QUESTIONS ON MOD p REPRESENTATIONS OF REDUCTIVE p-ADIC GROUPS

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0) **Introduction**

We compiled these questions for the workshop *Geometric methods in the mod p local Langlands correspondence* held in June 2016 at the Centro di Ricerca Matematica Ennio de Giorgi in Pisa. We thank the organizers, Michael Harris and Peter Schneider, for inviting us.

The following is a preliminary and a bit informal discussion of questions raised by our work on modulo p admissible smooth representations of reductive p-adic groups, for which some kind of answer is required if we want to venture into derived algebraic geometry for further study.

1) **Admissibility questions**

The framework is the following: p is a prime number, F is a finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((T))$, G is a connected reductive F-group, R is the coefficient field, which unless otherwise stated, is algebraically closed of characteristic p. We examine smooth R-representations of $G(F)$, where a representation of $G(F)$ on an R-vector space V is *smooth* if the $G(F)$-stabilizer of any vector $v \in V$ is open. Such a representation is *admissible* if moreover the subspace V^J of fixed vectors fixed under any open subgroup J of $G(F)$ has finite dimension; actually it is enough to require that for one open pro-p subgroup of $G(F)$.

Our joint work [AHHV17] gives a complete classification of irreducible admissible R-representations of $G(F)$ in terms of supercuspidal R-representations of Levi subgroups of $G(F)$ – where a supercuspidal representation is an irreducible admissible representation which is not a subquotient of a representation obtained by parabolic induction of an irreducible admissible representation of a proper Levi subgroup.

The requirement of admissibility, both in the definition of supercuspidal and in the classification, is a bit awkward. Indeed, if C is an algebraically closed field of characteristic different from p, it is known that an irreducible smooth C-representation of $G(F)$ is admissible.

Question 1. *Is any irreducible smooth R-representation of $G(F)$ admissible?*

The answer is yes when $G(F) = \mathrm{GL}(2, \mathbb{Q}_p)$ [Ber12]. It is also yes when G is anisotropic modulo its centre: in that case, $G(F)$ divided by its centre is compact and all smooth irreducible representations of $G(F)$ are finite-dimensional.

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An affirmative answer to Question 1 has a number of desirable consequences. In an irreducible admissible R-representation π of $G(F)$ the centre of $G(F)$ acts via a character, called the central character of π. The following is weaker than Question 1.

Question 2. *Does any irreducible smooth R-representation of $G(F)$ possess a central character?*

The answer is yes if R is uncountable but it is unknown if R is an algebraic closure of $\mathbb{F}_p$, for example. Answering Question 2 might be the first step in answering Question 1.

Question 3. *Does $G(F)$ possess supercuspidal R-representations?*

All supercuspidal representations of $\mathrm{GL}(2, \mathbb{Q}_p)$ are known ([BP12], building on [BL94]). Otherwise, many are constructed, though not in an explicit way, when $G = \mathrm{GL}_2$ and $F/\mathbb{Q}_p$ is an unramified extension ([Pas04], [BP12]) and in a few other low rank cases. A local-global construction yields such supercuspidal representations for $G = \mathrm{GL}_n$, provided F has characteristic 0.

For the following questions, the answer may depend on the characteristic of F, 0 or p.

When F has characteristic 0, it is known that a quotient of an admissible R-representation of $G(F)$ is still admissible. However, when F has characteristic p, we can construct an admissible R-representation of $\mathbb{G}_m(F) = F^\times$ with a quotient which is not admissible. That quotient has infinite length so we might ask:

Question 4. *Assume* $\operatorname{char} F = p$. *Let V be a finite length admissible R-representation of $G(F)$. Is every quotient of V admissible?*

Again, this is weaker than Question 1.

If P is a parabolic F-subgroup of G and M a Levi component of P, the parabolic induction functor Ind_P^G from smooth R-representations of $M(F)$ to smooth R-representations of $G(F)$ preserves admissibility. The functor has a left adjoint, the usual Jacquet functor taking coinvariants under the unipotent radical $N(F)$ of $P(F)$, and also a right adjoint. We can prove that the right adjoint respects admissibility. On the other hand, the Jacquet functor respects admissibility, provided F has characteristic 0 ([Eme10] when R is finite). We did not yet check if this remains true when R is an algebraically closed field of characteristic p.

Question 5. *Assume* $\operatorname{char} F = p$. *Let V be a finite length admissible R-representation of $G(F)$. Is the representation $V_{N(F)}$ of $M(F)$ admissible?*

In a different direction, let $\varphi : G' \to G$ be a central isogeny of connected reductive F-groups and let Z_G the center of G. If F has characteristic 0, the quotient $G(F)/Z_G(F)G'(F)$ is finite, whereas it is only compact when F has characteristic p.

Question 6. *Assume* $\operatorname{char} F = p$. *Let V be a finite length admissible R-representation of $G(F)$. If we view V as a representation of $G'(F)$ via φ, do we get a finite length R-representation of $G'(F)$ with admissible subquotients?*

The first case to look at is that of $\mathrm{SL}(2) \to \mathrm{PGL}(2)$ when $p = 2$.

When $\operatorname{char} F = 0$, Jan Kohlhaase [Koh] has investigated contragredients for irreducible admissible representations of $G(F)$. In particular, he proved that the contragredient of such a representation V is 0 unless V has finite dimension. We have extended that result to the case where $\operatorname{char} F = p$. But Kohlhaase went further to study the derived functors of the contragredient functor.

Question 7. *Assume* $\operatorname{char} F = p$. *Do the derived functors of the contragredient functor lead to some kind of duality for admissible representations of* $G(F)$*?*

Aside questions Before turning to questions centering on weights and eigenvalues, we mention some of our current explorations. If R has characteristic p but is not necessarily algebraically closed, can we still get a classification of irreducible admissible R-representations of $G(F)$? Is any such representation actually defined over a field of finite type over $\mathbb{F}_p$? Is any supercuspidal R-representation of $G(F)$, whose central character has finite order, definable over a finite extension of $\mathbb{F}_p$?

2) **Weights and eigenvalues**

Our classification of irreducible admissible representations of $G(F)$ uses weights and eigenvalues, to which we now turn as they raise their own set of questions.

We choose a special parahoric subgroup K of $G(F)$. Let V be an irreducible R-representation of K, $\operatorname{ind}_K^{G(F)} V$ the compactly induced (smooth) representation of $G(F)$, and let π be an irreducible smooth R-representation of $G(F)$.

We say that V is a *weight* of π if π is isomorphic to a quotient of $\operatorname{ind}_K^{G(F)} V$. Every irreducible smooth R-representation of $G(F)$ admits a weight.

Let $Z(G,K,V)$ be the center of the R-algebra $H(G,K,V)$ of $G(F)$-intertwiners of $\operatorname{ind}_K^{G(F)} V$. A homomorphism $\chi : Z(G,K,V) \to R$ is called a *Hecke eigenvalue* of V in π if π is isomorphic to a quotient of

$$\pi(V,\chi) = \chi \otimes_{Z(G,K,V)} \operatorname{ind}_K^{G(F)} V.$$

Every weight V of an irreducible admissible R-representation π of $G(F)$ admits a Hecke eigenvalue.

Question 8. *Let* π *be an irreducible smooth* R*-representation of* $G(F)$. *Is it true that* π *admits a Hecke eigenvalue* χ *for some weight* V*?*

If Question 1 has a negative answer, Question 8 could still have a positive one. Irreducible quotients of $\pi(V,\chi)$ are still amenable to the techniques of our work. On the other hand, one could try and answer Question 1 positively by dealing first with Question 8, and then proving that the irreducible quotients of $\pi(V,\chi)$ are admissible. Note that $\pi(V,\chi)$ has a central character, and hence also its subquotients.

In previous work, for a parabolic subgroup $P = MN$ of G in good position with respect to K, we constructed an algebra homomorphism (called *Satake homomorphism*) $Z(G,K,V) \to Z(M, K \cap M(F), V_{K\cap N(F)})$ – here $V_{K\cap N(F)}$ is an irreducible representation of the parahoric subgroup $K \cap M(F)$ of $M(F)$. We defined a *supersingular* R-representation of $G(F)$ to be an irreducible admissible representation π such that for all weights V of π all Hecke eigenvalues $\chi : Z(G,K,V) \to R$ of V in π are supersingular, i.e. never factor through the Satake homomorphism when P is a proper parabolic subgroup of G. We proved that if π is an irreducible admissible representation, then π is supersingular if and only if it is supercuspidal, and that if π admits a supersingular Hecke eigenvalue, it is supersingular.

Question 9. *Let π be an irreducible smooth R-representation of $G(F)$ admitting a supersingular Hecke eigenvalue. Is it true that all its Hecke eigenvalues are supersingular?*

The kernel of the natural map

$$\operatorname{ind}_K^{G(F)} V \to \prod_\chi \pi(V,\chi)$$

is a $Z(G,K,V)$-representation $I_1(G,K,V)$ of $G(F)$. By induction, for any $n \geq 1$, we define $I_{n+1}(G,K,V)$ as the kernel of the natural map $I_n(G,K,V) \to \prod_\chi \pi_n(V,\chi)$ where $\pi_n(V,\chi) = \chi \otimes I_n(G,K,V)$.

Question 10. *Is the decreasing filtration $(I_n(G,K,V))$ of $\operatorname{ind}_K^{G(F)} V$ finite?*

Let I be a pro-p Iwahori subgroup of $G(F)$, chosen in K and in good position. Let H be its Hecke algebra, that is the algebra of endomorphisms of the R-representation $\operatorname{ind}_I^{G(F)} 1_R$ compactly induced by the trivial representation of I on R. If V is an irreducible smooth R-representation of K, the action of $H(G,K,V)$ on $\operatorname{ind}_K^{G(F)} V$ commutes with the action of $G(F)$. It is known (Ollivier, Vignéras) that the H-module of I-invariants is a cyclic H-module and that the action of $H(G,K,V)$ on $(\operatorname{ind}_K^{G(F)} V)^I$ induces an isomorphism of $H(G,K,V)$ onto the H-endomorphisms of $(\operatorname{ind}_K^{G(F)} V)^I$. The action of $H(G,K,V)$ and of the center of H on $(\operatorname{ind}_K^{G(F)} V)^I$ are almost the same. As H is a finite module over its center, the space $\chi \otimes (\operatorname{ind}_K^{G(F)} V)^I$ is finite-dimensional for any Hecke eigenvalue χ of V.

Question 11. *Let χ be a Hecke eigenvalue for the weight V. Is the natural map $\chi \otimes (\operatorname{ind}_K^{G(F)} V)^I \to \pi(V,\chi)$ injective?*

The map $\chi \otimes (\operatorname{ind}_K^{G(F)} V)^I \to \pi(V,\chi)^I$ cannot be surjective when $\pi(V,\chi)^I$ is infinite-dimensional. This is the case when F is a proper unramified extension of $\mathbb{Q}_p$, $G = \mathrm{GL}(2)$ and χ is supersingular (Breuil, Morra).

For any connected reductive F-group G, any special parahoric subgroup K and any weight V, there exist supersingular Hecke eigenvalues $\chi : Z(G,K,V) \to R$.

Question 12. *If $\chi : Z(G,K,V) \to R$ is supersingular, is $\pi(V,\chi)$ non-zero?*

Note that Question 1 and Question 12 (for one choice of (V,χ)) together imply Question 3. To see this, pick (V,χ) such that χ is supersingular and $\pi(V,\chi)$ is non-zero. The representation $\pi(V,\chi)$ admits an irreducible quotient π, since it is finitely generated. If Question 1 holds, or in fact even if Question 1 holds only for irreducible representations with a central character, then π is admissible, and π is supercuspidal by our results discussed above.

(In [AHHV17, §V] we defined a non-zero quotient R_G of $Z(G,K,V)$ which has the property that a character $\chi : Z(G,K,V) \to R$ is supersingular if and only if χ factors through R_G, and we showed that $\pi(V) := R_G \otimes_{Z(G,K,V)} \operatorname{ind}_K^{G(F)} V$ is a free R_G-module. It follows that Question 12 is equivalent to asking whether $\pi(V) \neq 0$. In particular, the answer to Question 12 is independent of χ, for any fixed weight V.)

The following question may be related. There is a definition of what it means for a finite-dimensional H-module to be supersingular.

Question 13. *If X is a supersingular H-module, is $X \otimes_H \operatorname{ind}_I^{G(F)} 1_R$ non-zero?*

Aside question If π is an irreducible admissible R-representation of $G(F)$ it is equivalent to say that π is supersingular or that the space of I-fixed vectors is supersingular as H-module (Ollivier, Vignéras). Can we extend this equivalence to the case where π is only an irreducible quotient of some $\pi(V,\chi)$?

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