w') = \ell - 1$ and $s = s_\beta$ for some $\beta \in J$ – note that $w'(\beta) \in \Phi^+$ since $\ell(w's) = \ell(w') + 1$. In particular $n_w = n_{w'} n_s$ and $T^*(n_w) = T^*(n_{w'})T^*(n_s)$.

We need to investigate $T^*(n_s)E_{o_J}(\lambda)$ for $\lambda \in {}_1\Lambda$. Suppose we can prove

$$(*) \qquad f_J T^*(n_{w'})T^*(n_s)E_{o_J}(\lambda) = f_J T^*(n_{w'})E_{o_J}(n_s \cdot \lambda)T^*(n_s);$$

then the desired formula follows from the induction hypothesis. So we need to compare $E_{o_J}(n_s \cdot \lambda)T^*(n_s)$ and $T^*(n_s)E_{o_J}(\lambda)$. By [Vig3, Corollary 5.53] we have, for any orientation o such that $\operatorname{Ker}\beta$ is a wall of the Weyl chamber corresponding to o:

$$\text{(IV.16.1)} \quad \begin{array}{lll} \text{If} & \beta \circ \nu(\lambda) = 0, & E_o(n_s \cdot \lambda)E_o(n_s) = E_o(n_s)E_o(\lambda); \\ \text{if} & \beta \circ \nu(\lambda) > 0, & E_o(n_s \cdot \lambda)E_o(n_s) = E_{o\cdot s}(n_s)E_o(\lambda); \\ \text{if} & \beta \circ \nu(\lambda) < 0, & E_o(n_s \cdot \lambda)E_{o\cdot s}(n_s) = E_o(n_s)E_o(\lambda). \end{array}$$

We now apply the results in [Vig3, §5.4] to our case, where $o = o_J$. (We need to point out that since $\beta \in J$, $\operatorname{Ker}(\beta)$ is a wall of the Weyl chamber corresponding to o_J; also [Vig3] uses the notation s for an element of ${}_1W_0$, where we use n_s, but we do have $n_s^2 \in Z_k$ as required by [Vig3, 5.35 and 5.36].) Since $\beta \in J$, we have $E_{o_J}(n_s) = T(n_s)$ (IV.12) and $E_{o_J\cdot s}(n_s) = T^*(n_s)$ by [Vig3, Example 5.32]. So we get:

$$\text{(IV.16.2)} \quad \begin{array}{lll} \text{If} & \beta \circ \nu(\lambda) = 0, & E_{o_J}(n_s \cdot \lambda)T(n_s) = T(n_s)E_{o_J}(\lambda); \\ \text{if} & \beta \circ \nu(\lambda) > 0, & E_{o_J}(n_s \cdot \lambda)T(n_s) = T^*(n_s)E_{o_J}(\lambda); \\ \text{if} & \beta \circ \nu(\lambda) < 0, & E_{o_J}(n_s \cdot \lambda)T^*(n_s) = T(n_s)E_{o_J}(\lambda). \end{array}$$

IV.17. Accordingly we distinguish the three cases.

Assume first $\beta \circ \nu(\lambda) = 0$; then formula $(*)$ of IV.16 follows from (IV.16.2) and the following lemma.

Lemma *Assume $\beta \circ \nu(\lambda) = 0$. Then $E_{o_J}(n_s \cdot \lambda)c_{n_s} = c_{n_s}E_{o_J}(\lambda)$.*

Proof We work within the Levi subgroup M_β of G. As $\beta \circ \nu(\lambda) = 0$, λ normalizes $K \cap M_\beta$ (III.7 Corollary). (Note that $K \cap M_\beta$ is the parahoric subgroup of M_β attached to our special point $\mathbf{x}_0$; λ also normalizes the pro-p radical $K(1) \cap M_\beta$ of $K \cap M_\beta$.) Consequently, λ acts via conjugation on $M_{\beta,k}$; that action stabilizes $U_{\beta,k}$ and $U_{\beta,k}^{\mathrm{op}}$, so it also stabilizes the subgroup $M'_{\beta,k}$ they generate. Consequently, λ acts via conjugation on $Z_{k,s} = Z_k \cap M'_{\beta,k}$. On the other hand, an element t in $Z_{k,s}$ has length 0, implying $E_{o_J}(n_s \cdot \lambda)T(t) = E_{o_J}((n_s \cdot \lambda)t)$ and $T(t)E_{o_J}(\lambda) = E_{o_J}(t\lambda)$. Now, computing in ${}_1W$, $(n_s \cdot \lambda)t\lambda^{-1} = (n_s\lambda n_s^{-1}\lambda^{-1})(\lambda t\lambda^{-1})$. As t runs through $Z_{k,s}$, so does $\lambda t \lambda^{-1}$; on the other hand, by construction n_s belongs to $M'_{\beta,k}$ so $n_s\lambda n_s^{-1}\lambda^{-1}$ belongs to $Z_{k,s}$. The result follows. $\square$

IV.18. Assume now that $\beta\circ\nu(\lambda) < 0$. Since $w'(\beta)$ is positive, $(w_J w's)(\beta) = -w_J w'(\beta)$ is positive too. But $((w_J w's)(\beta))\circ\nu$, evaluated on $\nu_{w_J} n_{w'}(\lambda)$ gives $(s(\beta)\circ\nu)(\lambda) = -\beta\circ\nu(\lambda) > 0$ so $\nu_{w_J} n_{w'}(\lambda)$ is not in ${}_1\Lambda^+$, and consequently $f_J T^*(n_{w'})E_{o_J}(\lambda) = 0$ by the induction hypothesis. But by (IV.16.2)

$$f_J T^*(n_{w'})[T^*(n_s)E_{o_J}(\lambda) - E_{o_J}(n_s \cdot \lambda)T^*(n_s)] = -f_J T^*(n_{w'})c_{n_s}E_{o_J}(\lambda).$$

Since $f_J T^*(n_{w'})c_{n_s} = -f_J T^*(n_{w'})$ by IV.8 Proposition, $-f_J T^*(n_{w'})c_{n_s}E_{o_J}(\lambda)$ is equal to $f_J T^*(n_{w'})E_{o_J}(\lambda)$, which is 0 by the above, and $(*)$ is true in that case too.

The case where $\beta \circ \nu(\lambda) > 0$ is dealt with similarly: in that case we find

$$f_J T^*(n_{w'})[T^*(n_s)E_{o_J}(\lambda) - E_{o_J}(n_s \cdot \lambda)T^*(n_s)] = f_J T^*(n_{w'})E_{o_J}(n_s \cdot \lambda)c_{n_s}$$

by (IV.16.2) and that is 0 by induction because $\nu_{w_J} n_w(\lambda)$ is not in ${}_1\Lambda^+$ (as $(w_J w(\beta)) \circ \nu$ is positive on it). This completes the proof of IV.14 Theorem. □

IV.19. We now reach the easier part of our change of weight (IV.19 Theorem (i)), which is a consequence of the following theorem.

Theorem *Assume that $\lambda \in {}_1\Lambda$ normalizes ψ. Then*

$$f' E_{o_{J'}}(\lambda n^{-1}_{w_J w_{J'}}) T^*(n_{w_J w_{J'}}) = \begin{cases} \tau(\lambda) f & \lambda \in {}_1\Lambda^+ \text{ and } \alpha \circ \nu(\lambda) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $z \in Z^+$, normalizing ψ, and with $|\alpha|(z) < 1$, we get $f'T = \tau_z f$ for some T in $\mathcal{H}$, which gives IV.1 Theorem (i). To prove the theorem, we first prove:

Lemma $f' = f_J T^*(n_{w_J w_{J'} w_J}) T(n_{w_J w_{J'}})$.

Proof By IV.7 Corollary, $f' = f_0 \sum_{w \in w_0 W_{J'}} T(n_w)$, which can also be written as $f' = f_0 \sum_{v \in W_{J'}} T(n_{w_0 v w_{J'}})$. For v in $W_{J'}$, write $w_0 v w_{J'} = w^J (w_J v w_J)(w_J w_{J'})$. We have

$$\ell(w_0 v w_{J'}) = \ell(w_0) - \ell(v w_{J'}) = \ell(w_0) - \ell(w_{J'}) + \ell(v) \text{ (since } v \in W_{J'}),$$

and

$$\ell(w^J) = \ell(w_0) - \ell(w_J),\ \ell(w_J v w_J) = \ell(v),\ \ell(w_J w_{J'}) = \ell(w_J) - \ell(w_{J'}),$$

so $\ell(w_0 v w_{J'}) = \ell(w^J) + \ell(w_J v w_J) + \ell(w_J w_{J'})$. Consequently,

$$\sum_{v \in W_{J'}} T(n_{w_0 v w_{J'}}) = T(n_{w^J}) \big(\sum_{v \in W_{J'}} T(n_{w_J v w_J}) \big) T(n_{w_J w_{J'}})$$

and $f' = f_J (\sum_{v \in W_{J'}} T(n_{w_J v w_J})) T(n_{w_J w_{J'}})$.

Now $J'' = -w_J(J')$ is a subset of J and $w_J W_{J'} w_J = W_{J''}$; the element $w_J w_{J'} w_J$ is the longest element of that group, hence

$$\sum_{v \in W_{J'}} T(n_{w_J v w_J}) = \sum_{v \leq w_J w_{J'} w_J} T(n_v).$$

By IV.9 Proposition

$$f_J \big(\sum_{v \leq w_J w_{J'} w_J} T(n_v) \big) = f_J T^*(n_{w_J w_{J'} w_J})$$

so $f' = f_J T^*(n_{w_J w_{J'} w_J}) T(n_{w_J w_{J'}})$. □

Proof of the theorem Put $v = w_J w_{J'}$. Note that since $v \in W_J$, $n_v \cdot \lambda$ normalizes ψ, see IV.14 Remark 2).

By the relations (IV.12.1) we get

$$q_{n_v, \lambda n_v^{-1}} E_{o_J}(n_v \cdot \lambda) = E_{o_J}(n_v) E_{o_J \cdot v}(\lambda n_v^{-1}).$$

On the other hand $E_{o_J}(n_w) = T(n_w)$ for $w \in W_J$, so we get

$$T(n_v)E_{o_J\cdot v}(\lambda n_v^{-1}) = q_{n_v,\lambda n_v^{-1}} E_{o_J}(n_v \cdot \lambda).$$

We now compute

$$\begin{aligned} f' E_{o_J\cdot v}(\lambda n_v^{-1}) &= f_J T^*(n_{w_J w_{J'} w_J}) T(n_v) E_{o_J\cdot v}(\lambda n_v^{-1}) \\ &= q_{n_v,\lambda n_v^{-1}} f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v \cdot \lambda). \end{aligned}$$

By IV.14 Theorem we see that $f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v\cdot\lambda)$ is 0 if $\lambda \notin {}_1\Lambda^+$. If $\alpha\circ\nu(\lambda) = 0$, since $v(\alpha) \in \Phi^-$, $\ell(\lambda v^{-1}) > \ell(\lambda) - \ell(v^{-1}) = \ell(v\cdot\lambda) - \ell(v)$ by IV.13, so $q_{n_v,\lambda n_v^{-1}} = 0$.

Assume $\alpha \circ \nu(\lambda) < 0$ and $\lambda \in {}_1\Lambda^+$. Let $\beta \in \Phi^+$ with $v(\beta) \in \Phi^-$. Since $v \in W_J$, β is a linear combination of roots in J (with non-negative integer coefficients). Moreover $w_{J'}(\beta) \in \Phi^+$, so the coefficient of α in β is positive. Then for $\beta \in \Phi^+ \cap v^{-1}(\Phi^-)$ we have $\beta \circ \nu(\lambda) \le \alpha \circ \nu(\lambda)$ by the above, so $\beta \circ \nu(\lambda) < 0$, which implies by IV.13 that $\ell(\lambda v^{-1}) = \ell(v\cdot\lambda) - \ell(v^{-1})$ and $f_J T^*(n_{w_J w_{J'} w_J}) E_{o_J}(n_v\cdot\lambda) = \tau(\lambda) f_J T^*(n_{w_J w_{J'} w_J})$ by IV.14 Theorem (indeed, $\ell(w_J w_{J'} w_J) + \ell(v) = \ell(w_J)$ implies $n_{w_J w_{J'} w_J} n_v = n_{w_J}$, so $\nu_{w_J} n_{w_J w_{J'} w_J} n_v = 1$). The theorem follows on multiplying by $T^*(n_v)$, noting $T^*(n_{w_J w_{J'} w_J}) T^*(n_v) = T^*(n_{w_J})$ and $o_{J'} = o_J \cdot v$. $\square$

IV.20. We now turn to the other part of the change of weight theorem (IV.19 Theorem (ii), (iii)), which is harder. From now on, we put $s = s_\alpha$.

Lemma $f' = f - f_J T^*(n_{w_J s}) = f_J T^*(n_{w_J s}) T(n_s)$.

Proof By IV.9 we have $f = f_J\big(\sum_{w\le w_J} T(n_w)\big)$ and $f_J T^*(n_{w_J s}) = f_J\Big(\sum_{w \le w_J s} T_w\Big)$ so $f - f_J T^*(n_{w_J s}) = f_J(\sum T(n_w))$ where the sum runs over w in W_J with $w \not\le w_J s$; but for $w \in W_J$, $w \le w_J s$ is equivalent to $s \le w_J w$, so $w \not\le w_J s$ means that $w_J w$ belongs to $W_{J'}$. Consequently, $f - f_J T^*(n_{w_J s}) = f_J\big(\sum_{w\in W_{J'}} T(n_{w_J w_{J'} w})\big) = f_0 T(n_{w^J})\big(\sum_{w\in W_{J'}} T(n_{w_J w_{J'} w})\big)$. For w in $W_{J'}$, $\ell(w_J w_{J'} w) = \ell(w_J) - \ell(w_{J'} w) = \ell(w_J) - \ell(w_{J'}) + \ell(w) = \ell(w_J w_{J'}) + \ell(w)$ so $T(n_{w_J w_{J'} w}) = T(n_{w_J w_{J'}}) T(n_w)$. On the other hand, $\ell(w^J) + \ell(w_J w_{J'}) = \ell(w_0) - \ell(w_J) + \ell(w_J) - \ell(w_{J'}) = \ell(w^{J'})$ so $T(n_{w^{J'}}) = T(n_{w^J}) T(n_{w_J w_{J'}})$. It follows that $f - f_J T^*(n_{w_J s}) = f_0 T(n_{w^{J'}})\big(\sum_{w\in W_{J'}} T(n_w)\big) = f'$ by IV.7 Corollary applied to J'. Moreover, as $\ell(w_J s) + \ell(s) = \ell(w_J)$ we have $T^*(n_{w_J}) = T^*(n_{w_J s}) T^*(n_s)$ and $f = f_J T^*(n_{w_J}) = f_J T^*(n_{w_J s})(T(n_s) + 1)$, as seen in IV.9 above, so $f' = f_J T^*(n_{w_J s}) T(n_s)$. $\square$

IV.21. Let now $\lambda \in {}_1\Lambda^+$ and put $\lambda' = n_s\cdot\lambda$. It is the element $f E_{o_J\cdot s}(n_s\lambda')$ that we want to relate to f'. To get an expression for it, we again need to distinguish cases, according to the integer $r = -\alpha_a \circ \nu(\lambda) \ge 0$ (recall that α_a is the simple root in Φ_a corresponding to α). We first deal with the "easy" relations in $\mathcal{H}$.

Lemma *(i)* $\lambda'(n_s \cdot \lambda') = n_s \cdot (\lambda\lambda') \in {}_1\Lambda^+$.

(ii) If $r > 0$, $\ell(n_s\lambda') = \ell(\lambda') - 1$ *and* $T(n_s)E_{o_J\cdot s}(n_s\lambda') = T(n_s^2)E_{o_J}(\lambda')$.

(iii) If $r \ge 2$, *then* $E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = 0$.

(iv) If $r = 1$, *then* $E_{o_J\cdot s}(n_s\lambda') = E_{o_J}(n_s\lambda')$ *and*

$$E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = E_{o_J}(\lambda'(n_s\cdot\lambda'))T(n_s).$$

Proof (i) The first equality is clear. Let us prove that $\lambda'(n_s \cdot \lambda')$ is in ${}_1\Lambda^+$. We have $\alpha_a \circ \nu(\lambda'(n_s \cdot \lambda')) = 0$. For $\beta \in \Delta$, $\beta \neq \alpha$ we compute

$$\begin{aligned}\beta_a \circ \nu(\lambda'(n_s \cdot \lambda')) &= \beta_a \circ \nu((n_s \cdot \lambda)(n_s^2 \cdot \lambda)) \\ &= (\beta_a + s(\beta_a))(\nu(\lambda)).\end{aligned}$$

It is ≤ 0 since $\beta_a, s(\beta_a) > 0$ and λ is in ${}_1\Lambda^+$. So we get (i).

(ii) Assume $r > 0$. We need to work in $\mathcal{H}_{\mathbb{Z}}$, and then specialize to $\mathcal{H}$. By (IV.13.2), we get $\ell(n_s\lambda') = \ell(\lambda') - 1$ because $\alpha \circ \nu(\lambda') > 0$. So the relation (IV.12.1) gives $E_{o_J \cdot s}(n_s)E_{o_J}(\lambda') = q_s E_{o_J \cdot s}(n_s\lambda')$. We also have $E_{o_J}(n_s)E_{o_J \cdot s}(n_s) = q_s E_{o_J}(n_s^2)$, which gives

$$q_s E_{o_J}(n_s)E_{o_J \cdot s}(n_s\lambda') = q_s E_{o_J}(n_s^2)E_{o_J}(\lambda').$$

Cancelling q_s, using $E_{o_J}(n_s^2) = T(n_s^2)$, and specializing to $\mathcal{H}$ we get (ii).

(iii) We proved $\ell(n_s\lambda') = \ell(\lambda') - 1$ in (ii), so $\ell(\lambda') + \ell(n_s\lambda') = 2\ell(\lambda') - 1$. On the other hand $\lambda' n_s \lambda' = \lambda'(n_s \cdot \lambda')n_s$ and $\alpha_a \circ \nu(\lambda'(n_s \cdot \lambda')) = 0$ so $\ell(\lambda' n_s \lambda') = \ell(\lambda'(n_s \cdot \lambda')) + 1$ by (IV.13.3). But $\ell(\lambda'(n_s \cdot \lambda')) = \ell(\lambda') + \ell(n_s \cdot \lambda') - 2r$ by (IV.13.1) so we get $\ell(\lambda') + \ell(n_s\lambda') - \ell(\lambda' n_s \lambda') = 2r - 2$. This is > 0 if $r \geq 2$, so in that case $E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = 0$ by the relations (IV.12.1).

(iv) Assume now that $r = 1$. The first formula is given by [Vig3, Lemma 5.34]. In the proof of (ii) we have seen that $\ell(\lambda') + \ell(n_s\lambda') = \ell(\lambda' n_s \lambda')$ so we get $E_{o_J}(\lambda')E_{o_J}(n_s\lambda') = E_{o_J}(\lambda' n_s \lambda')$. On the other hand $\lambda' n_s \lambda' = \lambda'(n_s \cdot \lambda')n_s$ and we have seen $\ell(\lambda' n_s \lambda') = \ell(\lambda'(n_s \cdot \lambda')) + 1$, so $E_{o_J}(\lambda' n_s \lambda') = E_{o_J}(\lambda'(n_s \cdot \lambda'))E_{o_J}(n_s) = E_{o_J}(\lambda'(n_s \cdot \lambda'))T(n_s)$. $\square$

IV.22. In the sequel it is convenient to put $\varphi = f_J T^*(n_{w_J s})$ so that $f' = \varphi T(n_s)$, $f = \varphi + f'$. From IV.14 Theorem, we get the following: for $\mu \in {}_1\Lambda$,

$$\text{(IV.22.1)} \qquad \varphi E_{o_J}(n_s \cdot \mu) = \begin{cases} \tau(\mu)\varphi & \text{if } \mu \in {}_1\Lambda^+ \text{ and normalizes } \psi, \\ 0 & \text{if } \mu \notin {}_1\Lambda^+. \end{cases}$$

Put $E = E_{o_J \cdot s}(n_s\lambda')$ with $\lambda' = n_s \cdot \lambda$ as in IV.21 – note that λ' also normalizes ψ.

By (ii) of IV.21 Lemma, $T(n_s)E = T(n_s^2)E_{o_J}(\lambda')$, so $\varphi T(n_s)E = \tau(n_s^2)\varphi E_{o_J}(\lambda')$ by (IV.22.1). But $\tau(n_s^2)\varphi = \varphi$ because n_s^2, which belongs to $Z_k \cap M'_{\alpha,k}$, acts trivially on φ by IV.7 Lemma. We deduce $\varphi T(n_s)E = \varphi E_{o_J}(\lambda') = \tau(\lambda)\varphi$, again by (IV.22.1).

We are now ready to prove a change of weight formula, in the special case where $\lambda \in {}_1\Lambda^+$ normalizes ψ and $\alpha_a \circ \nu(\lambda) = -1$. Indeed, by (IV.22.1) and (iv) of IV.21 Lemma we get $\tau(\lambda)\varphi E = \varphi E_{o_J}(\lambda')E = \varphi E_{o_J}(\lambda'(n_s \cdot \lambda'))T(n_s)$, hence $\tau(\lambda)\varphi E = \tau(\lambda\lambda')\varphi T(n_s)$, using again (IV.22.1). We deduce that $\varphi E = \tau(\lambda')\varphi T(n_s)$, as $\tau(\lambda)$ is invertible in $\mathcal{H}_Z(\psi)$.

Consequently, $fE = \varphi E + \varphi T(n_s)E = \tau(\lambda)\varphi + \tau(\lambda')\varphi T(n_s) = \tau(\lambda)(f - f') + \tau(\lambda')f'$. We have proved:

Proposition *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $\alpha_a \circ \nu(\lambda) = -1$. Then*

$$\tau(\lambda)f - fE = (\tau(\lambda) - \tau(\lambda'))f'.$$

Remark Note that $\tau(\lambda)f$ belongs to $\operatorname{ind}_K^G V$ because $\lambda \in {}_1\Lambda^+$, so we see that $\operatorname{ind}_K^G V$ contains $(\tau(\lambda) - \tau(\lambda'))(\operatorname{ind}_K^G V')$. Note also that $\tau(\lambda)f'$ belongs to $\operatorname{ind}_K^G V'$ for the same reason; but $\tau(\lambda')f'$ does not necessarily belong to $\operatorname{ind}_K^G V'$ because λ' is not in ${}_1\Lambda^+$.

IV.23. We now seek a similar formula in the case where $\lambda \in {}_1\Lambda^+$ normalizes ψ, $r = -\alpha_a \circ \nu(\lambda) \geq 2$, still with $\lambda' = n_s \cdot \lambda$ and $E = E_{o_J \cdot s}(n_s \lambda')$. By [Vig3, Proposition 5.48] we have, in $\mathcal{H}_{\mathbb{Z}}$, an identity

$$(*) \qquad E_{o_J \cdot s}(n_s \lambda') - E_{o_J}(n_s \lambda') = \sum_{k=1}^{r-1} q(k, \lambda') q_s^{-1} c(k, \lambda') E_{o_J}(\mu(k, \lambda'))$$

and by [Vig3, Proposition 5.49], in $\mathcal{H}$ only the terms for $k = 1$ and $k = r - 1$ may be non-zero, so we get, in $\mathcal{H}$,

$$E = E_{o_J}(n_s \lambda') + c_1 E_{o_J}(\mu_1 \lambda') + c_{r-1} E_{o_J}(\mu_{r-1} \lambda'),$$

where the last term disappears if $r = 2$. For the moment we need not know what c_1, c_{r-1} are in $C[Z_k]$, nor what μ_1 and μ_{r-1} are in ${}_1\Lambda$ except that they do not depend on λ and $\nu(\mu_k) = -k\alpha_a^\vee$ by [Vig3, formula (87)], so $\nu(n_s^{-1} \cdot \mu_k) = k\alpha_a^\vee$. From that it follows that $(n_s^{-1} \cdot \mu_1)\lambda$ is in ${}_1\Lambda^+$, but not $(n_s^{-1} \cdot \mu_{r-1})\lambda$ if $r > 2$. Also by [Vig3, 5.49], the q-terms in the identity $(*)$ above give 1 in C for $k = 1$ or $k = r - 1$. Indeed, we have to show that $\ell(\lambda') - \ell(\mu_{-\alpha_a}^{-1} \lambda') = 2$: remarking that $\nu(\mu_{-\alpha_a}^{-1} \lambda') = \nu(\lambda') - \alpha_a^\vee$, that comes from the length formula in IV.13. As in IV.22 we have $\varphi T(n_s) E = \tau(\lambda)\varphi$. On the other hand

$$\varphi E = \varphi E_{o_J}(n_s \lambda') + \varphi c_1 E_{o_J}(\mu_1 \lambda') + \varphi c_{r-1} E_{o_J}(\mu_{r-1} \lambda')$$

where the last term disappears if $r = 2$.

But $\tau(\lambda)\varphi = \varphi E_{o_J}(\lambda')$ by (IV.22.1), so $\tau(\lambda)\varphi E_{o_J}(n_s \lambda') = \varphi E_{o_J}(\lambda') E_{o_J}(n_s \lambda')$ which is 0 by IV.21 Lemma (iii), and hence $\varphi E_{o_J}(n_s \lambda') = 0$. For $z \in Z_k$ we have $\varphi E_{o_J}(n_s \cdot z) = \tau_z \varphi = \psi(z^{-1})\varphi$ so we get $\varphi c_1 E_{o_J}(\mu_1 \lambda') = \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda')$, with the obvious notation for the conjugation action on $C[Z_k]$, and the obvious extension of ψ^{-1} from Z_k to $C[Z_k]$. Similarly, if $r \geq 3$, $\varphi c_{r-1} E_{o_J}(\mu_{r-1} \lambda') = \psi^{-1}(n_s^{-1} \cdot c_{r-1})\varphi E_{o_J}(\mu_{r-1} \lambda')$, which is 0 by (IV.22.1) because $(n_s^{-1} \cdot \mu_{r-1})\lambda$ is not in ${}_1\Lambda^+$. Thus for $r \geq 2$,

$$\varphi E = \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda').$$

As $\varphi T(n_s) E = \tau(\lambda)\varphi$ we obtain:

Proposition *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $-\alpha_a \circ \nu(\lambda) \geq 2$. Then*

$$fE = \tau(\lambda)\varphi + \psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1 \lambda').$$

IV.24. We now apply the formulas given by IV.22 Proposition and IV.23 Proposition to the case where $\lambda \in {}_1\Lambda^+$ normalizes ψ, and deduce IV.1 Theorem (ii) and (iii). We first assume $\alpha_a \circ \nu(\lambda) = -1$. As we have seen in IV.22 Remark, λ' normalizes ψ and $(\tau(\lambda) - \tau(\lambda'))(\operatorname{ind}_K^G V') \subset \operatorname{ind}_K^G V$.

Proposition *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $\alpha_a \circ \nu(\lambda) = -1$. Then ψ is trivial on $Z^0 \cap M'_\alpha$ and $\tau(\lambda') = \tau(\lambda)\tau_\alpha$.*

Proof We work within M_α. The semisimple Bruhat-Tits building of M_α is a tree, the apartment corresponding to $\mathbf{S}$ is the line in V_{ad} generated by $\alpha_a^\vee$; the group Z acts on that line via its quotient Λ, and $\lambda \in \Lambda$ acts via translation by v with $\alpha_a \circ \nu(\lambda) = \alpha_a(v)$ and as $\alpha_a \circ \nu(\lambda) = -1$, λ sends the (special) vertex $\mathbf{x}_0$ to the adjacent (special) vertex $\mathbf{x}_1 = \mathbf{x}_0 - \frac{1}{2}\alpha_a^\vee$ in the apartment. We shall later prove the following claim.

For the claim the situation is the following:

Assumption Assume that $\mathbf{G}$ has relative semisimple rank 1, and let $\mathbf{x}_1$ be a vertex in V_{ad} (a line) adjacent to $\mathbf{x}_0$, and K_1 the corresponding (special) parahoric subgroup of G. Let $\mathbf{G}_{1,k}$ be the group over k attached to the parahoric subgroup K_1. (Note that both $K = K_0$ and K_1 contain Z^0 and G_k, $G_{1,k}$ contain Z_k.)

Claim *The subgroup of Z_k generated by $Z_k \cap G'_k$ and $Z_k \cap G'_{1,k}$ is the image of $Z^0 \cap G'$ in Z_k.*

We apply the claim to $\mathbf{M}_\alpha$. Since λ sends $\mathbf{x}_0$ to $\mathbf{x}_1$, it conjugates K_0 to K_1, and conjugation by λ induces an isomorphism of $M_{\alpha,k}$ onto $M_{\alpha,1,k}$ and of $M'_{\alpha,k}$ onto $M'_{\alpha,1,k}$. As ψ is trivial on $Z_k \cap M'_{\alpha,k}$ by hypothesis, and λ stabilizes ψ, ψ is also trivial on $Z_k \cap M'_{\alpha,1,k}$ and by the claim ψ is trivial on $Z^0 \cap M'_\alpha$. By the second line after formula (87) in [Vig3], from $\alpha_a \circ \nu(\lambda) = -1$ we get $\nu(\lambda^{-1}\lambda') = \alpha_a^\vee$; but $\lambda' = n_s \cdot \lambda$ by definition, so $\lambda^{-1}\lambda' = \lambda^{-1} n_s \lambda n_s^{-1}$. Take $z \in Z$ with image λ in ${}_1\Lambda$ and $\tilde{n}_s$ in $K \cap M'_\alpha \cap \mathcal{N}$ with image n_s in $M_{\alpha,k}$ (the existence follows from III.7 Lemma, for instance). Since M'_α is normal in M_α, $z^{-1}\tilde{n}_s z$ is in M'_α so $\lambda^{-1} n_s \lambda n_s^{-1}$ is the image in ${}_1\Lambda$ of an element of $Z \cap M'_\alpha$. It follows that we can take $\lambda^{-1}\lambda'$ as the image in ${}_1\Lambda$ of a_α of III.16 Notation (which verifies $\nu(a_\alpha) = \alpha_a^\vee$, cf. IV.11 Example 3), and then $\tau(\lambda') = \tau(\lambda)\tau_\alpha$. □

From the above proposition and IV.22 Proposition, we get case (iii) of IV.1 Theorem when $\alpha_a \circ \nu(\lambda) = -1$.

Corollary *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $\alpha_a \circ \nu(\lambda) = -1$. Then ψ is trivial on $Z^0 \cap M'_\alpha$ and $\tau(\lambda)(1-\tau_\alpha)\operatorname{ind}_K^G V' \subset \operatorname{ind}_K^G V$.*

We note that $\lambda a_\alpha \notin Z^+$ so in particular $\tau(\lambda)(1-\tau_\alpha) \notin \mathcal{Z}_G$.

IV.25. We investigate the term $\psi^{-1}(n_s^{-1} \cdot c_1)\varphi E_{o_J}(\mu_1\lambda')$ in IV.23 Proposition.

Proposition *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $-\alpha_a \circ \nu(\lambda) \geq 2$.*
(i) The element $n_s^{-1} \cdot \mu_1 \in {}_1\Lambda$ is in the image of $Z \cap M'_\alpha$.
(ii) If ψ is not trivial on $Z^0 \cap M'_\alpha$, then $\psi^{-1}(n_s^{-1} \cdot c_1) = 0$.
(iii) If ψ is trivial on $Z^0 \cap M'_\alpha$, then $\psi^{-1}(n_s^{-1} \cdot c_1) = -1$ and $\tau((n_s^{-1} \cdot \mu_1)\lambda) = \tau(\lambda)\tau_\alpha$.

Note that from (i) and III.16 Proposition (i), $n_s^{-1} \cdot \mu_1$ normalizes ψ if ψ is trivial on $Z^0 \cap M'_\alpha$. In particular, in (iii) the element $\tau((n_s^{-1} \cdot \mu_1)\lambda)$ is defined. Using IV.23 Proposition and (IV.22.1) we get

$$fE = \begin{cases} \tau(\lambda)(f - f') & \text{if } \psi \text{ is not trivial on } Z^0 \cap M'_\alpha, \\ \tau(\lambda)(1-\tau_\alpha)(f - f') & \text{if } \psi \text{ is trivial on } Z^0 \cap M'_\alpha. \end{cases}$$

This formula immediately yields IV.1 Theorem (ii), (iii) when $-\alpha_a \circ \nu(\lambda) \geq 2$ (note that this implies $\lambda a_\alpha \in {}_1\Lambda^+$):

Corollary *Let $\lambda \in {}_1\Lambda^+$ normalize ψ, and assume $-\alpha_a \circ \nu(\lambda) \geq 2$.*
(i) If ψ is not trivial on $Z^0 \cap M'_\alpha$ then $\tau(\lambda)\operatorname{ind}_K^G V' \subset \operatorname{ind}_K^G V$.
(ii) If ψ is trivial on $Z^0 \cap M'_\alpha$ then

$$\tau(\lambda)(1-\tau_\alpha)\operatorname{ind}_K^G V' \subset \operatorname{ind}_K^G V.$$

To prove the proposition we need to know precisely what c_1 and μ_1 are. We have to distinguish cases: $\alpha_a \circ \nu(\Lambda) = \delta\mathbb{Z}$ for $\delta = 1$ or 2 [Vig3, Remark 5.3]. The generic case is $\delta = 1$, which we tackle first. In that case choose $\lambda_s \in \Lambda$ with $\alpha_a \circ \nu(\lambda_s) = 1$; then

$\mu_1 = (n_s \cdot \lambda_s)\lambda_s^{-1}$ and $c_1 = (n_s \cdot \lambda_s) \cdot c_{n_s}$. Recall that $c_{n_s} = \frac{-1}{|Z_{k,s}|} \sum_{z \in Z_{k,s}} z$ in $C[Z_k]$. In particular, $\psi^{-1}(n_s^{-1} \cdot c_1) = \frac{-1}{|Z_{k,s}|} \sum_{z \in Z_{k,s}} \psi^{-1}(\lambda_s \cdot z)$. So we see that $\psi^{-1}(n_s^{-1} \cdot c_1)$ is non-zero if and only if ψ is trivial on $\lambda_s Z_{k,s} \lambda_s^{-1}$, in which case it is equal to -1. Reasoning as in IV.24 with λ_s instead of λ we see that $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$ if and only if ψ is trivial on $Z^0 \cap M'_\alpha$ and the other assertions of the proposition are obtained as in IV.24 as well (when $\delta = 1$), noting that τ_α is in the centre of $\mathcal{H}_Z(\psi)$.

IV.26. We continue the proof of IV.25 Proposition. Now assume that $\delta = 2$. One situation where this may happen is when $\mathbf{G}$ has relative semisimple rank 1, or more generally when the connected component of the relative Dynkin diagram of $\mathbf{G}$ containing α has rank 1. In that case, let $\tilde{s}$ be the reflection in the affine Weyl group of $\mathbf{M}_\alpha$ corresponding to the affine root $\alpha_a + 1$; it corresponds to a vertex $\mathbf{x}_1$ in the semisimple Bruhat-Tits building of $\mathbf{M}_\alpha$ (a tree) adjacent to the vertex $\mathbf{x}_0$. As in IV.24 we let K_1 be the parahoric subgroup of M_α corresponding to the vertex $\mathbf{x}_1$ (which is special), and $K_1(1)$ its pro-p radical. Then $Z \cap K_1 = Z^0$, $Z \cap K_1(1) = Z(1)$. The image of $\mathcal{N} \cap K_1$ in $K_1/K_1(1) = M_{\alpha,1,k}$ is the group $\mathcal{N}_{1,k}$ of k-points of the normalizer of $\mathbf{Z}_k$ in $\mathbf{M}_{\alpha,1,k}$ and we can choose in $\mathcal{N}_{1,k}$ a lift $n_{\tilde{s}}$ of $\tilde{s}$ which actually belongs to $M'_{\alpha,1,k}$ – note that $\tilde{s}$ generates $(\mathcal{N} \cap K_1)/Z^0$ which we identify, via reduction with $\mathcal{N}_{1,k}/Z_k$. Then, inside ${}_1W = \mathcal{N}/Z(1)$, we can take (cf. [Vig3, Notation 5.37]) $\lambda_s = n_s n_{\tilde{s}}$, $\mu_1 = \lambda_s^{-1}$, $c_1 = c_{\tilde{s}} n_s^2$, where $c_{\tilde{s}} = \frac{-1}{|Z_{k,\tilde{s}}|} \sum_{z \in Z_{k,\tilde{s}}} z$, with $Z_{k,\tilde{s}} = Z_k \cap M'_{\alpha,1,k}$[14].
We see that $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$ if and only if ψ is trivial on $Z_{k,\tilde{s}}$. As ψ is already trivial on $Z_{k,s}$, we get by IV.24 Claim that $\psi^{-1}(n_s^{-1} \cdot c_1) \neq 0$ if and only if ψ is trivial on $Z^0 \cap M'_\alpha$, in which case $\psi^{-1}(n_s^{-1} \cdot c_1) = -1$. On the other hand, $n_s^{-1} \cdot \mu_1$ is in the image of $Z \cap M'_\alpha$ (by lifting n_s and $n_{\tilde{s}}$ to $\mathcal{N} \cap M'_\alpha$ as in IV.24). Moreover, by construction $\nu(\mu_1) = -\alpha_a^\vee$ and as in IV.24 we deduce that we can take the image of a_α in ${}_1\Lambda$ to be $n_s^{-1} \cdot \mu_1$ and that $\tau((n_s^{-1} \cdot \mu_1)\lambda) = \tau(\lambda)\tau_\alpha$ if ψ is trivial on $Z^0 \cap M'_\alpha$.

IV.27. The only other case when $\delta = 2$ may happen is when the connected component of the Dynkin diagram of Φ_a containing α has type C_n, $n \geq 2$, and α is a long root [Vig3, Proposition 5.14]. Let then $\tilde{\alpha}_a$ be the highest root in Φ_a^+ lying in the same component as α, and $\tilde{s}$ be the reflection associated with $\tilde{\alpha}_a + 1$. Then (cf. [Vig3, Lemma 5.15 and Notation 5.37]) $\mu_{-\alpha_a} = s w \tilde{s} w^{-1}$ for some $w \in W^a$ such that $\ell(\mu_{-\alpha_a}) = 2\ell(w) + 2$ and $w\tilde{s}w^{-1}$ is the reflection s' associated with the affine root $\alpha_a + 1$ (whereas s is associated with α_a). Moreover $\mu_{-\alpha_a} = ss'$ satisfies $\nu(\mu_{-\alpha_a}) = \alpha_a^\vee$. In that case (cf. [Vig3]) $c_1 = (w \cdot c_{\tilde{s}}) n_s^2$ and $\lambda_s = n_s(w \cdot n_{\tilde{s}})$, $\mu_1 = \lambda_s^{-1}$ with $n_{\tilde{s}}$, $c_{\tilde{s}}$ defined similarly as before [Vig3, §4]; but conjugating by w yields $w \cdot c_{\tilde{s}} = c_{s'}$ and $w \cdot n_{\tilde{s}} = n_{s'}$ where now $c_{s'}, n_{s'}$ have a similar meaning, but in the relative semisimple rank 1 group M_α. The same reasoning as in IV.26 then gives the desired result.

IV.28. To finish the proof of IV.25 Proposition we need only prove IV.24 Claim. It is convenient to deal first with the case where $\mathbf{G} = \mathbf{G}^{\mathrm{is}}$. Then $W = W^a$ is generated by the involutions s_0 (generating $\mathcal{N}^0/Z^0$) and s_1 (generating $(\mathcal{N} \cap K_1)/Z^0$). As $s_0 s_1$ acts as a non-trivial translation on the apartment, $s_0 s_1$ has infinite order.

[14] In principle those elements are defined in [Vig3] with respect to $\mathbf{G}$, not $\mathbf{M}_\alpha$, but the above choices in M_α also work in G. The same remark applies in IV.27.

Identify $\mathcal{N}^0/Z(1)$ with $\mathcal{N}_k$ and similarly $(\mathcal{N} \cap K_1)/Z(1)$ with the group $\mathcal{N}_{1,k}$ of k-points of the normalizer of $\mathbf{Z}_k$ in $\mathbf{G}_{1,k}$. Choose a lifting n_0 of s_0 in $\mathcal{N}_k \cap G'_k \subset {}_1W$ and a lifting n_1 of s_1 in $\mathcal{N}_{1,k} \cap G'_{1,k} \subset {}_1W$. An element w of W has a unique reduced expression $w = \sigma_1 \cdots \sigma_h$ with $\sigma_i = s_0$ or s_1 and we put $n_w = x_1 \cdots x_h$ with $x_i = n_0$ if $\sigma_i = s_0$, $x_i = n_1$ if $\sigma_i = s_1$. We let X be the subgroup of Z_k generated by $Z_k \cap G'_k$ and $Z_k \cap G'_{1,k}$, and put $Y = \{n_w x \mid w \in W, x \in X\}$.

Lemma 1 *X and Y are normal subgroups of ${}_1W$.*

Proof Let $x \in Z_k$; then $n_0^{-1} x n_0 x^{-1}$ belongs to Z_k; but Z_k normalizes G'_k so $n_0^{-1} x n_0 x^{-1}$ belongs to $Z_k \cap G'_k$. Similarly $n_1^{-1} x n_1 x^{-1}$ belongs to $Z_k \cap G'_{1,k}$. In particular, n_0 and n_1 normalize X. Since Z_k also normalizes X, so ${}_1W$ itself normalizes X. As n_0^2 and n_1^2 belong to X, we deduce that for w, $w' \in W$ $n_w n_{w'} \in n_{ww'} X$ and $n_w^{-1} \in n_{w^{-1}} X$, so Y is indeed a normal subgroup of ${}_1W$. □

Now let $H = IYI$ with the usual abuse of notation.

Lemma 2 *H is a normal subgroup of G and $(H \cap Z^0)/Z(1) = X$*

Proof We first prove that H is a subgroup of G. By Lemma 1, H is closed under inverses. Working in $\mathcal{H}_{\mathbb{Z}}$, it is enough to show that for y, y' in Y, the product $T(y)T(y')$ in $\mathcal{H}_{\mathbb{Z}}$ is a linear combination of $T(y'')$ for y'' in Y. But that is given by the relations in $\mathcal{H}_{\mathbb{Z}}$: the braid relations and the two quadratic relations $T(n_i)^2 = q_i T(n_i^2) + c_i T(n_i)$ where $q_i \in \mathbb{Z}$ and $c_i \in \mathbb{Z}[Z_k \cap G'_{i,k}]$ for $i = 0, 1$.

As Z^0 normalizes I, and Z_k normalizes Y, Z^0 normalizes H. The normalizer of H contains n_0, n_1 (which belong to H), Z^0 and I, so it is G itself. If an element x of H in a class IyI, $y \in Y$, is in Z^0 then y has to belong to Z_k so by the very definition of Y, y belongs to X and x itself has image y in $Z^0/Z(1) = Z_k$. That gives the last assertion of the lemma. □

Clearly H is not central in G, so $H = G$ because the only non-central normal subgroup of G is G itself (II.3 Proposition). But then $H \cap Z^0 = G \cap Z^0 = Z^0$ so $X = Z_k$, which gives the claim for $\mathbf{G} = \mathbf{G}^{\text{is}}$.

Let us now prove IV.24 Claim in the general case. We show first that the claim is equivalent to

$$(*) \qquad Z(1)(Z^0 \cap G') = Z(1)\big\langle Z^0 \cap \langle U^0, U^0_{\text{op}}\rangle, Z^0 \cap \langle U \cap K_1, U_{\text{op}} \cap K_1\rangle\big\rangle.$$

It suffices to show that the image of $Z^0 \cap \langle U^0, U^0_{\text{op}}\rangle$ in Z_k equals $Z_k \cap G'_k$ (and similarly for the other term). It is clear that an arbitrary element of $Z_k \cap G'_k$ lifts to an element of $\langle U^0, U^0_{\text{op}}\rangle \cap Z^0 K(1)$. Using the Iwahori decomposition of $K(1)$ (III.7) we can modify the lift so that it is contained in $Z^0 \cap \langle U^0, U^0_{\text{op}}\rangle$.

The only non-trivial part of the equality $(*)$ is the inclusion $\subset$. The inclusion is true for G^{is}, and we deduce it for G by applying the natural homomorphism $\iota : G^{\text{is}} \to G$, using that $(Z^{\text{is}})^0 = \iota^{-1}(Z^0)$ (III.19 Proposition) and that $Z(1)$ is the pro-p Sylow of Z^0. This completes the proof of IV.24 Claim and hence of IV.1 Theorem. □

V. Universal modules

V.1. In this chapter our goal is, for an irreducible representation V of K, to study the "universal" representation $\operatorname{ind}_K^G V$ as a module over the centre $\mathcal{Z}_G(V)$ of the Hecke algebra

$\mathcal{H}_G(V)$. In fact that structure is difficult to elucidate, so we consider various algebra homomorphisms $\chi : \mathcal{Z}_G(V) \to A$ and the corresponding A-module $A \otimes_\chi \operatorname{ind}_K^G V$. As an application, for a character $\chi : \mathcal{Z}_G(V) \to C$, we prove Theorem 6 of the introduction – used in Chapter III at the end of our classification – which gives a nice filtration of $C \otimes_\chi \operatorname{ind}_K^G V$ as a representation of G. In this chapter we fix an irreducible representation V of K and let $(\psi, \Delta(V))$ be its parameter as defined in III.9.

A) Freeness of the supersingular quotient of $\operatorname{ind}_K^G V$

V.2. Until V.11 we fix a parabolic subgroup $P = MN$ of G containing B. Recall from III.4 the subgroup $Z^\perp_{\Delta_M}$ of Z consisting of those $z \in Z$ with $|\beta|(z) = 1$ for all $\beta \in \Delta_M$. We write Z^{+M} for the set of $z \in Z$ with $|\beta|(z) \leq 1$ for $\beta \in \Delta_M$. Recall from III.4 that $\mathcal{Z}_Z(V_{U^0})$ is spanned by the τ_z for $z \in Z_\psi$, and that the natural image of $\mathcal{Z}_M(V_{N^0})$ in $\mathcal{Z}_Z(V_{U^0})$ (via $\mathcal{S}_Z^M$) is spanned by the τ_z for $z \in Z^{+M} \cap Z_\psi$ – we identify $\mathcal{Z}_M(V_{N^0})$ with that image.

Notation We let R_M be the quotient of $\mathcal{Z}_M(V_{N^0})$ by the ideal of elements supported on $(Z^{+M} \cap Z_\psi) - Z^\perp_{\Delta_M}$.

As $\mathcal{Z}_M(V_{N^0})$ is viewed as a subset of $\mathcal{Z}_Z(V_{U^0})$, we emphasize that the supports above are subsets of Z. Note that the elements of $\mathcal{Z}_M(V_{N^0})$ supported on $Z^\perp_{\Delta_M}$ form a subalgebra which maps isomorphically onto R_M.

Our first main result in this chapter is:

Theorem *Let $P = MN$ be a parabolic subgroup of G containing B. Then $R_M \otimes_{\mathcal{Z}_G(V)} \operatorname{ind}_K^G V$ is free over R_M, where the tensor product is via the composite map $\mathcal{Z}_G(V) \to \mathcal{Z}_M(V_{N^0}) \to R_M$.*

We call $R_M \otimes_{\mathcal{Z}_G(V)} \operatorname{ind}_K^G V$ the supersingular quotient of $\operatorname{ind}_K^G V$ (cf. III.4 Corollary).

The proof of that theorem is rather long (V.3 to V.11). We first treat the case where $P = G$ (V.3 Proposition). The proof then proceeds by comparing with situations with a more regular weight (i.e. smaller $\Delta(V)$). Using the change of weight results of Chapter IV, we reduce the proof in general to a special case where, in particular, Δ_M is orthogonal to $\Delta - \Delta_M$ (V.7). Finally, we use a filtration argument (V.8 to V.11).

V.3. Proposition *$R_G \otimes_{\mathcal{Z}_G(V)} \operatorname{ind}_K^G V$ is free over R_G.*

The proof in V.4 requires several lemmas. We use again the Kottwitz homomorphism w_G and the map v_G (III.16).

Lemma 1 *Let z, z_1, z_2 in Z. If $zz_1z_2 \in Kz_1Kz_2K$, then $w_G(z) = 0$.*

Proof The Kottwitz homomorphism w_G is a homomorphism of G into a commutative group; the result follows from $w_G(K) = 0$. $\square$

Lemma 2 *Let $z_1 \in Z^+$ normalizing ψ, and $f \in \mathcal{H}_G(V)$ with support in Kz_1K. Then $\mathcal{S}_Z^G(f) \in \mathcal{H}_Z(V_{U^0})$ has support in $(Z \cap \operatorname{Ker} w_G)z_1$.*

Proof That is immediate from (III.3.2), once we note that w_G is trivial on U. $\square$

Lemma 3 *Let $z_1 \in Z^+$ normalizing ψ, and $z_2 \in Z$. If $f \in \operatorname{ind}_K^G V$ has support in Kz_2K, then $\tau_{z_1} * f$ has support in $K(Z \cap \operatorname{Ker} w_G)z_1z_2K$.*

Proof By definition τ_{z_1}, as an element of $\mathcal{H}_Z(V_{U^0})$, has support $Z^0 z_1$. From Lemma 2, τ_{z_1}, as an element of $\mathcal{H}_G(V)$, has support in $K(Z \cap \operatorname{Ker} w_G) z_1 K$. The result then follows from the convolution formula in $\mathcal{H}_G(V)$ and Lemma 1. □

Lemma 4 $Z_\Delta^\perp \cap \operatorname{Ker} w_G = Z^0$.

Proof Let $z \in \operatorname{Ker} w_G$. Then $v_G(z) = 0$. If moreover $z \in Z_\Delta^\perp$, then $v_Z(z) = 0$ for the analogous map v_Z, cf. [HV1, 6.3 Remark 1]; from [HV1, 6.2 Lemma], (ii) and (iii), it follows that $z \in Z^0$. Conversely $Z^0 \subset Z_\Delta^\perp \cap \operatorname{Ker} w_G$ is clear. □

V.4. We prove V.3 Proposition. We decompose $\operatorname{ind}_K^G V$ as $\oplus I(x)$, $x \in Z/(Z \cap \operatorname{Ker} w_G)$, where $I(x)$ consists of the functions in $\operatorname{ind}_K^G V$ with support in $Kx(Z \cap \operatorname{Ker} w_G)K$. For z in Z^+ normalizing ψ, we have $\tau_z * I(x) \subset I(zx)$ by V.3 Lemma 3, with equality if $z \in Z_\Delta^\perp$ since then τ_z has inverse $\tau_{z^{-1}}$. For $x \in Z/(Z \cap \operatorname{Ker} w_G)$, let $I^+(x)$ be the sum of the subspaces $\tau_z * I(y)$ of $I(x)$, where $z \in Z^+ \cap Z_\psi$, $z \notin Z_\Delta^\perp$, $y \in Z/(Z \cap \operatorname{Ker} w_G)$ and $zy = x$ in $Z/(Z \cap \operatorname{Ker} w_G)$. By definition $R_G \otimes_{\mathcal{Z}_G(V)} \operatorname{ind}_K^G V$ is the quotient of $\operatorname{ind}_K^G V$ obtained by killing all the subspaces $I^+(x)$; thus it appears as $\oplus_{x \in Z/(Z \cap \operatorname{Ker} w_G)} (I(x)/I^+(x))$. Let $z \in Z_\Delta^\perp \cap Z_\psi$; then $\tau_z * I(x) = I(zx)$, $\tau_z * I^+(x) = I^+(zx)$ for $x \in Z/(Z \cap \operatorname{Ker} w_G)$, hence the corresponding element in R_G, still written τ_z, sends $I(x)/I^+(x)$ isomorphically onto $I(zx)/I^+(zx)$. As $Z_\Delta^\perp \cap \operatorname{Ker} w_G = Z^0$ by V.3 Lemma 4, the image of $Z_\Delta^\perp \cap Z_\psi$ in $Z/(Z \cap \operatorname{Ker} w_G)$ acts by multiplication without fixed points on $Z/(Z \cap \operatorname{Ker} w_G)$; choosing a set of representatives Ω for the orbits, we deduce that $R_G \otimes_{\mathcal{Z}_G(V)} \operatorname{ind}_K^G V$ is isomorphic to the free R_G-module $R_G \otimes_C (\bigoplus_{x \in \Omega} I(x)/I^+(x))$. □

For further use, we state a result proved in a similar manner.

Lemma *Let* $z \in Z^+ \cap Z_\psi$.

(i) If $v_G(z) \neq 0$, $\tau_z - 1$ *acts injectively on* $\operatorname{ind}_K^G V$*; if moreover* $z \in Z_\Delta^\perp$ *then* $\tau_z - 1$ *is not a divisor of* 0 *in* R_G.

(ii) Let $T \in \mathcal{Z}_G(V)$*; if* $v_G(z)$ *is linearly independent from* $v_G(\operatorname{Supp}(T))$*, then* $(\tau_z - 1) \operatorname{ind}_K^G V \cap T \operatorname{ind}_K^G V = (\tau_z - 1) T \operatorname{ind}_K^G V$.

Remark The condition $v_G(z) = 0$ is equivalent to $v_Z(z) \in \mathbb{R}\Delta^\vee \subset X_*(\mathbf{S}) \otimes \mathbb{R}$.

Proof (i) Let $f \in \operatorname{ind}_K^G V$, and write as above $f = \sum f_x$, $x \in Z/(Z \cap \operatorname{Ker} w_G)$, $f_x \in I(x)$. Then for $z \in Z^+ \cap Z_\psi$, $\tau_z * f = \sum_x \tau_z * f_x$ with $\tau_z * f_x \in I(zx)$. The equality $\tau_z * f = f$ amounts to $\tau_z * f_x = f_{zx}$ for all $x \in Z/(Z \cap \operatorname{Ker} w_G)$. If $v_G(z) \neq 0$ then the image of z in $Z/(Z \cap \operatorname{Ker} w_G)$ has infinite order; since $f_x = 0$ for all but a finite number of x's, $\tau_z * f = f$ implies $f = 0$, and $\tau_z - 1$ acts injectively on $\operatorname{ind}_K^G V$; in particular, as $\mathcal{Z}_G(V)$ acts faithfully on $\operatorname{ind}_K^G V$, $\tau_z - 1$ is not a divisor of 0 in $\mathcal{Z}_G(V)$. If moreover $z \in Z_\Delta^\perp$ then $\tau_z - 1$ is not a divisor of 0 in the subalgebra of $\mathcal{Z}_G(V)$ which maps isomorphically onto R_G.

(ii) Let Γ be the subgroup of Z generated by the elements ξ with $v_G(\xi)$ in $v_G(\operatorname{Supp} T)$. For $y \in Z/\Gamma$, let $J(y)$ be the space of functions in $\operatorname{ind}_K^G V$ with support in $Ky\Gamma K$; then $TJ(y) \subset J(y)$ and for $z \in Z^+ \cap Z_\psi$, $\tau_z * J(y) \subset J(zy)$. Let f, f' in $\operatorname{ind}_K^G V$ with $(\tau_z - 1) f = f'$. We have $\operatorname{ind}_K^G V = \oplus_{y \in Z/\Gamma} J(y)$ and decomposing accordingly $f = \sum f_y$ and $f' = \sum f'_y$ we get $\tau_z * f_y = f_{zy} + f'_{zy}$ for $y \in Z/\Gamma$. Let $f' \in T \operatorname{ind}_K^G V$; then $f'_y \in T \operatorname{ind}_K^G V$ for all $y \in Z/\Gamma$ so if f_y belongs to $T \operatorname{ind}_K^G V$, then so do $\tau_z * f_y$ and f_{zy}. The hypothesis on z in (ii) implies that its image in Z/Γ has infinite order, so $f_{z^{-r}y}$ is 0 for large r. So we get, using descending induction on r, that f_y does indeed belong to $T \operatorname{ind}_K^G V$. □

V.5. We now turn to the general case of V.2 Theorem. For each parabolic subgroup $P_1 = M_1 N_1$ of G containing P, we let V_{P_1} be the irreducible representation of K with parameter $(\psi, \Delta_{P_1} \cap \Delta(V))$ – for $P_1 = G$ we have $V_G = V$; we choose a basis vector for $(V_{P_1})_{U^0}$.

For such a P_1 consider the sequence of canonical (injective) intertwiners:

$$\operatorname{ind}_K^G V_{P_1} \to \operatorname{Ind}_{P_1}^G \operatorname{ind}_{M_1^0}^{M_1} (V_{P_1})_{N_1^0} \to \operatorname{Ind}_P^G \operatorname{ind}_{M^0}^M (V_{P_1})_{N^0} \to \operatorname{Ind}_B^G \operatorname{ind}_{Z^0}^Z (V_{P_1})_{U^0}. \tag{V.5.1}$$

As $(V_{P_1})_{N_1^0}$ has the same parameter as $V_{N_1^0}$, there is a unique isomorphism between them that is compatible with the choice of basis vectors in $(V_{P_1})_{U^0}$ and V_{U^0}; it induces an isomorphism of $(V_{P_1})_{N^0}$ onto V_{N^0}. Using those isomorphisms we identify the sequence (V.5.1) with

$$\operatorname{ind}_K^G V_{P_1} \to \operatorname{Ind}_{P_1}^G \operatorname{ind}_{M_1^0}^{M_1} V_{N_1^0} \to \operatorname{Ind}_P^G \operatorname{ind}_{M^0}^M V_{N^0} \to \operatorname{Ind}_B^G \operatorname{ind}_{Z^0}^Z \psi. \tag{V.5.2}$$

The sequence (V.5.1) is equivariant for the sequence of Hecke algebras

$$\mathcal{H}_G(V_{P_1}) \to \mathcal{H}_{M_1}((V_{P_1})_{N_1^0}) \to \mathcal{H}_M((V_{P_1})_{N^0}) \to \mathcal{H}_Z((V_{P_1})_{U^0}) \tag{V.5.3}$$

given by the (injective) Satake homomorphisms. The choice of basis vectors gives an isomorphism $\mathcal{H}_Z((V_{P_1})_{U^0}) \simeq \mathcal{H}_Z(\psi)$, actually independent of that choice, and inside $\mathcal{H}_Z(\psi)$ the Hecke algebras in (V.5.3) do not depend on P_1; accordingly we write $\mathcal{H}_G$, $\mathcal{H}_{M_1}$, $\mathcal{H}_M$, $\mathcal{H}_Z$, and similarly for the centres. The sequence (V.5.2) is then equivariant for the sequence of algebras $\mathcal{H}_G \to \mathcal{H}_{M_1} \to \mathcal{H}_M \to \mathcal{H}_Z$.

As in Chapter IV we identify the spaces in (V.5.2) with their images in $\operatorname{Ind}_B^G \operatorname{ind}_{Z^0}^Z \psi$, and similarly $\mathcal{H}_G$, $\mathcal{H}_{M_1}$, $\mathcal{H}_M$ with their images in $\mathcal{H}_Z$.

Notation For P_1 as above containing P, we let π_{P_1} be the $\mathcal{Z}_M[G]$-submodule $\mathcal{Z}_M \otimes_{\mathcal{Z}_G} \operatorname{ind}_K^G V_{P_1}$ of $\operatorname{Ind}_P^G \operatorname{ind}_{M^0}^M V_{N^0}$ (which is then π_P).

Remark By III.14 Theorem, π_{P_1} is also $\mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \operatorname{Ind}_{P_1}^G \operatorname{ind}_{M_1^0}^{M_1} V_{N_1^0}$, which we also see as $\operatorname{Ind}_{P_1}^G (\mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \operatorname{ind}_{M_1^0}^{M_1} V_{N_1^0})$, cf. [HV2] Corollary 1.3.

For further use, let us recall some useful facts. Let X be a locally profinite space with a countable basis. Then the functor $X \mapsto C_c^\infty(X, A)$ is exact on $\mathbb{Z}$-modules A, $C_c^\infty(X, \mathbb{Z})$ is free and $C_c^\infty(X, \mathbb{Z}) \otimes A \to C_c^\infty(X, A)$ is an isomorphism; if A is a free module over some ring R, then so is $C_c^\infty(X, A)$ and if $R \to R'$ is a ring homomorphism, then $R' \otimes_R C_c^\infty(X, A) \to C_c^\infty(X, R' \otimes_R A)$ is an isomorphism of R'-modules. If Y is an open subset of X, we have an exact sequence $0 \to C_c^\infty(Y, \mathbb{Z}) \to C_c^\infty(X, \mathbb{Z}) \to C_c^\infty(X - Y, \mathbb{Z}) \to 0$ of free $\mathbb{Z}$-modules. We are particularly interested in the case $X = J\backslash H$ where H is a locally profinite second countable group, and J a closed subgroup of H. If A is a smooth $R[J]$-module for some ring R, choosing a continuous section of $H \to J\backslash H$ gives an isomorphism of R-modules $C_c^\infty(J\backslash H, A) \simeq \operatorname{ind}_J^H A$, so we deduce that ind_J^H is an exact functor on smooth $R[J]$-modules, that $\operatorname{ind}_J^H A$ is free over R if A is, and that $R' \otimes_R \operatorname{ind}_J^H A \to \operatorname{ind}_J^H (R' \otimes_R A)$ is an isomorphism of $R'[H]$-modules for any ring homomorphism $R \to R'$.

V.6. We gather consequences of the change of weight results of Chapter IV.

Proposition *Let P_1, P_2 be parabolic subgroups of G containing P, with $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$.*

(i) $\pi_{P_2} \subset \pi_{P_1}$ with equality if $\alpha \notin \Delta(V)$ or if ψ is not trivial on $Z^0 \cap M'_\alpha$.

(ii) If $\alpha \in \Delta(V)$ and ψ is trivial on $Z^0 \cap M'_\alpha$, then $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2}$ (with τ_α as in III.16 *Notation). If moreover α is not orthogonal to Δ_M, the inclusion $\pi_{P_2} \subset \pi_{P_1}$ induces an isomorphism $R_M \otimes_{\mathcal{Z}_M} \pi_{P_2} \xrightarrow{\sim} R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$.*

Proof First note that if $\alpha \notin \Delta(V)$ then V_{P_1} and V_{P_2} are isomorphic, so $\pi_{P_2} = \pi_{P_1}$ is immediate. Assume $\alpha \in \Delta(V)$. We apply IV.1 Theorem to V_{P_2} (in lieu of V) and V_{P_1} (in lieu of V'). Choose $z \in Z_\psi$ with $|\alpha|(z) < 1$ and $|\beta|(z) = 0$ for $\beta \in \Delta$, $\beta \neq \alpha$; thus τ_z is an invertible element of $\mathcal{Z}_M$. By IV.1 Theorem (i), we have the inclusion $\tau_z \operatorname{ind}_K^G V_{P_2} \subset \operatorname{ind}_K^G V_{P_1}$ of subspaces of $\operatorname{Ind}_B^G(\operatorname{ind}_{Z^0}^Z \psi)$. As τ_z is invertible in $\mathcal{Z}_M$, we get $\pi_{P_2} \subset \pi_{P_1}$. If ψ is not trivial on $Z^0 \cap M'_\alpha$ then IV.1 Theorem (ii) gives $\tau_z \operatorname{ind}_K^G V_{P_1} \subset \operatorname{ind}_K^G V_{P_2}$ hence $\pi_{P_2} = \pi_{P_1}$. If ψ is trivial on $Z^0 \cap M'_\alpha$, IV.1 Theorem (ii) gives $\tau_z(1-\tau_\alpha) \operatorname{ind}_K^G V_{P_1} \subset \operatorname{ind}_K^G V_{P_2}$ so $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2}$. Now $\tau_\alpha = \tau_{a_\alpha}$ for $a_\alpha \in Z_\psi$ with $\nu(a_\alpha) = r\alpha^\vee$ with some positive rational number r (III.16 Proposition (i), IV.12 Example). If α is not orthogonal to Δ_M, we have $|\beta|(a_\alpha) < 1$ for some $\beta \in \Delta_M$; but τ_α is sent to 0 in R_M. This implies the last assertion. □

V.7. We deduce a reduction for the proof of V.2 Theorem. Let $\Delta_1 = \Delta_M \cup \{\alpha \in \Delta(V), \alpha \perp \Delta_M, \psi(Z^0 \cap M'_\alpha) = 1\}$ and let $P_1 = M_1 N_1$ be the corresponding parabolic subgroup of G. By V.6 Proposition, the inclusion $\pi_G \subset \pi_{P_1}$ induces an isomorphism $R_M \otimes_{\mathcal{Z}_M} \pi_G \simeq R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$. But $R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$ is the same as $\operatorname{Ind}_{P_1}^G (R_M \otimes_{\mathcal{Z}_{M_1}} \operatorname{ind}_{M_1^0}^{M_1} V_{N_1^0})$ (V.5 Remark); if the R_M-module inside the induction is free, then so is $R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$ (V.5 Remark). As a consequence, it is enough to prove V.2 when $\Delta_1 = \Delta$.

Assumption (until V.11): $\Delta = \Delta_M \cup \Delta(V)$, $(\Delta - \Delta_M) \perp \Delta_M$ and $\psi(Z^0 \cap M'_\alpha) = 1$ for $\alpha \in \Delta - \Delta_M$.

Notation We put $\sigma = \operatorname{ind}_{M^0}^M V_{N^0}$, so $\pi_P = \operatorname{Ind}_P^G \sigma$. We also put $W(M) = \{w \in W_0, w^{-1}(\Delta_M) \subset \Phi^+\}$. The above assumption allows us to define τ_α as in III.16 Notation.

By V.3 Proposition, we know that $R_M \otimes_{\mathcal{Z}_M} \sigma$ is free over R_M, and so is $R_M \otimes_{\mathcal{Z}_M} \pi_P$ (V.5 Remark). We want to deduce the same for $R_M \otimes_{\mathcal{Z}_M} \pi_G$. For that we filter π_P according to the double cosets PwB for $w \in W(M)$ (recall that G is the disjoint union of the double cosets PwB, $w \in W(M)$).

We consider upper sets in $W(M)$, i.e. subsets A such that $v \in A$, $v' \in W(M)$ and $v' \geq v$ (in the Bruhat order) imply $v' \in A$. For an upper set A, $PAB = \bigcup_{v \in A} PvB$ is open in G and we let $\pi_{P,A}$ be the subspace of functions in $\pi_P = \operatorname{Ind}_P^G \sigma$ with support in PAB; it is a $\mathcal{Z}_M$-submodule of π_P.

Let A be non-empty upper set in $W(M)$ and choose a minimal element w in A. Put $A' = A - \{w\}$; then A' is an upper set in $W(M)$ and we have the submodule $\pi_{P,A'}$ of $\pi_{P,A}$.

Let $\bar{A}$, $\bar{A}'$ be the (open) images of PAB, $PA'B$ in $P\backslash G$. We have an exact sequence of free $\mathbb{Z}$-modules

$$0 \longrightarrow C_c^\infty(\bar{A}', \mathbb{Z}) \longrightarrow C_c^\infty(\bar{A}, \mathbb{Z}) \longrightarrow C_c^\infty(\bar{A} - \bar{A}', \mathbb{Z}) \longrightarrow 0 \quad \text{(cf. V.5 Remark).}$$

Choosing a continuous section of $G \to P\backslash G$, and noting that $\bar{A} - \bar{A}'$ is the image of PwB in $P\backslash G$, we get from V.5 Remark that evaluating functions on PwB gives an isomorphism of

$\pi_{P,A}/\pi_{P,A'}$ with the $\mathcal{Z}_M$-module of locally constant functions $f : PwB \to \sigma$ with $f(pg) = pf(g)$ for $p \in P$, $g \in PwB$, and with compact support in $P\backslash PwB$; equivalently evaluating on wU gives an isomorphism with the compactly induced representation $\mathrm{ind}_{w^{-1}Pw\cap U}^{U} {}^w\sigma$.

Lemma *The inclusion $\pi_{P,A} \to \pi_P$ induces an isomorphism of $R_M \otimes_{\mathcal{Z}_M} \pi_{P,A}$ onto the subspace of $R_M \otimes_{\mathcal{Z}_M} \pi_P = \mathrm{Ind}_P^G(R_M \otimes_{\mathcal{Z}_M} \sigma)$ consisting of functions with support in PAB. The sequence*

$$0 \longrightarrow R_M \otimes_{\mathcal{Z}_M} \pi_{P,A'} \longrightarrow R_M \otimes_{\mathcal{Z}_M} \pi_{P,A} \longrightarrow R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'}) \longrightarrow 0$$

is exact, and all three terms are free over R_M.

Proof Choosing a continuous section of $G \to P\backslash G$, $\pi_{P,A}$ appears as $C_c^\infty(\bar{A}, \mathbb{Z}) \otimes \sigma$, $R_M \otimes_{\mathcal{Z}_M} \pi_{P,A}$ as $C_c^\infty(\bar{A}, \mathbb{Z}) \otimes (R_M \otimes_{\mathcal{Z}_M} \sigma)$, so the result follows from V.5 Remark via the exact sequence $0 \to C_c^\infty(\bar{A}', \mathbb{Z}) \to C_c^\infty(\bar{A}, \mathbb{Z}) \to C_c^\infty(\bar{A} - \bar{A}', \mathbb{Z}) \to 0$. □

V.8. Let A, w, A' be as in V.7, and let Q be a parabolic subgroup of G containing P. Then $\pi_Q \subset \pi_P$ and we let $\pi_{Q,A} = \pi_{P,A} \cap \pi_Q$, similarly for A', so we get an inclusion of $\mathcal{Z}_M$-modules

$$\pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

Notation Set $c_{Q,w} = \Pi_{\alpha \in \Delta_Q, w^{-1}(\alpha)<0}(\tau_\alpha - 1) \in \mathcal{Z}_M$.

Remarks 1) For $\alpha \in \Delta$, $w^{-1}(\alpha) < 0$ is equivalent to $s_\alpha w < w$ and it implies $\alpha \notin \Delta_M$ since $w \in W(M)$. In particular for such an α we have $v_M(a_\alpha) \neq 0$ by V.4 Remark.

2) By V.4 Lemma (i) (applied to M) $c_{Q,w}$ acts injectively on σ hence on $\pi_{P,A}/\pi_{P,A'}$; moreover, $c_{Q,w}$ does not divide 0 in R_M.

Proposition $\pi_{Q,A}/\pi_{Q,A'} = c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ inside $\pi_{P,A}/\pi_{P,A'}$.

Before we give the proof, we derive consequences, in particular V.2 Theorem.

Corollary 1 *$R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'}) \to R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'})$ is injective, and $R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'})$ is free over R_M.*

Proof By the proposition, multiplication by $c_{Q,w}$ induces maps

$$\pi_{P,A}/\pi_{P,A'} \twoheadrightarrow \pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

Tensoring with R_M over $\mathcal{Z}_M$ gives

$$R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'}) \twoheadrightarrow R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'}) \longrightarrow R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'})$$

whose composite is multiplication by $c_{Q,w}$. By the above remark 2) $c_{Q,w}$ does not divide 0 in R_M; since $R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'})$ is free over R_M by V.7 Lemma, multiplication by $c_{Q,w}$ is injective on it so we get an isomorphism $R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'}) \simeq R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'})$, thus proving Corollary 1. □

Corollary 2 *$R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A} \to R_M \otimes_{\mathcal{Z}_M} \pi_{P,A}$ is injective (in particular, for $A = W(M)$, $R_M \otimes_{\mathcal{Z}_M} \pi_Q \to R_M \otimes_{\mathcal{Z}_M} \pi_P$ is injective).*

Proof By induction on $\#A$, $R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A'} \to R_M \otimes_{\mathcal{Z}_M} \pi_{P,A'}$ is injective. By V.7 Lemma, $R_M \otimes_{\mathcal{Z}_M} \pi_{P,A'} \to R_M \otimes_{\mathcal{Z}_M} \pi_{P,A}$ is injective and by Corollary 1, $R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'}) \to$

$R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'})$ is injective too. The result follows from the snake lemma applied to the commutative diagram (with exact rows)

$$\begin{array}{ccccccccc} & & R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A'} & \to & R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A} & \to & R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'}) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & R_M \otimes_{\mathcal{Z}_M} \pi_{P,A'} & \to & R_M \otimes_{\mathcal{Z}_M} \pi_{P,A} & \to & R_M \otimes_{\mathcal{Z}_M} (\pi_{P,A}/\pi_{P,A'}) & \to & 0 \end{array} \quad \square$$

Corollary 3 *$R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A'} \to R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A}$ and $R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A} \to R_M \otimes_{\mathcal{Z}_M} \pi_Q$ are injective, and $(R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A})/(R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A'}) \to R_M \otimes_{\mathcal{Z}_M} (\pi_{Q,A}/\pi_{Q,A'})$ is an isomorphism.*

Proof In the left hand square of the previous diagram, the two vertical maps and the bottom horizontal one are injective, hence so is the top horizontal one, giving the first assertion, which immediately implies the last one. The second one follows from the first by descending induction on $\#A$. $\square$

Now V.2 Theorem follows from the corollaries. Indeed, by Corollary 1 and Corollary 3, $R_M \otimes_{\mathcal{Z}_M} \pi_Q$ is a successive extension of free modules. Therefore $R_M \otimes_{\mathcal{Z}_M} \pi_Q$ is free.

V.9. The proof of V.8 Proposition will involve an induction argument on $\dim G$. For this, a further corollary is necessary.

Corollary 4 *Let $z \in Z^{+M} \cap Z_\psi$, and assume $v_M(z) \neq 0$. Then $\pi_Q \cap (\tau_z - 1)\pi_P = (\tau_z - 1)\pi_Q$.*

The proof is given after a lemma. Let A, w, A' be as in V.7, and use the notation $\pi_{P,A}$, $\pi_{Q,A}$ of V.7, V.8.

Lemma $(\tau_z - 1)\pi_{P,A} = (\tau_z - 1)\pi_P \cap \pi_{P,A}$.

Proof By descending induction on $\#A$, the case $A = W(M)$ being trivial. By V.4 Lemma (i), $\tau_z - 1$ acts injectively on σ, hence also on $\pi_{P,A}/\pi_{P,A'}$ which is a direct sum of copies of σ (V.5 Remark). By the snake lemma $\pi_{P,A'}/(\tau_z - 1)\pi_{P,A'}$ injects into $\pi_{P,A}/(\tau_z - 1)\pi_{P,A}$ i.e. $(\tau_z - 1)\pi_{P,A} \cap \pi_{P,A'} = (\tau_z - 1)\pi_{P,A'}$. The assertion $(\tau_z - 1)\pi_{P,A} = (\tau_z - 1)\pi_P \cap \pi_{P,A}$ then implies the similar assertion for A'. $\square$

Proof of Corollary 4 Applying V.4 Lemma (ii) to $T = c_{Q,w} \in \mathcal{Z}_M$ whose support is in $\operatorname{Ker} v_M$ we get

$$(\tau_z - 1)\sigma \cap c_{Q,w}\sigma = (\tau_z - 1)c_{Q,w}\sigma,$$

hence

$$(\tau_z - 1)(\pi_{P,A}/\pi_{P,A'}) \cap c_{Q,w}(\pi_{P,A}/\pi_{P,A'}) = (\tau_z - 1)c_{Q,w}(\pi_{P,A}/\pi_{P,A'}).$$

But by V.8 Proposition $c_{Q,w}(\pi_{P,A}/\pi_{P,A'}) = \pi_{Q,A}/\pi_{Q,A'}$, so we obtain $(\tau_z - 1)\pi_{P,A} \cap \pi_{Q,A} \subset (\tau_z - 1)\pi_{Q,A} + \pi_{P,A'}$. By the lemma we get $(\tau_z - 1)\pi_P \cap \pi_{Q,A} \subset [(\tau_z - 1)\pi_Q \cap \pi_{P,A}] + \pi_{P,A'}$. As $(\tau_z - 1)\pi_P \cap \pi_Q$ contains $(\tau_z - 1)\pi_Q$, both give the same contribution to $\pi_{P,A}/\pi_{P,A'}$. Their equality now follows by induction on $\#A$. $\square$

V.10. We now proceed to the proof of V.8 Proposition, keeping its notation. We first deal with the (easier) statement that $\pi_{Q,A}/\pi_{Q,A'}$ contains $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$.

Notation Let $\Delta_w = \{\alpha \in \Delta,\ w^{-1}(\alpha) > 0\}$ and let $P_w = M_w N_w$ be the corresponding parabolic subgroup of G; it contains P, and w is in $W(M_w)$.

Lemma *Let A, w, A' be as in* V.7. Then $\pi_{P_w,A} \to \pi_{P,A}/\pi_{P,A'}$ is surjective.

Assume that lemma for a moment. Since $\pi_{Q\cap P_w,A}$ contains $\pi_{P_w,A}$, the map $\pi_{Q\cap P_w,A} \to \pi_{P,A}/\pi_{P,A'}$ is surjective as well. But by V.6 Proposition π_Q contains $c_{Q,w}\,\pi_{Q\cap P_w}$, so the image of $\pi_{Q,A}$ in $\pi_{P,A}/\pi_{P,A'}$ contains $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$, i.e. the quotient $\pi_{Q,A}/\pi_{Q,A'}$ contains $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$.

Proof Let $A_{\geq w} = \{v \in W(M),\, v \geq w\}$ and $A_{>w} = \{v \in W(M), v > w\}$. We use the abbreviations $\pi_{P,\geq w} = \pi_{P,A_{\geq w}}$, $\pi_{P,>w} = \pi_{P,A_{>w}}$. Then $\pi_{P,A} \supset \pi_{P,\geq w}$ and $\pi_{P,A'} \supset \pi_{P,>w}$; moreover $\pi_{P,A'} \cap \pi_{P,\geq w} = \pi_{P,>w}$, so $\pi_{P,\geq w}/\pi_{P,>w}$ injects into $\pi_{P,A}/\pi_{P,A'}$. But evaluation on PwB identifies both quotients with the same space of functions, so the injection is an isomorphism. Hence it is enough to prove the lemma for $A = A_{\geq w}$.

Sublemma *(i)* $w^{-1}Pw \cap U = w^{-1}Uw \cap U = w^{-1}P_w w \cap U$.
(ii) $PA_{\geq w}B = \sqcup_{v \in W(M_w), v \geq w} P_w v B$.

Proof (i) The first equality comes from $w \in W(M)$, the second one from $w \in W(M_w)$.
(ii) By [Abe, Lemma 4.20], $w \in W(M)$ implies $W_M A_{\geq w} = \{v \in W_0, v \geq w\}$ and similarly $w \in W(M_w)$ implies $W_{M_w}\{v \in W(M_w), v \geq w\} = \{v \in W_0, v \geq w\}$. The result follows on taking B-double cosets. □

To prove the lemma (for $A = A_{\geq w}$) we need to consider closely the inclusion $\pi_{P_w} \hookrightarrow \pi_P$. Both are parabolically induced from P_w, and the inclusion comes from the injective map $\Phi : \mathcal{Z}_M \otimes_{\mathcal{Z}_{M_w}} \operatorname{ind}_{M_w^0}^{M_w} V_{N_w^0} \to \operatorname{Ind}_{P\cap M_w}^{M_w} \sigma$ obtained from the canonical intertwiner (III.13.1), so π_{P_w} is simply the subspace $\operatorname{Ind}_{P_w}^G(\operatorname{Im}\Phi)$ of $\pi_P = \operatorname{Ind}_{P_w}^G(\operatorname{Ind}_{P\cap M_w}^{M_w}\sigma)$. Seeing π_P as induced from P_w, we let $\pi'_{P,\geq w}$ be the subspace of functions with support in $\bigcup_{v\in W(M_w), v\geq w} P_w v B$, and similarly $\pi'_{P,>w}$. An element f of $\pi_P = \operatorname{Ind}_P^G \sigma$ is seen as the function f' in $\operatorname{Ind}_{P_w}^G(\operatorname{Ind}_{P\cap M_w}^{M_w}\sigma)$ given by $f'(g) : m \mapsto f(mg)$ for $g \in G$, $m \in M_w$. Hence by (ii) of the sublemma $\pi_{P,\geq w} = \pi'_{P,\geq w}$ and $\pi_{P,>w} \supset \pi'_{P,>w}$. By (i) of the sublemma (and V.5 Remark), choosing a continuous section of $U \to w^{-1}Uw \cap U\backslash U$ gives a $\mathcal{Z}_M$-linear isomorphism ι of $\pi'_{P,\geq w}/\pi'_{P,>w}$ with $C_c^\infty(w^{-1}Uw \cap U\backslash U, \mathbb{Z}) \otimes \operatorname{Ind}_{P\cap M_w}^{M_w}\sigma$, a similar isomorphism of $\pi_{P,\geq w}/\pi_{P,>w}$ with $C_c^\infty(w^{-1}Uw\cap U\backslash U,\mathbb{Z}) \otimes \sigma$, and the quotient map $\pi'_{P,\geq w}/\pi'_{P,>w} \twoheadrightarrow \pi_{P,\geq w}/\pi_{P,>w}$ corresponds to evaluation at $1 : \operatorname{Ind}_{P\cap M_w}^{M_w}\sigma \to \sigma$. But $(\pi_{P_w} \cap \pi'_{P,\geq w})/(\pi_{P_w} \cap \pi'_{P,\geq w})$ is sent by ι to $C_c^\infty(w^{-1}Uw \cap U\backslash U, \mathbb{Z}) \otimes \operatorname{Im}\Phi$ so to get the surjectivity of $\pi_{P_w} \cap \pi'_{P,\geq w} \to \pi_{P,\geq w}/\pi_{P,>w}$ it suffices to see that evaluation at $1 : \operatorname{Im}\Phi \to \sigma$ is surjective. But for $x \in V_{N_w^0}$ the function in $\operatorname{ind}_{M_w^0}^{M_w} V_{N_w^0}$ with support M_w^0 and value x at 1, is sent in $\operatorname{Ind}_{P\cap M_w}^{M_w}\sigma$ to a function with value at 1 the function in σ with support M^0 and value at 1 the projection of x in V_{N^0}; as those last functions, for varying x, generate σ as a representation of M, and $\operatorname{Im}\Phi \to \sigma$ is M-equivariant, it is surjective. □

V.11. We turn to the inclusion $\pi_{Q,A}/\pi_{Q,A'} \subset c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$ in V.8 Proposition. We need auxiliary lemmas, where $\alpha \in \Delta - \Delta_M$ is fixed; we let $P^\alpha = M^\alpha N^\alpha$ be the parabolic subgroup corresponding to $\Delta_M \cup \{\alpha\}$ and we put $\bar\sigma = \sigma/(\tau_\alpha - 1)\sigma$. Note that Hypothesis (H) of III.15 holds with the map $\varphi : V_{N^0} \to \sigma \to \bar\sigma$. We also note that $\varphi\tau_\alpha = \tau_\alpha\varphi = \varphi$.

Lemma 1 *$\bar\sigma$ extends to P^α, trivially on N.*

Proof By II.7 it suffices to prove that $\bar\sigma$ is trivial on $Z \cap M'_\alpha$. Since α is orthogonal to Δ_M and ψ is trivial on $Z^0 \cap M'_\alpha$ by assumption, that comes from the fact that τ_α acts trivially on $\bar\sigma$ (III.17). □

We write ${}^e\bar\sigma$ for the extension of $\bar\sigma$ to P^α. Inside of $\pi_P/(\tau_\alpha - 1)\pi_P \simeq \mathrm{Ind}_P^G \bar\sigma$ we have the subspace $\mathrm{Ind}_{P^\alpha}^G {}^e\bar\sigma$, cf. III.22 Lemma 2.

Lemma 2 *The image of* $\pi_{P^\alpha} \to \pi_P \to \pi_P/(\tau_\alpha - 1)\tau_P$ *is contained in* $\mathrm{Ind}_{P^\alpha}^G {}^e\bar\sigma$.

Proof Since $\pi_{P^\alpha} \to \mathrm{Ind}_P^G \bar\sigma$ is $\mathcal{Z}_M[G]$-equivariant and π_{P^α} is generated as a $\mathcal{Z}_M[G]$-module by V_{P^α} it is enough to prove that the inclusion of $\mathrm{Hom}_K(V_{P^\alpha}, \mathrm{Ind}_{P^\alpha}^G {}^e\bar\sigma)$ into $\mathrm{Hom}_K(V_{P^\alpha}, \mathrm{Ind}_P^G \bar\sigma)$ is an isomorphism. By Frobenius reciprocity, this means that

$$\mathrm{Hom}_{M^{\alpha 0}}((V_{P^\alpha})_{N^{\alpha 0}}, {}^e\bar\sigma) \hookrightarrow \mathrm{Hom}_{M^{\alpha 0}}((V_{P^\alpha})_{N^{\alpha 0}}, \mathrm{Ind}_{P\cap M^\alpha}^{M^\alpha} \bar\sigma)$$

is an isomorphism. The quotient of $\mathrm{Ind}_{P\cap M^\alpha}^{M^\alpha} \bar\sigma$ by ${}^e\bar\sigma$ is the representation ${}^e\bar\sigma \otimes \mathrm{St}_{P\cap M^\alpha}^{M^\alpha}$ and it is enough to show that $(V_{P^\alpha})_{N^{\alpha 0}}$ is not a weight of that representation. But the parameter for $(V_{P^\alpha})_{N^{\alpha 0}}$ is $(\psi, (\Delta_M \cup \{\alpha\}) \cap \Delta(V))$ and $\alpha \in \Delta(V)$ whereas by III.18 the weights of ${}^e\bar\sigma \otimes \mathrm{St}_{P\cap M^\alpha}^{M^\alpha} = I(P\cap M^\alpha, \bar\sigma, P\cap M^\alpha)$ have parameters (ψ', I) where $\alpha \notin I$. □

Lemma 3 *Let* $P_1 = M_1N_1$ *and* $P_2 = M_2N_2$ *be parabolic subgroups of* G *containing* P*, and assume* $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$*. Let* A*,* w*,* A' *be as in* V.7, and assume that $w^{-1}(\beta) < 0$ for all $\beta \in \Delta_{P_2} - \Delta_M$. Then

$$\pi_{P_2,A} \subset (\tau_\alpha - 1)\pi_{P_1,A} + \pi_{P,A'}.$$

Proof Let $f \in \pi_{P_2,A}$ and let $\bar f$ be its image in $\mathrm{Ind}_P^G \bar\sigma$. As $\pi_{P_2} \subset \pi_{P^\alpha}$, we get $\bar f \in \mathrm{Ind}_{P^\alpha}^G {}^e\bar\sigma$ by Lemma 2. If $\bar f$ does not vanish on PwB, its support, being P^α invariant, contains $Ps_\alpha wB$. But $w^{-1}(\alpha) < 0$ means $s_\alpha w < w$ and w being minimal in A, that contradicts $f \in \pi_{P,A}$. Hence $\bar f$ vanishes on PwB and there exist $f_1 \in \pi_P$, $f_2 \in \pi_{P,A'}$ with $f = (\tau_\alpha - 1)f_1 + f_2$. The point is to prove that we can take f_1 in $\pi_{P_1,A}$. View π_{P_1} as $\mathrm{Ind}_{P_1}^G \sigma_1$ with $\sigma_1 = \mathcal{Z}_M \otimes_{\mathcal{Z}_{M_1}} \mathrm{ind}_{M_1^0}^{M_1} V_{N_1^0}$ and π_P as $\mathrm{Ind}_{P_1}^G \mathrm{Ind}_{P\cap M_1}^{M_1} \sigma$, the inclusion $\pi_{P_1} \hookrightarrow \pi_P$ being induced by the natural intertwiner $\mathrm{ind}_{M_1^0}^{M_1} V_{N_1^0} \to \mathrm{Ind}_{P\cap M_1}^{M_1} \sigma$.

Sublemma *For* $v \in A'$ *we have* $P_1wB \cap PvB = \emptyset$.

Proof Indeed, if $P_1wB \cap PvB \neq \emptyset$ there exists v' in W_{M_1} with $v'w = v$. Since $\Delta - \Delta_M$ is orthogonal to Δ_M, W_{M_1} is the direct product of W_M and the subgroup W_1 generated by the s_β for $\beta \in \Delta_{P_1} - \Delta_M$. For such a β we have $s_\beta w < w$ and it follows (using [Deo] as in IV.9 Lemma 2), by induction on length, that $v_1 w \leq w$ for any $v_1 \in W_1$. Writing v' as $v_2^{-1}v_1$ with $v_1 \in W_1$ and $v_2 \in W_M$ we get $v_1w = v_2v$. But $v_2v \geq v$ and $v_1w \leq w$ so $v \leq w$ contrary to the assumption $v \in A'$. □

Let us pursue the proof of Lemma 3.

Since $f_2 \in \pi_{P,A'}$, it follows from the sublemma that, seen as an element of $\mathrm{Ind}_P^G \sigma$, it vanishes on P_1wB; but then, seen as an element of $\mathrm{Ind}_{P_1}^G \mathrm{Ind}_{P\cap M_1}^{M_1} \sigma$, it also vanishes on P_1wB. So for any $x \in P_1wB$, $f(x) = (\tau_\alpha - 1)f_1(x)$ in $\mathrm{Ind}_{P\cap M_1}^{M_1} \sigma$. Now $\dim P_1 < \dim G$ so V.8 Proposition and all its corollaries are true for M_1. As $v_{M_1}(a_\alpha) \neq 0$ we conclude from V.9 Corollary 4 that there exists $y \in \sigma_1$ with $(\tau_\alpha - 1)f_1(x) = (\tau_\alpha - 1)y$. But $\tau_\alpha - 1$ does not kill any element of $\mathrm{Ind}_{P\cap M_1}^{M_1} \sigma$, by V.4 Lemma (i), so $f_1(x) = y$ belongs to σ_1. We can choose f_1' in $\mathrm{Ind}_{P_1}^G \sigma_1 \cap \pi_{P_1,A}$ with the same restriction as f_1 on P_1wB (use V.5 Remark). Put $f_2' = f - (\tau_\alpha - 1)f_1' = (\tau_\alpha - 1)(f_1 - f_1') + f_2$. Then $f_1 - f_1'$ vanishes on P_1wB, f_2 vanishes on PwB so f_2' belongs to $\pi_{P,A'}$ and $f = (\tau_\alpha - 1)f_1' + f_2'$ belongs to $(\tau_\alpha - 1)\pi_{P_1,A} + \pi_{P,A'}$. □

We now finish the proof of V.8 Proposition. Let R be the parabolic subgroup between P and Q with $\Delta_R - \Delta_M = \{\alpha \in \Delta_Q, w^{-1}(\alpha) < 0\}$. Applying Lemma 3 successively we get $\pi_{R,A} \subset c_{R,w}\pi_{P,A} + \pi_{P,A'}$, hence the result since $\pi_{Q,A} \subset \pi_{R,A}$ and $c_{R,w} = c_{Q,w}$. □

We can get more out of that:

Lemma 4 *Let A, w, A' be as in* V.7. Then $\pi_{Q,A} \subset c_{Q,w}\pi_{P_w,A} + \pi_{Q,A'}$.

Proof V.10 Lemma gives $\pi_{P,A} \subset \pi_{P_w,A} + \pi_{P,A'}$ so from V.8 Proposition we get $\pi_{Q,A} \subset c_{Q,w}\pi_{P_w,A} + \pi_{P,A'}$. But $\pi_{P_w} \subset \pi_{Q\cap P_w}$ and $c_{Q,w}\pi_{Q\cap P_w} \subset \pi_Q$ by V.6 Proposition so $c_{Q,w}\pi_{P_w,A} \subset \pi_{Q,A}$ and the result follows. □

B) Filtration theorem for $\chi \otimes_{\mathcal{Z}_G} \mathrm{ind}_K^G V$

V.12. We now turn to the filtration theorem (I.6). For that, as before, an irreducible representation V of K is fixed, with parameter $(\psi, \Delta(V))$, but we also fix a character χ of $\mathcal{Z}_G = \mathcal{Z}_G(V)$. We let $P = MN$ be the parabolic subgroup with $\Delta_P = \Delta_0(\chi)$, so P is the smallest parabolic subgroup of G containing B such that χ extends to a character – still written χ – of $\mathcal{Z}_M = \mathcal{Z}_M(V_{N^0})$, and that character further factors through $\mathcal{Z}_M \to R_M$.

Notation For a $\mathcal{Z}_M$-module W, we put $W^\chi = \chi \otimes_{\mathcal{Z}_M} W$.

Recall that for each parabolic subgroup Q of G containing P, V_Q denotes the irreducible representation of K of parameter $(\psi, \Delta_Q \cap \Delta(V))$; we make the same identifications as in V.5. In particular we get a $\mathcal{Z}_M[G]$-submodule π_Q of $\pi_P = \mathrm{Ind}_P^G \sigma$ – we keep writing $\sigma = \mathrm{ind}_{M^0}^M V_{N^0}$. Our main interest is in π_G^χ, but its analysis goes through the π_Q^χ, in particular π_P^χ.

As σ^χ satisfies property (H) of III.15, the maximal parabolic subgroup of G to which σ^χ extends, trivially on N, has associated set of roots $\Delta_M \sqcup \Theta_{\max}$ where $\Theta_{\max}$ is the set of $\alpha \in \Delta - \Delta_M$, orthogonal to Δ_M and such that $\psi(Z^0 \cap M'_\alpha) = 1$ and $\chi(\tau_\alpha) = 1$ (III.17 Corollary).

Notation We let $\Theta = \Theta_{\max} \cap \Delta(V)$, $P_e = P_{\Delta_M \sqcup \Theta}$ and write ${}^e\sigma^\chi$ for the extension of σ^χ to P_e, trivial on N. (Note that III.22 Lemma 2 gives an identification of π_P^χ with $\mathrm{Ind}_{P_e}^G({}^e\sigma^\chi \otimes \mathrm{Ind}_P^{P_e} 1)$.)

Lemma *The inclusion $\pi_G \to \pi_{P_e}$ induces an isomorphism $\pi_G^\chi \to \pi_{P_e}^\chi$*

Proof It suffices to show that for $P_e \subset P_1 \subset P_2$ with $\Delta_{P_2} = \Delta_{P_1} \sqcup \{\alpha\}$, the natural map $\pi_{P_2}^\chi \to \pi_{P_1}^\chi$ is an isomorphism. If $\alpha \notin \Delta(V)$ or if ψ is not trivial on $Z^0 \cap M'_\alpha$ or α not orthogonal to Δ_M, then by V.6 we even have an isomorphism $R_M \otimes_{\mathcal{Z}_M} \pi_{P_2} \xrightarrow{\sim} R_M \otimes_{\mathcal{Z}_M} \pi_{P_1}$. Otherwise, $\chi(\tau_\alpha) \neq 1$ and since $(\tau_\alpha - 1)\pi_{P_1} \subset \pi_{P_2} \subset \pi_{P_1}$ by V.6, we have an isomorphism $\pi_{P_2}^\chi \xrightarrow{\sim} \pi_{P_1}^\chi$. □

V.13. **Notation** Let $\mathcal{D}$ be the set of parabolic subgroups of G between P and P_e.

- For Q, Q_1 in $\mathcal{D}$, $Q \supset Q_1$, put $c_{Q,Q_1} = \prod_{\alpha \in \Delta_Q - \Delta_{Q_1}} (\tau_\alpha - 1)$ (then $c_{Q,Q_1}\pi_{Q_1} \subset \pi_Q$ by V.6 Proposition).
- For $Q \in \mathcal{D}$, let τ_Q be the image of $\pi_Q \xrightarrow{c_{P_e,Q}} \pi_{P_e} \to \pi_{P_e}^\chi (= \pi_G^\chi)$, and let ρ_Q be the image of $\pi_Q \hookrightarrow \pi_P \to \pi_P^\chi$.

• For $Q \in \mathcal{D}$, let $Q^c \in \mathcal{D}$ be the parabolic subgroup such that $\Delta_{Q^c} - \Delta_M = \Delta_{P_e} - \Delta_Q$, and let Φ_Q, Ψ_Q be the G-equivariant maps

$$\begin{array}{lllllll}\Phi_Q & : & \tau_Q & \hookrightarrow & \pi^\chi_{P_e} & \longrightarrow & \pi^\chi_{Q^c}, \\ \Psi_Q & : & \rho_Q & \hookrightarrow & \pi^\chi_P & \stackrel{c_{Q^c,P}}{\longrightarrow} & \pi^\chi_{Q^c}.\end{array}$$

Here, the last map is obtained from $\pi_P \stackrel{c_{Q^c,P}}{\longrightarrow} \pi_{Q^c}$ by tensoring by χ.

• Let I_Q the submodule $\mathrm{Ind}^G_{P_e}({}^e\sigma^\chi \otimes \mathrm{Ind}^{P_e}_Q 1)$ of π^χ_P. In particular, $I_P = \rho_P = \pi^\chi_P$. Note also that $\tau_{P_e} = \pi^\chi_{P_e}$.

Remark 1 The maps $\pi_Q \stackrel{c_{P_e,Q}}{\longrightarrow} \pi_{P_e} \hookrightarrow \pi_{Q^c}$ and $\pi_Q \hookrightarrow \pi_P \stackrel{c_{Q^c,P}}{\longrightarrow} \pi_{Q^c}$ are equal because $c_{Q^c,P} = c_{P_e,Q}$. Therefore $\mathrm{Im}\,\Phi_Q = \mathrm{Im}\,\Psi_Q$.

Remark 2 For Q, Q_1 in $\mathcal{D}$, $Q \supset Q_1$, we have $\tau_{Q_1} \subset \tau_Q$ and $\rho_{Q_1} \supset \rho_Q$.

Our second main result in this chapter is:

Theorem *Let $Q \in \mathcal{D}$.*
(i) $\rho_Q = I_Q$.
(ii) Ker $\Psi_Q = \sum_{Q_1 \in \mathcal{D}, Q_1 \supsetneq Q} \rho_{Q_1}$.
(iii) $\mathrm{Ker}\,\Phi_Q = \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$.
(iv) Let $\mathcal{P} \subset \mathcal{D}$; then $\tau_Q \cap \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1} = \sum_{Q_1 \in \mathcal{P}} \tau_{Q \cap Q_1}$.

It implies I.6 Theorem 6:

Corollary 1 For $Q \in \mathcal{D}$, $\tau_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$ is isomorphic to $I_e(P, \sigma^\chi, Q)$.

Proof By Remark 1 we have that $\tau_Q/\,\mathrm{Ker}\,\Phi_Q$ is isomorphic to $\rho_Q/$ Ker Ψ_Q. But $\rho_Q = I_Q$ by (i) so we get by (ii) and (iii) a G-isomorphism between $\tau_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$ and $I_Q / \sum_{Q_1 \in \mathcal{D}, Q_1 \supsetneq Q} I_{Q_1}$ which is $I_e(P, \sigma^\chi, Q)$. □

Corollary 2 *Enumerate the parabolic subgroups in $\mathcal{D}$ as $P = Q_1, \ldots, Q_r = P_e$, so that $i \leq j$ if $Q_i \subset Q_j$. For $i = 0, \ldots, r$, put $I_i = \sum_{1 \leq j \leq i} \tau_{Q_j}$. Then for $i = 1, \ldots, r$, $I_i/I_{i-1} \simeq I_e(P, \sigma^\chi, Q_i)$.*

Proof For $i = 1, \ldots, r$ $I_i/I_{i-1} = \tau_{Q_i}/(\tau_{Q_i} \cap \sum_{1 \leq j < i} \tau_{Q_j})$ is also $\tau_{Q_i}/\sum_{1 \leq j < i} \tau_{Q_i \cap Q_j}$ by (iv). The assertion follows from Corollary 1. □

Remark 3 The proofs below are in fact valid more generally: it would suffice, for a given parabolic subgroup $P = MN$ of G containing B, to tensor $\mathrm{ind}^G_K V$ with the quotient of R_M in which all $\tau_\alpha - 1$ for $\alpha \in \Theta$ are killed.

Since we consider only parabolic subgroups in $\mathcal{D}$, and all the representations we consider are parabolically induced from analogously defined representations of the Levi quotient of P_e, it is enough to prove the theorem when $P_e = G$, i.e. $\Delta = \Delta_M \sqcup \Theta$, which we assume from now on.

V.14. Under that assumption $\Delta = \Delta_M \sqcup \Theta$, we prove V.13 Theorem in a succession of lemmas.

We fix $Q \in \mathcal{D}$ and let M_Q be its Levi subgroup containing M.

Lemma 1 $\rho_Q \subset I_Q$.

Proof Equality is clear when $Q = P$, so we assume $Q \supsetneq P$. For each $\alpha \in \Delta_Q - \Delta_P$, let P^α be as in V.11. By V.11 Lemma 2, ρ_{P^α} is included in I_{P^α} so a fortiori $\rho_Q \subset I_{P^\alpha}$. But the subgroup of G generated by the P^α's for $\alpha \in \Delta_Q - \Delta_P$ is Q, so $\cap_{\alpha\in\Delta_Q-\Delta_P} I_{P^\alpha} = I_Q$, and $\rho_Q \subset I_Q$. □

To prove equality in Lemma 1, we resort to filtration arguments. In the following A, w, A' are as in V.7 and $\pi_{P,A}$, $\pi_{Q,A}$ as in V.7, V.8.

Remark 1 We can also filter π_P^χ by support yielding $(\pi_P^\chi)_A \subset \pi_P^\chi$. But from V.7 Lemma we get, after tensoring with $\chi : R_M \to C$, that $\pi_{P,A} \to \pi_P^\chi$ induces an isomorphism $(\pi_{P,A})^\chi \simeq (\pi_P^\chi)_A$. We let $\pi_{P,A}^\chi$ denote $(\pi_P^\chi)_A$.

We put $\rho_{Q,A} = \rho_Q \cap \pi_{P,A}^\chi$, $I_{Q,A} = I_Q \cap \pi_{P,A}^\chi$, so $\rho_{Q,A} = \rho_Q \cap I_{Q,A}$.

Remark 2 By V.8 Corollary 2 and Corollary 3,

$$0 \to R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A} \to R_M \otimes_{\mathcal{Z}_M} \pi_Q \to R_M \otimes_{\mathcal{Z}_M} (\pi_Q/\pi_{Q,A}) \to 0$$

is an exact sequence of free R_M-modules (an extension of free R_M-modules is free and by induction $R_M \otimes_{\mathcal{Z}_M} \pi_{Q,A}$ and $R_M \otimes_{\mathcal{Z}_M} \pi_Q$ are free R_M-modules). Therefore the map $(\pi_{Q,A})^\chi \to \pi_Q^\chi$ is injective.

Lemma 2 *(i) If $w \notin W(M_Q)$ then $I_{Q,A} = I_{Q,A'}$, and $\rho_{Q,A} = \rho_{Q,A'}$.*

(ii) If $w \in W(M_Q)$ the maps $\pi_Q \to \rho_Q \to I_Q \to \pi_P^\chi$ induces isomorphisms

$$(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \simeq \rho_{Q,A}/\rho_{Q,A'} \simeq I_{Q,A}/I_{Q,A'} \simeq \pi_{P,A}^\chi/\pi_{P,A'}^\chi.$$

(iii) $\rho_{Q,A}$ is the image of $\pi_{Q,A}$ in π_P^χ.

Note $w \in W(M_Q)$ means that for $\alpha \in \Delta_Q$, $w^{-1}(\alpha) > 0$; it is equivalent to $c_{Q,w} = 1$ (V.8).

Proof (i) Let $f \in I_{Q,A} - I_{Q,A'}$; then f is not identically 0 on PwB, but its support is left Q-equivariant, so for any $v \in W_{M_Q}$, f is not identically 0 on $PvwB$. If $w \notin W(M_Q)$ we can choose $v \in W_{M_Q}$ so that $vw < w$. That implies $vw \notin A$ by minimality of w, a contradiction. So $I_{Q,A} = I_{Q,A'}$ and $\rho_{Q,A} = \rho_{Q,A'}$ follows by intersecting with ρ_Q.

(ii) Let $w \in W(M_Q)$. Then $c_{Q,w} = 1$ and V.8 Proposition gives that the map $\pi_{Q,A} \to \pi_{P,A}$ induces an isomorphism $\pi_{Q,A}/\pi_{Q,A'} \simeq \pi_{P,A}/\pi_{P,A'}$. Tensoring with χ gives an isomorphism of $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi$ onto $(\pi_{P,A})^\chi/(\pi_{P,A'})^\chi$ which is $\pi_{P,A}^\chi/\pi_{P,A'}^\chi$ by Remark 1; since the image of that isomorphism is contained in $\rho_{Q,A}/\rho_{Q,A'}$, itself contained in $I_{Q,A}/I_{Q,A'}$, we get (ii).

(iii) We prove it by descending induction on $\#A$, the case $A = W(M)$ being true by definition of ρ_Q. We assume that the result is true for A and prove it for A'. By V.11 Lemma 4 we have

$$\pi_{Q,A} \subset c_{Q,w}\pi_{P_w,A} + \pi_{Q,A'}.$$

If $w \notin W(M_Q)$ then $\chi(c_{Q,w}) = 0$. Hence $\pi_{Q,A}$ and $\pi_{Q,A'}$ have the same image in π_P^χ, which is $\rho_{Q,A}$ by induction and $\rho_{Q,A'}$ by (i). If $w \in W(M_Q)$ we use the isomorphism $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \simeq \rho_{Q,A}/\rho_{Q,A'}$ in (ii). Since $(\pi_{Q,A})^\chi \to \rho_{Q,A}$ is surjective by induction, $(\pi_{Q,A'})^\chi \to \rho_{Q,A'}$ has to be surjective too. □

Lemma 3 $\rho_Q = I_Q$.

Proof By induction on $\#A$: if $\rho_{Q,A'} = I_{Q,A'}$, then Lemma 2 (i), (ii), and Lemma 1 give $\rho_{Q,A} = I_{Q,A}$. □

Lemma 4 *For $Q_1 \in \mathcal{D}$, $Q_1 \supsetneq Q$,* Ker Ψ_Q *contains ρ_{Q_1}.*

Proof It enough to show that the composite map $\pi^\chi_{Q_1} \to \pi^\chi_Q \to \rho_Q \xrightarrow{\Psi_Q} \pi^\chi_{Q^c}$ is 0. But it factors as $\pi^\chi_{Q_1} \to \pi^\chi_Q \xrightarrow{c_{G,Q}} \pi^\chi_G \to \pi^\chi_{Q^c}$ since $c_{Q^c,P} = c_{G,Q}$. From $c_{G,Q} = c_{Q_1,Q}c_{G,Q_1}$ we get $c_{G,Q}\pi^\chi_{Q_1} = c_{Q_1,Q}c_{G,Q_1}\pi^\chi_{Q_1} \subset c_{Q_1,Q}\pi^\chi_G$ which is 0 since $\chi(c_{Q_1,Q}) = 0$. □

Lemma 5 Ker $\Psi_Q \subset \sum_{Q_1\in\mathcal{D},Q_1\supsetneq Q} \rho_{Q_1}$.

Proof We show by induction on $\#A$ that

$$(*) \qquad \text{Ker } \Psi_Q \cap \pi^\chi_{P,A} \subset \sum_{Q_1\in\mathcal{D},Q_1\supsetneq Q} \rho_{Q_1}.$$

We assume that $(*)$ is true for A' and prove it for A. Note that Ker $\Psi_Q \subset \rho_Q \subset \pi^\chi_P$ so Ker $\Psi_Q \cap \pi^\chi_{P,A} =$ Ker $\Psi_Q \cap \rho_{Q,A}$. If $w \notin W(M_Q)$ then $\rho_{Q,A} = \rho_{Q,A'}$ by Lemma 2 (i), so the result is immediate. Assume $w \in W(M_Q)$. On $\rho_{Q,A}/\rho_{Q,A'}$, Ψ_Q induces $\bar\Psi_Q : \rho_{Q,A}/\rho_{Q,A'} \to \pi^\chi_{P,A}/\pi^\chi_{P,A'} \xrightarrow{c_{Q^c,P}} (\pi_{Q^c,A})^\chi/(\pi_{Q^c,A'})^\chi$. By Lemma 2(ii), the first map is an isomorphism, so we focus on the second map.

Notation Put $d^Q_w = \prod_{\alpha\in\Delta-\Delta_Q,w^{-1}(\alpha)>0} (\tau_\alpha - 1)$, so that $c_{Q^c,P} = d^Q_w c_{Q^c,w}$ because $\Delta - \Delta_Q = \Delta_{Q^c} - \Delta_M$.

By V.8 Proposition and the remark before it, $c_{Q^c,w}$ gives an isomorphism $\pi_{P,A}/\pi_{P,A'} \xrightarrow{\sim} \pi_{Q^c,A}/\pi_{Q^c,A'}$. If $d^Q_w = 1$ then $\bar\Psi_Q$ is injective and Ker $\Psi_Q \cap \pi^\chi_{P,A} =$ Ker $\Psi_Q \cap \pi^\chi_{P,A'}$ so $(*)$ follows from the induction hypothesis. Let $d^Q_w \neq 1$, choose $\alpha \in \Delta - \Delta_Q$ with $w^{-1}(\alpha) > 0$ and let Q^α be the parabolic subgroup of G corresponding to $\Delta_Q \cup \{\alpha\}$. Then $w \in W(M_{Q^\alpha})$ and Lemma 2 (ii) gives the isomorphism

$$\rho_{Q^\alpha,A}/\rho_{Q^\alpha,A'} \xrightarrow{\sim} \pi^\chi_{P,A}/\pi^\chi_{P,A'}.$$

Let $f \in$ Ker $\Psi_Q \cap \pi^\chi_{P,A}$, and choose $f' \in \rho_{Q^\alpha,A}$ with $f - f' \in \pi^\chi_{P,A'}$. As $f' \in$ Ker Ψ_Q by Lemma 4, $f - f' \in$ Ker Ψ_Q so $f - f'$ belongs to $\sum_{Q_1\in\mathcal{D},Q_1\supsetneq Q} \rho_{Q_1}$ by induction; as f' also belongs to that space, the result follows. □

V.15. We have proved (i) and (ii) in V.13 Theorem, and now we turn to part (iii). Describing Ker Φ_Q is analogous to describing Ker Ψ_Q. We let A, w, A' be as before, and let $\tau_{Q,A} \subset \tau_Q$ be the image of $\pi_{Q,A}$ (or $(\pi_{Q,A})^\chi$) in $\pi^\chi_G = \tau_G$, via the map $\pi^\chi_Q \xrightarrow{c_{G,Q}} \pi^\chi_G$. We observe that $\tau_{Q,A'} \subset \tau_{Q,A}$ and $\tau_{Q_1,A} \subset \tau_{Q,A}$ if $Q_1 \subset Q$ in $\mathcal{D}$. We note also that by V.14 Remark 2 we have $(\pi_{G,A})^\chi = \tau_{G,A} \subset \pi^\chi_G$.

Lemma 6 *(i) If for some $\alpha \in \Delta - \Delta_Q$, $w^{-1}(\alpha) > 0$ then $\tau_{Q,A} = \tau_{Q,A'}$. Otherwise, the natural maps $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \twoheadrightarrow \tau_{Q,A}/\tau_{Q,A'} \to \tau_{G,A}/\tau_{G,A'}$ are isomorphisms.*

(ii) $\tau_{Q,A} = \tau_{G,A} \cap \tau_Q$.

Proof (i) Let $\phi \in \pi_{Q,A}$. With P_w as in V.10, V.11 Lemma 4 implies that we can write $\phi = c_{Q,w}\phi_w + \phi'$ with $\phi_w \in \pi_{P_w,A}$ and $\phi' \in \pi_{Q,A'}$. Since $d_w^Q c_{G,w} = c_{G,Q}c_{Q,w}$ we get $c_{G,Q}\phi = d_w^Q(c_{G,w}\phi_w) + c_{G,Q}\phi'$. But $c_{G,w} = c_{G,P_w}$ so $c_{G,w}\phi_w$ belongs to π_G by V.6 Proposition. In the first case of (i), $\chi(d_w^Q) = 0$, so ϕ has the same image as ϕ' in τ_Q; this implies $\tau_{Q,A} = \tau_{Q,A'}$. Let us assume we are in the second case of (i), so $d_w^Q = 1$. Consider the natural inclusions

$$\pi_{G,A}/\pi_{G,A'} \hookrightarrow \pi_{Q,A}/\pi_{Q,A'} \hookrightarrow \pi_{P,A}/\pi_{P,A'}.$$

By V.8 Proposition, the first space is $c_{G,w}(\pi_{P,A}/\pi_{P,A'})$ and the second is $c_{Q,w}(\pi_{P,A}/\pi_{P,A'})$. Consequently, $c_{G,Q}(\pi_{Q,A}/\pi_{Q,A'}) = \pi_{G,A}/\pi_{G,A'}$ since $d_w^Q = 1$. Thus $c_{G,Q}$ induces a surjective map of $\pi_{Q,A}/\pi_{Q,A'}$ onto $\pi_{G,A}/\pi_{G,A'}$. But by V.4 Lemma (i) (applied to M) $c_{G,Q}$ acts injectively on σ hence on $\pi_{P,A}/\pi_{P,A'}$, so we actually get an isomorphism. Tensoring with χ we get an isomorphism $(\pi_{Q,A})^\chi/(\pi_{Q,A'})^\chi \to \tau_{G,A}/\tau_{G,A'}$; but this factors as in the statement of (i), so (i) follows again.

(ii) We proceed by descending induction on $\#A$, the case $A = W(M)$ being obvious. The containment $\tau_{Q,A'} \subset \tau_{G,A'} \cap \tau_Q$ is clear, and we have $\tau_{Q,A} = \tau_{G,A} \cap \tau_Q$ by induction. In the first case of (i) $\tau_{Q,A'} = \tau_{Q,A} = \tau_{G,A} \cap \tau_Q \supset \tau_{G,A'} \cap \tau_Q$ so $\tau_{Q,A'} = \tau_{G,A'} \cap \tau_Q$. In the second case of (i), $\tau_{Q,A}/\tau_{Q,A'} \to \tau_{G,A}/\tau_{G,A'}$ is an isomorphism; as moreover $\tau_{G,A'} \cap \tau_Q \subset \tau_{Q,A}$ by induction, the result follows. □

Lemma 7 *For $Q_1 \in \mathcal{D}$, $Q_1 \subsetneq Q$, then $\tau_{Q_1} \subset \operatorname{Ker} \Phi_Q$.*

Proof Let P_1 be the parabolic subgroup corresponding to $\Delta_{Q_1} \sqcup (\Delta - \Delta_Q) = \Delta_{Q_1} \cup \Delta_{Q^c}$. Since $Q_1 \subsetneq Q$, we get $P_1 \subsetneq G$. We have $c_{P_1,Q_1}\pi_{Q_1} \subset \pi_{P_1} \subset \pi_{Q^c}$ so $c_{G,Q_1}\pi_{Q_1} \subset c_{G,P_1}\pi_{Q^c}$. As $\chi(c_{G,P_1}) = 0$ the image of $\pi_{Q_1} \xrightarrow{c_{Q^c,Q_1}} \pi_{Q^c} \to \pi^\chi_{Q^c}$ is 0; but that image is $\Phi_Q(\tau_{Q_1})$. □

Lemma 8 $\operatorname{Ker} \Phi_Q \subset \sum_{Q_1 \in \mathcal{D}, Q_1 \subsetneq Q} \tau_{Q_1}$.

Proof We prove that Ker $\Phi_Q \cap \tau_{G,A}$ is contained in the right-hand side, by induction on $\#A$. In the first case of Lemma 6 (i), $\tau_{Q,A} = \tau_{Q,A'}$, so $\tau_{G,A} \cap \tau_Q = \tau_{G,A'} \cap \tau_Q$ by Lemma 6 (ii). Consequently, Ker $\Phi_Q \cap \tau_{G,A} = \operatorname{Ker} \Phi_Q \cap \tau_{G,A'}$ and we are done. So we assume that for all $\alpha \in \Delta - \Delta_Q = \Delta_{Q^c} - \Delta_P$ we have $w^{-1}(\alpha) < 0$. On $\tau_{Q,A}/\tau_{Q,A'}$, Φ_Q induces $\bar{\Phi}_Q : \tau_{Q,A}/\tau_{Q,A'} \to (\pi_{G,A})^\chi/(\pi_{G,A'})^\chi \longrightarrow (\pi_{Q^c,A})^\chi/(\pi_{Q^c,A'})^\chi$, where the first map is an isomorphism by Lemma 6 (i), and the second comes, upon tensoring with χ, from the inclusion of $\pi_{G,A}/\pi_{G,A'}$ into $\pi_{Q^c,A}/\pi_{Q^c,A'}$. By V.8 Proposition, we have, inside $\pi_{P,A}/\pi_{P,A'}$, $\pi_{G,A}/\pi_{G,A'} = c_{G,w}(\pi_{P,A}/\pi_{P,A'})$, and $\pi_{Q^c,A}/\pi_{Q^c,A'} = c_{Q^c,w}(\pi_{P,A}/\pi_{P,A'})$. If for all $\alpha \in \Delta - \Delta_{Q^c}$ we have $w^{-1}(\alpha) > 0$, then $c_{G,w} = c_{Q^c,w}$, and $\pi_{G,A}/\pi_{G,A'} = \pi_{Q^c,A}/\pi_{Q^c,A'}$; thus Ker $\Phi_Q \cap \tau_{G,A} = \operatorname{Ker} \Phi_Q \cap \tau_{G,A'}$, so we conclude by induction. In the opposite case, choose $\alpha \in \Delta - \Delta_{Q^c} = \Delta_Q - \Delta_P$ with $w^{-1}(\alpha) < 0$, and let Q_α correspond to $\Delta_Q - \{\alpha\}$. Then $\tau_{Q_\alpha,A}/\tau_{Q_\alpha,A'} \to \tau_{G,A}/\tau_{G,A'}$ is an isomorphism by Lemma 6 (i). If $f \in \operatorname{Ker} \Phi_Q \cap \tau_{Q,A}$, there is $f' \in \tau_{Q_\alpha,A}$ with $f - f' \in \tau_{G,A'}$. As $\tau_{Q_\alpha} \subset \tau_Q$ we have $f - f' \in \tau_{G,A'} \cap \tau_Q = \tau_{Q,A'}$ by Lemma 6 (ii). Lemma 7 gives $\Phi_Q(f') = 0$, so $\Phi_Q(f - f') = 0$ and by induction $f - f'$ belongs to the right-hand side of Lemma 8; since $f' \in \tau_{Q_\alpha}$ also belongs to that space, so does f. □

V.16. It remains to prove (iv) of V.13 Theorem.

Lemma 9 *Let $\mathcal{P} \subset \mathcal{D}$. Then* $\left(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1}\right) \cap \tau_{G,A} = \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A}$.

Proof The containment $\supset$ is clear; we prove the other direction by descending induction on $\#A$. Let $\mathcal{P}^- = \{Q_1 \in \mathcal{P} \mid w^{-1}(\alpha) < 0 \text{ for any } \alpha \in \Delta - \Delta_{Q_1}\}$. If $\mathcal{P}^-$ is empty then $\tau_{Q_1,A} = \tau_{Q_1,A'}$ for any $Q_1 \in \mathcal{P}$ (Lemma 6 (i)), and we have nothing to prove. Assume $\mathcal{P}^-$ is not empty, and put $Q_\cap = \bigcap_{Q_1 \in \mathcal{P}^-} Q_1$. Then for $\alpha \in \Delta - \Delta_{Q_\cap}$ we have $w^{-1}(\alpha) < 0$ so by Lemma 6 (i) the map $\tau_{Q_\cap,A} \to \tau_{G,A}/\tau_{G,A'}$ is surjective. For $Q_1 \in \mathcal{P}$ let $f_{Q_1} \in \tau_{Q_1}$ be chosen so that $\sum_{Q_1 \in \mathcal{P}} f_{Q_1} \in \tau_{G,A'}$; by the inductive hypothesis we may assume that all $f_{Q_1} \in \tau_{Q_1,A}$. For $Q_1 \in \mathcal{P} - \mathcal{P}^-$, we even have $f_{Q_1} \in \tau_{Q_1,A'}$ by Lemma 6 (i). Fix $Q_2 \in \mathcal{P}^-$; for $Q_1 \in \mathcal{P}^-$, $Q_1 \neq Q_2$ choose $f'_{Q_1} \in \tau_{Q_\cap,A}$ with $f_{Q_1} - f'_{Q_1} \in \tau_{G,A'}$. Since $\tau_{Q_\cap,A} \subset \tau_{Q_1,A}$, $f_{Q_1} - f'_{Q_1}$ belongs to $\tau_{G,A'} \cap \tau_{Q_1} = \tau_{Q_1,A'}$. So $\sum_{Q_1 \in \mathcal{P}} f_{Q_1}$ appears as $f_{Q_2} + \sum_{Q_1 \in \mathcal{P}^-, Q_1 \neq Q_2} f'_{Q_1}$ plus terms in $\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}$. But for $Q_1 \in \mathcal{P}^-$, $Q_1 \neq Q_2$, f'_{Q_1} belongs to $\tau_{Q_\cap,A} \subset \tau_{Q_2,A}$ so $f_{Q_2} + \sum_{Q_1 \in \mathcal{P}^-, Q_1 \neq Q_2} f'_{Q_1}$ belongs to $\tau_{Q_2} \cap \tau_{G,A'} = \tau_{Q_2,A'} \subset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}$. $\square$

We finally prove (iv) of V.13 Theorem. Fix $Q \in \mathcal{D}$ and let $\mathcal{P} \subset \mathcal{D}$. It is clear that

$$\Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1}\Big) \cap \tau_Q \supset \sum_{Q_1 \in \mathcal{P}} (\tau_{Q_1} \cap \tau_Q) \supset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q}.$$

We prove now

$$\Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1}\Big) \cap \tau_{Q,A} \subset \sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q} \text{ by induction on } \#A.$$

If there is $\alpha \in \Delta - \Delta_Q$ with $w^{-1}(\alpha) > 0$ then $\tau_{Q,A} = \tau_{Q,A'}$ (Lemma 6 (i)) and there is nothing to prove, so we assume the contrary. By Lemma 9

$$\Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1}\Big) \cap \tau_{Q,A} = \Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A}\Big) \cap \tau_{Q,A}.$$

Let $\mathcal{P}^- \subset \mathcal{P}$ be the same subset as in the proof of Lemma 9. If $\mathcal{P}^-$ is empty, then $\tau_{Q_1,A} = \tau_{Q_1,A'}$ for any Q_1 in $\mathcal{P}$. Hence

$$\Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1}\Big) \cap \tau_{Q,A} = \Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}\Big) \cap \tau_{Q,A} = \Big(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}\Big) \cap \tau_{Q,A'},$$

and the result follows from Lemma 9 and the induction hypothesis. Now assume $\mathcal{P}^- \neq \emptyset$, and write $Q_\cap = Q \cap \bigcap_{Q_1 \in \mathcal{P}^-} Q_1$; then for $\alpha \in \Delta - \Delta_{Q_\cap}$, $w^{-1}(\alpha) < 0$ and again $\tau_{Q_\cap,A} \to \tau_{G,A}/\tau_{G,A'}$ is surjective. For $Q_1 \in \mathcal{P}$ let $f_{Q_1} \in \tau_{Q_1}$ be chosen so that $\sum_{Q_1 \in \mathcal{P}} f_{Q_1} \in \tau_{Q,A}$. By Lemma 9 we may assume $f_{Q_1} \in \tau_{Q,A}$. For $Q_1 \in \mathcal{P}^-$, choose $f'_{Q_1} \in \tau_{Q_\cap,A}$ with $f_{Q_1} - f'_{Q_1} \in \tau_{G,A'}$ (then $f_{Q_1} - f'_{Q_1} \in \tau_{Q_1,A'}$). Write

$$\sum_{Q_1 \in \mathcal{P}} f_{Q_1} = \sum_{Q_1 \in \mathcal{P}^-} (f_{Q_1} - f'_{Q_1}) + \sum_{Q_1 \in \mathcal{P} - \mathcal{P}^-} f_{Q_1} + \sum_{Q_1 \in \mathcal{P}^-} f'_{Q_1}.$$

We examine the right hand side. The last term belongs to $\tau_{Q_\cap,A} \subset \tau_{Q,A}$, so the sum of the first two belongs to τ_Q. As each summand in those two terms indexed by Q_1 is in $\tau_{Q_1,A'}$, their sum belongs to $(\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1,A'}) \cap \tau_Q$, which is in $\sum_{Q_1 \in \mathcal{P}} \tau_{Q_1 \cap Q}$ by the induction hypothesis. But for $Q_1 \in \mathcal{P}^-$, $f'_{Q_1} \in \tau_{Q_\cap, A}$, and $\tau_{Q_\cap} \subset \tau_{Q_1 \cap Q}$ since $Q_\cap \subset Q_1 \cap Q$. Thus the third term also belongs to $\sum_{Q_1 \in \mathcal{P}^-} \tau_{Q_1 \cap Q}$. □

VI. Consequences of the classification

VI.1. We recall from I.3 that a representation of G is supercuspidal if it is irreducible, admissible, and does not appear as a subquotient of a parabolically induced representation $\mathrm{Ind}_P^G \sigma$, where P is a proper parabolic subgroup of G and σ an irreducible admissible representation of the Levi quotient of P.

It is well known [BL1, Br] that there exist irreducible admissible supercuspidal representations when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, therefore the following proposition shows that we cannot drop the condition that σ be irreducible admissible in the definition of supercuspidality, unlike for representations of G over a field of characteristic different from p.

Proposition *Any irreducible representation π of G is a subquotient of $\mathrm{Ind}_B^G \sigma$ for some representation σ of Z.*

Proof The smoothness of π implies that π has a weight V. The irreducibility of π implies that π is a quotient of $\mathrm{ind}_K^G V$. The representation $\mathrm{ind}_K^G V$ embeds in $\mathrm{Ind}_B^G(\mathrm{ind}_{Z^0}^Z V_{U^0})$ by the intertwiner $\mathcal{I}$ of III.13. □

VI.2. We derive the desired consequences of I.5 Theorem 4. Mostly we follow the pattern of [He2].

We now prove I.5 Theorem 5, which we recall.

Theorem *Let π be an irreducible admissible representation of G. Then π is supercuspidal if and only if π is supersingular.*

As observed in the introduction, this theorem shows that the notion of supersingularity, for an irreducible admissible representation of G, is independent of the choices of $\mathbf{S}, \mathbf{B}, K$.

Proof Let π be supercuspidal. By I.5 Theorem 4, there is a supersingular B-triple (P, σ, Q) such that $\pi \simeq I(P, \sigma, Q)$. By III.24 Proposition, $I(P, \sigma, Q)$ is a component of $\mathrm{Ind}_P^G \sigma$, so $P = G$ and $\pi \simeq \sigma$ is supersingular.

Let π be supersingular. Assume it occurs as a subquotient of $\mathrm{Ind}_P^G \sigma$ for a parabolic subgroup P of G and an irreducible admissible representation σ of the Levi quotient M of P; we may and do assume that P contains B. By I.5 Theorem 4, III.24 Proposition, and transitivity of parabolic induction, we may assume that σ is supersingular. By III.24 Proposition, π is isomorphic to some $I(P, \sigma, Q)$ and I.5 Theorem 4 implies that $P = G$, so that π is indeed supercuspidal. □

Theorems 1 to 3 in Section I.3 are now rather immediate. They follow from I.5 Theorem 4 and the following elementary observations:

(i) Any triple is G-conjugate to a B-triple.
(ii) A B-triple is supersingular if and only if it is supercuspidal (by the theorem).
(iii) $I(P, \sigma, Q) \simeq I(P', \sigma', Q')$ if the triples (P, σ, Q), (P', σ', Q') are G-conjugate.

VI.3. We also have the desired consequence about supercuspidal support.

Proposition *Let π be an irreducible admissible representation of G. Then there is a parabolic subgroup P of G and a supercuspidal representation σ of the Levi quotient of P such that π is a subquotient of $\operatorname{Ind}_P^G \sigma$. If P_1 is a parabolic subgroup of G and σ_1 a supercuspidal representation of the Levi quotient of P_1 such that π is a subquotient of $\operatorname{Ind}_{P_1}^G \sigma_1$, then there is g in G such that $P_1 = gPg^{-1}$ and that σ_1 is equivalent to $x \mapsto \sigma(g^{-1}xg)$.*

Proof By I.3 Theorem 3, π has the form $I(P, \sigma, Q)$ for some supercuspidal triple (P, σ, Q) and the first assertion comes from III.24 Proposition. The uniqueness assertion is derived in the same way from I.3 Theorem 2. □

We say that the **supercuspidal support** of π is the class of (P, σ) for the equivalence relation appearing in the proposition.

VI.4. We give one more consequence mentioned in the introduction.

Proposition *Let (P, σ, Q) be a B-triple. Assume that σ is a supercuspidal (or equivalently, supersingular) representation of M. Then $I(P, \sigma, Q)$ is finite-dimensional if and only if $P = B$ and $Q = G$.*

Proof As Z is compact mod centre, any irreducible representation τ of Z is finite dimensional [Hn, Vig1] and consequently supercuspidal. If $P(\tau) = G$ then $I(B, \tau, G) = {}^e\tau$ is finite dimensional. Conversely, let π be a finite-dimensional irreducible representation of G. Then its kernel is an open normal subgroup of G. Considering $\iota : G^{\text{is}} \to G$ as in Chapter II, $\operatorname{Ker}(\sigma \circ \iota)$ is an open normal subgroup of G^{is} which by II.3 Proposition has to be G^{is} itself. Thus π is trivial on G' and since $G = ZG'$, π restricts to an irreducible (supercuspidal) representation τ of Z; we have $P(\tau) = G$ and ${}^e\tau = \pi$, $\pi = I(B, \tau, G)$. □

VI.5. It is worth noting that our results recover the classifications obtained previously in special cases. Keep the notation of Chapter III. When $\mathbf{G}$ is split, then for $\alpha \in \Delta$, $Z \cap M'_\alpha$ is simply the image in $Z = S$ of the coroot $\alpha^\vee$, so our classification is the same as that of [Abe]; it also gives the classification of [He2] for $\mathbf{G} = \mathrm{GL}_n$. Other special cases are worth mentioning: groups of semisimple rank 1 and inner forms of GL_n. Of course if $\mathbf{G}$ has relative rank 0, all irreducible representations of G are finite dimensional and supercuspidal, and our classification theorem says nothing. If $\mathbf{G}$ has relative semisimple rank 1, the classification is rather simple (see also [BL1, BL2, Abd, Che, Ko, Ly2]). An irreducible admissible representation π of G falls into one (and only one) of the following cases:

1) π is supercuspidal (hence infinite dimensional), i.e. $\pi \simeq I(G, \pi, G)$.
2) π is finite dimensional; it is then trivial on G' and restricts to an irreducible representation τ of Z, trivial on $Z \cap G'$, and $\pi \simeq I(B, \tau, G)$.
3) $\pi \simeq \sigma \otimes \mathrm{St}_B^G$ where σ is as in 2), i.e. $\pi \simeq I(B, \sigma|_Z, B)$.
4) $\pi \simeq I(B, \tau, B)$ where τ is an irreducible representation of Z (hence finite dimensional and supercuspidal) which is not trivial on $Z \cap G'$.

VI.6. Let us briefly consider the case of inner forms of general linear groups. Thus $\mathbf{G} = \mathrm{GL}_{n/D}$ where D is a central division algebra of finite degree over F. We take for S the diagonal subgroup $(F^\times)^n$ (so that Z is the diagonal subgroup $(D^\times)^n$), and for B the upper triangular subgroup. We can take $K = \mathrm{GL}_n(\mathcal{O}_D)$ where $\mathcal{O}_D$ is the ring of integers of D; all other special parahoric subgroups of G are conjugate to K.

A parabolic subgroup P of G containing B is an upper triangular block subgroup, and the corresponding Levi subgroup M is the block diagonal subgroup. If the successive blocks down the diagonal have size $n_1, \dots, n_r$, then M appears as $M_1 \times \cdots \times M_r$, $M_i = \mathrm{GL}_{n_i}(D)$ and an irreducible admissible representation of M factors as a tensor product $\pi_1 \otimes \cdots \otimes \pi_r$, where π_i is an irreducible admissible representation of M_i for $i = 1, \dots, r$ determined up to isomorphism. (Conversely such a tensor product is an irreducible admissible representation of M: the reader can devise a proof as suggested in [He2], perhaps using [HV2, 7.10 Lemma].) Note that the group G' is the kernel of the non-commutative determinant $\det : G \to F^\times$. Parameters for the irreducible admissible representations of G can then be described in a way entirely parallel to the case $D = F$ obtained in [He2]. (The cases of $\mathrm{GL}_n(D)$ where $n \leq 3$ are treated in T. Ly's Ph.D. thesis [Ly2], [Ly3, Chapter 3].)

We simply state the results, leaving to the reader the translation from our classification in this paper.

For $i = 1, \dots, r$ let π_i be a representation of M_i which is either supercuspidal or of the form $\chi_i \circ \det$ for some character $\chi_i : F^\times \to C^\times$; if for two consecutive indices i, $i+1$ we have $\pi_i = \chi_i \circ \det$ and $\pi_{i+1} = \chi_{i+1} \circ \det$, assume $\chi_i \neq \chi_{i+1}$.

For each index i such that $\pi_i = \chi_i \circ \det$, choose an upper (block) triangular parabolic subgroup Q_i of M_i, and put $\sigma_i = (\chi_i \circ \det) \otimes \mathrm{St}_{Q_i}^{M_i}$; for other indices i put $\sigma_i = \pi_i$. Then $\mathrm{Ind}_P^G(\sigma_1 \otimes \cdots \otimes \sigma_r)$ is irreducible and admissible. Conversely any irreducible admissible representation of G has such a shape, where the integers $n_1, \dots, n_r$, the parabolic subgroups Q_i of $M_i = \mathrm{GL}_{n_i}(D)$, and the isomorphism classes of the π_i, are determined.

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