Recall : Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function $f^{-1} : Y \to X$ are continuous, then f is called a ***homeomorphism***.

§19 Product Topology (general case)

$X_{\lambda\in\Lambda}$ be topological spaces, where Λ is index set.
Caresian product $\prod_{\lambda\in\Lambda} X_\lambda = \{(x_\lambda)_{\lambda\in\Lambda} : x_\lambda \in X_\lambda,\ \forall\,\lambda\}$

$$= \left\{\text{functions } h : \Lambda \to \bigcup_{\lambda\in\Lambda} X_\lambda \,\middle|\, h(\lambda) \in X_\lambda,\ \lambda \in \Lambda\right\}$$

Projection maps $p_\mu : \prod_{\lambda\in\Lambda} X_\lambda \to X_\mu\ \ (\mu \in \Lambda)$

$$(x_\lambda)_{\lambda\in\Lambda} \to x_\mu$$

If $X_\lambda = X$ for all λ, then $\prod_{\lambda\in\Lambda} X = \{f : \Lambda \to X\} = : X^\wedge$.

Definition The ***product topology*** on $\prod X_\lambda$ is the coarest topology such that all projection maps p_μ are continuous.

p_μ for topology τ on $\prod X_\lambda$ continuous $\Leftrightarrow p_\mu^{-1}(U_\mu) \in \tau,\ \forall\, U_\mu \subset X_\mu$ is open.

If $f = \left\{p_\mu^{-1}(U_\mu) : \mu \in \Lambda,\ U_\mu \subset X_\mu \text{ open}\right\}$, then the topology generated by it, is the coarsest topology containing subbasis $\mathcal{S}$.
i.e. it is the product topology.

A subbasis of the product topology = $\left\{p_\mu^{-1}(U_\mu) : \mu \in \Lambda,\ U_\mu \subset X_\mu \text{ open}\right\}$.

To get a basis, take all finite intersections of elements os $\mathcal{S}$.

i.e. basis = $\left\{p_{\mu_1}^{-1}(U_{\mu_1}) \cap p_{\mu_2}^{-1}(U_{\mu_2}) \cap \ldots \cap p_{\mu_n}^{-1}(U_{\mu_n}),\right.$ where $\mu_i \in \Lambda$ and $U_{\mu_i} \subset X_{\mu_i}$ open.

If some μ_i's are the same, we don't get a new basis element.

$p_\mu^{-1}(U_1) \cap p_\mu^{-1}(U_2) \cap \ldots \cap p_\mu^{-1}(U_r) = p_\mu^{-1}(U_1 \cap U_2 \cap \ldots \cap U_r)$ for U_i open.

So we can assume μ_i's are pairwise distinct.

Then $\bigcap_{i=1}^{n} p_{\mu_i}^{-1}(U_{\mu_i}) = \prod_{\lambda\in\Lambda} V_\lambda$, where $V_\lambda = \begin{cases} U_\lambda, & \text{if } \lambda \in \{\mu_1, \ldots, \mu_n\} \\ X_\lambda, & \text{otherwise} \end{cases}$.

Another description is therefore

$\{\prod_{\lambda\in\Lambda} V_\lambda : V_\lambda \subset X_\lambda$ open, $V_\lambda \subset X_\lambda$ for all but finitely many λ's.

Definition The box topology on $\prod X_\lambda$ is the topology generated by the basis $\{\prod_\lambda V_\lambda : V_\lambda \subset X_\lambda$ open for all $\lambda\}$. ("open boxes")
This is clearly a basis.

Remark The box topology is finer than the product topology. If Λ is finite, they are the same! In general, they are different.

Example

Let $\mathbb{R}^w = \prod_{i=1}^{\infty} \mathbb{R}$. Then $\prod_{i=1}^{\infty} (-1,\ 1)$ is open in the box topology, but *not* in the product topology. The point $(0)_{i=1}^{\infty}$ has no basic open neighborhood $\subset \prod_{i=1}^{\infty} (-1,\ 1)$.

By default, on $\prod X_\lambda$ always take the product topology.

Aside $\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \ldots \times X_n$ with basis $U_1 \times U_2 \times \ldots \times U_n$ for $U_i \subset X_i$ open, $\forall\ i$.

$\prod_{i=1}^{\infty} X_i = X_1 \times X_2 \times \ldots = \{(x_i)_{i=1}^{\infty} : x_i \in X_i,\ \forall\ i\}$

basis for this product topology: $U_1 \times U_2 \times \ldots \times U_n \times X_{n+1} \times X_{n+2} \times X_{n+3} \times \ldots$ for each $U_i \subset X_i$ open.

Theorem 22 (continuous maps to a product)

Let $X_{\lambda \in \Lambda}$ and Y be topological spaces, a function $f : Y \to \prod_\Lambda X_\lambda$ is continuous $\Leftrightarrow$ $p_\mu \circ f : Y \to X_\mu$ is continous for all μ.
Declare $f_\lambda = p_\mu \circ f$. Then note $f(x) = (f_\lambda(x))_{\lambda \in \Lambda}$. This characterizes product topology.

Proof:

To check f is continuous, only need to check that all “coordinate functions” f_λ are continuous. Clearly, $p_\mu \circ f$ is continuous as a composition of two continuous functions. To demonstrate the reverse direction, continuity of $p_\mu \circ f$ implies $\left(p_\mu \circ f\right)^{-1} \left(U_\mu\right)$ open in $Y = f^{-1}\left(p_\mu^{-1}\left(U_\mu\right)\right)$. $\left(p_\mu^{-1}\left(U_\mu\right) \in\right.$ product topology. Consider $\tau = \left\{A \subset \prod X_\lambda : f^{-1}(A) \text{ open in } Y\right\}$

1) τ is a topology (easy check)
2) The above shows that $p_\mu^{-1}\left(U_\mu\right) \in \tau$, for all open $U_\mu \subset X_\mu$, for all μ.
The projection maps $p_\mu : \prod_\lambda X_\lambda \to X_\mu$ is continuous, where $\prod X_\lambda$ has topology τ. But the product topology is the coarsest topology on $\prod X_\lambda$ such that all p_μ are continuous. $\Rightarrow$ (product topology) $\subset \tau$, $f^{-1}(A)$ open in Y $\forall$ A open in the product topology
i.e. f is continuous. □

Remark One can show that the product topology is the unique topology on $\prod X_\lambda$ such that this theorem is true.

Example

$\prod_{\lambda \in \Lambda} X = X^\wedge$
The diagonal map $\triangle : X \to X^\wedge$, $(x \longmapsto (x)_{\lambda \in \Lambda})$ is continuous. Since each “cooridnate function” $x \longmapsto x$ is continuous.

Some useful facts (proofs are as for $X \times Y$)

(1) If $Y_\lambda \subset X_\lambda$ subset $\forall\ \lambda$, then on $\prod Y_\lambda$ the two natural topology coincide

i) subspace topology in $\prod X_\lambda$.
ii) product topology of subspace topologies.

(2) X_λ is T_2 $\forall\ \lambda \Rightarrow \prod X_\lambda$ is T_2.

(3) If $\mathcal{B}_\lambda$ is a basis for X_λ $\forall\, \lambda$, then

$$\mathcal{B} = \left\{ \prod_{\lambda \in \Lambda} B_\lambda \ = \begin{cases} B_\lambda \in \mathcal{B}_\lambda & \text{for finitely many } \lambda \\ B_\lambda = X_\lambda & \text{for all others} \end{cases} \right\}$$

(4) Projection maps $p_\mu : \prod X_\lambda \to X_\mu$ are open (as in problem §16 .4)

(5) Analogues (1)–(4) are true for the box topology, but in (3) we have to drop the condition "$B_\lambda = X_\lambda$".

<u>Example</u>

$\mathbb{R}^{\mathbb{R}} = \{ f : \mathbb{R} \to \mathbb{R}$ (don't have to be continuous)$\}$

A basic open in the product topology in above basis:

$\prod_{\alpha \in \mathbb{R}} V_\alpha :\ V_\alpha \in \mathbb{R}$ open, $V_\alpha = \mathbb{R}$ all but finitely many α's.

i.e. $\{ f : \mathbb{R} \to \mathbb{R} : f(\alpha_\iota) \in V_{\alpha_\iota},\ 1 \leq i \leq n \}$

Check $f_n \to f$ as $n \to \infty \iff f_n(x) \to f(x)$, $\forall\, x \in \mathbb{R}$.