CHARACTERIZATION OF THE APARTMENT. [2]

We first proceed with the construction of the space A; the set Φ_a and the X_α's will be defined in §§1.6 and 1.4. The relations (5) show us the way. The group $X^*(Z)$ of K-rational characters of Z can be identified with a subgroup of finite index of X^*. Let $\nu: Z(K) \to V$ be the homomorphism defined by

$$(1) \qquad \chi(\nu(z)) = -\,\omega(\chi(z)) \quad \text{for } z \in Z(K) \text{ and } \chi \in X^*(Z),$$

and let Z_c denote the kernel of ν. Then, $\Lambda = Z(K)/Z_c$ is a free abelian group of rank $\dim S = \dim V$. The quotient $\tilde{W} = N(K)/Z_c$ is an extension of the finite group ${}^v\tilde{W}$ by Λ. Therefore, there is an affine space A ($= A(G, S, K)$) under V and an extension of ν to a homomorphism, which we shall also denote by ν, of N in the group of affine transformations of A. If G is semisimple, the system (A, ν) is canonical, that is, unique up to unique isomorphism. Otherwise, it is only unique up to isomorphism, but one can, following G. Rousseau **[19]**, "canonify" it as follows: calling $\mathscr{D}G^\circ$ the derived group of G° and S_1 the maximal split torus of the center of G°, one takes for A the direct product of $A(\mathscr{D}G^\circ, G^\circ \cap S, K)$ (which is canonical) and $X_*(S_1) \otimes \boldsymbol{R}$ .The affine space A is called the *apartment of* S (relative to G and K). The group $N(K)$ operates on A through $\tilde{W}$.

This is one of the questions I got asked most often regarding Tits' article. What does this paragraph mean? It is not difficult, but not entirely obvious either. The detail can be found in Landvogt's book.

Explanation. The principle is this: suppose that you have an extension of groups

$$1 \to \Lambda \to N \to W \to 1,$$

where Λ is a free abelian group of finite rank, normal in N, and W is finite. Let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then we can pull out the above diagram with $\Lambda \hookrightarrow V$ and get

$$1 \to V \to N' \to W \to 1.$$

If you represent the first extension by a class in $H^2(W, \Lambda)$, the second extension is represented by the image of that class in $H^2(W, V)$. But $H^2(W, V) = 0$, so the second extension is trivial.

Therefore, $N' \simeq W \ltimes V$. The isomorphism may not be unique, but the obstruction to uniqueness lies in $H^1(W, V)$, which is again 0, so this isomorphism is actually unique.

We notice that the action map $a : W \to \mathrm{Aut}_{\mathrm{group}}(V)$ is induced from $W \to \mathrm{Aut}_{\mathrm{group}}(\Lambda) = \mathrm{GL}_{\mathbb{Z}}(\Lambda)$, hence factors through $\mathrm{GL}_{\mathbb{R}}(V)$. Thus it is obvious that there exists a pair (A, f) such that A affine space under V (i.e. a principal homogeneous space of V), $f : N \to \mathrm{Aut}_{\mathrm{affine}}(A)$ such that $f(\lambda)$ is translation by λ for all $\lambda \in \Lambda$, and $d\big(f(n)\big) = a(\bar{n})$ for all $n \in N$, where $\bar{n}$ is the image of n in W.

Now assume that $V^W = 0$. Suppose that both the pairs (A, f) and (A', f') have the above property. We claim that there is a unique isomorphism $(A, f) \simeq (A', f')$. Indeed, a simple analysis shows that the obstruction to uniqueness lies in $V^W = H^0(W, V) = 0$.

[2]This is the beginning of the 2nd lecture, which is the most fragmented one, offering various complements to Tits' article.

This, when applied to the context in [Tits, 1.2], shows that for semisimple G, $A(G, S, K)$ is uniquely characterized as an affine space (up to a unique isomorphism). For reductive G, the reduced apartment has the same uniqueness.

Extended building versus reduced building.

[Tits] deals exclusively with the extended building (aka enlarged building, reductive building). But throughout [BT], "building" usually mean the reduced building (aka semi-simple building).

We recall that the reduced building is really canonically defined. The extended building is canonical in the sense that we can "canonify" the definition. But this is somewhat artificial and hence the behavior is not ideal. The main reason to favor the extended building is this: *when the center of G is of split rank > 0, the stabilizer of points on $\mathcal{B}_{\text{red}}$ is no longer a compact subgroup of $G(K)$.*

We now recall that the construction of $\mathcal{B}_{\text{ext}}$ from $\mathcal{B}_{\text{red}}$. It is completely analogous to going from $\mathcal{S}_{\text{red}}$ to $\mathcal{S} = \mathcal{S}_{\text{ext}}$. We put

$$G(K)^1 = \{g \in G(K) : \operatorname{ord} \chi(g) = 0 \quad \forall \chi \in \operatorname{Hom}_K(G, \mathbb{G}_{\text{m}})\}.$$

Then $G(K)/G(K)^1$ is a finite generated abelian group, and there is an isomorphism

$$\left(G(K)/G(K)^1\right) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R},$$

where Z is the center of G. We let $G(K)$ acts on the vector space $X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R}$ by translations via the above isomorphism, and define $\mathcal{B}_{\text{ext}}$ as the product of two $G(K)$-sets: $\mathcal{B}_{\text{ext}} = \mathcal{B}_{\text{red}} \times (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R})$.

There is another way of dealing with this, and this viewpoint is also often used in [BT]: *instead of using $G(K)$ and $\mathcal{B}_{\text{ext}}$, we use $G(K)^1$ and $\mathcal{B}_{\text{red}}$.*

In fact, since most results in [BT] are stated using $\mathcal{B}_{\text{red}}$, it is often necessary to do the above. Notice that this means that the results in [BT] often actually apply to $G(K)^1$ instead of $G(K)$. Therefore, one gets decomposition theorems etc. for $G(K)^1$. Then one does a little bit more work to get the results for $G(K)$.

The maximal bounded subgroups. I want to bring attention to the fact that "maximal bounded subgroups", "stabilizer of vertices", and "maximal parahoric subgroups" are all different concepts. Although Tits' article is rather clear about this, this is still a common misconception (probably because people tend to extrapolate the easier case of BN-pairs). A few examples should impress you about this. For simplicity, assume that K is a locally compact non-archimedean field.

It is a consequence of Bruhat-Tits fixed-point theorem that every maximal compact subgroup of $G(K)$ is the stabilizer of some point on the building of G. But not all such stabilizers are maximal compact subgroups.