Abstract. Let $p$ be a prime number, $F$ a totally real number field unramified at places above $p$ and $D$ a quaternion algebra of center $F$ split at places above $p$ and at no more than one infinite place. Let $v$ be a fixed place of $F$ above $p$ and $\pi : \text{Gal}(\overline{F}/F) \to \text{GL}_2(\mathbb{F}_p)$ an irreducible modular continuous Galois representation which, at the place $v$, is semisimple and sufficiently generic (and satisfies some weak genericity conditions at a few other finite places). We prove that many of the admissible smooth representations of $\text{GL}_2(F_v)$ over $\mathbb{F}_p$ associated to $\pi$ in the corresponding Hecke-eigenspaces of the mod $p$ cohomology have Gelfand–Kirillov dimension $[F_v : \mathbb{Q}_p]$, as well as several related results.
7.3. Construction of the lattice 77
8. Global applications 81
8.1. Patching functors 81
8.2. Freeness for types 84
8.3. Freeness for projective envelopes 88
8.4. Gelfand–Kirillov dimensions 93
References 99
1. Introduction

Fix a prime number $p$, a totally real number field $F$ which is unramified at places above $p$, and a quaternion algebra $D$ of center $F$ which is split at places above $p$ and at exactly one infinite place. For $V$ a compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty})^\times$ denote by $X_V$ the associated smooth projective Shimura curve over $F$. Let $v$ be a fixed place of $F$ above $p$ and $\mathbb{F}$ a finite extension of $\mathbb{F}_p$ (“sufficiently large”, as usual). This paper is concerned with admissible smooth representations of $GL_2(F_v)$ over $\mathbb{F}$ of the form

$$\pi = \lim_{V'} \text{Hom}_{Gal(\mathbb{F}/F)}(\bar{\tau}, H^0_{dR}(X_{V'\cap}) \times F, \mathbb{F}),$$

where $V'$ is a fixed compact open subgroup of $(D \otimes_F \mathbb{A}_F^{\infty,v})^\times$, the inductive limit running over compact open subgroups $V'$ of $(D \otimes_F F_v)^\times \equiv GL_2(F_v)$ and $\tau : Gal(\mathbb{F}/F) \rightarrow GL_2(\mathbb{F})$ is a continuous absolutely irreducible Galois representation such that $\pi \neq 0$. Understanding such representations $\pi$ of $GL_2(F_v)$ attached to Galois representations is important, as it is hoped that they realize the mod $p$ Langlands correspondence. For instance, when $F = \mathbb{Q}$ and $X$ is the compactified modular curve, the representation $\pi$ of $GL_2(\mathbb{Q}_p)$ is well understood under weak assumptions on $\pi|_{Gal(\mathbb{Q}_p/\mathbb{Q}_p)}$, see [Emc]. In particular we have $\text{dim}_{GL_2(\mathbb{Q}_p)}(\pi) = 1$ (see below for the definition of this dimension). More generally, as soon as $F_v = \mathbb{Q}_p$, it seems reasonable to expect an analogous description of $\pi$ (see e.g. [CEG+18], Rk. 7.8 for a remark along these lines). When $F_v \neq \mathbb{Q}_p$, however, an explicit description of $\pi$ still seems to be out of reach despite a great deal of effort during the past 20 years.

The aim of this work is to lift a corner of the veil surrounding $\pi$ by proving that, when $\pi|_{Gal(F_v/F_v)}$ is semisimple sufficiently generic (and under some standard genericity conditions at a few other finite places), we have $\text{dim}_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]$. We also prove the same statement for the analog of $\pi$ when $D$ is totally definite. Although we did not check it carefully, the same method should also work in other global settings in which the group is $GL_2(F_v)$ at the place $v$, like for instance unitary groups which are forms of $GL_2$. Moreover, from exchanges with Koziol, we believe the same result applies when, in the global setup, the unitary group is a nonsplit unramified unitary group at $v$. In a companion paper (and the same global setup), Hu and Wang prove an analog of Theorem 1.3 below and apply our Theorem 1.4 to deduce $\text{dim}_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]$ when $\pi|_{Gal(F_v/F_v)}$ is not semisimple and sufficiently generic ([HW]).

In order to state our main theorem, let us first recall the definition of $\text{dim}_{GL_2(F_v)}(\pi)$, also called the Gelfand–Kirillov dimension\(^1\) of $\pi$. We let $f \overset{\text{def}}{=} [F_v : \mathbb{Q}_p]$, $K \overset{\text{def}}{=} GL_2(O_{F_v})$, $K_n \overset{\text{def}}{=} 1 + p^nM_2(O_{F_v}) \subseteq K$ for $n \geq 1$, $Z_1$ the center of $K_1$, and we assume $p > 2$. For $\pi$ a nonzero admissible smooth representation of $GL_2(F_v)$ over $\mathbb{F}$ with central character, we set (see §5.1)

$$\text{dim}_{GL_2(F_v)}(\pi) \overset{\text{def}}{=} 3f - \text{min}\{d \geq 0 : \text{Ext}^d_{\mathbb{F}[K_1/Z_1]}(\pi^\vee, \mathbb{F}[K_1/Z_1]) \neq 0\},$$

where $\mathbb{F}[K_1/Z_1]$ is the Iwasawa algebra of $K_1/Z_1$ and $\pi^\vee$ is the algebraic dual of $\pi$, considered as module over $\mathbb{F}[K_1/Z_1]$ (note that $Z_1$ acts trivially on $\pi$ and that $3f = \text{dim}(GL_2(F_v)/Z_1)$).

Another equivalent and maybe more intuitive definition of $\text{dim}_{GL_2(F_v)}(\pi)$ is the following: it is

\(^1\)Strictly speaking, this is not quite the Gelfand–Kirillov dimension of $\pi$, see Remark 5.1.1 in the text, but this is the only dimension we will consider.
the unique integer such that there exist \( a \leq b \) in \( \mathbb{R}_{>0} \) satisfying

\[
a \leq \frac{\dim \pi^K_n}{p^n \dim_{GL_2(F_v)}(\pi)} \leq b
\]

for all \( n \geq 1 \) (see Remark 5.1). So, roughly speaking, \( \dim_{GL_2(F_v)}(\pi) \) measures the growth of \( \pi^K_n \) when \( n \) grows. For instance it is 0 if and only if \( \dim_{F}(\pi) \) is finite and nonzero.

We now make the following additional assumptions on \( \pi \), where, for a finite place \( w \) of \( F \), \( I_{F_w} \) is the inertia subgroup at \( w \) and \( \omega_{w} \), \( f' \in \{f, 2f\} \) is Serre’s fundamental character of level \( f' \):

(i) \( \pi|_{G_F(\psi_T)} \) is absolutely irreducible;

(ii) for \( w \nmid p \) such that either \( D \) or \( \pi \) ramifies, the framed deformation ring \( R_{\pi_w} \) of \( \pi_w \) over the Witt vectors \( W(F) \) is formally smooth;

(iii) \( \pi|_{I_{F_w}} \) is generic in the sense of [BP12, Def. 11.7];

(iv) \( \pi|_{I_{F_w}} \) is semisimple of one of the following forms up to twist:

\[
\begin{pmatrix}
\omega_f^{(r_0+1)+\cdots+p^{-1}(r_f-1+1)} & 0 \\
0 & 1
\end{pmatrix} \quad 9 \leq r_i \leq p - 12, \tag{a}
\]

\[
\begin{pmatrix}
\omega_{2f}^{(r_0+1)+\cdots+p^{-1}(r_f-1+1)} & 0 \\
0 & \omega_{2f}^{p' let same})
\end{pmatrix} \quad 10 \leq r_0 \leq p - 11, 9 \leq r_i \leq p - 12 \text{ for } i > 0. \tag{b}
\]

Note that [iv] implies \( p > 19 \) and that [iii] can be made explicit ([Sho16]). We can now state our main result.

**Theorem 1.1** (Corollary 8.4.5). Keep all the above assumptions on \( F, D, \pi \). Let \( V^w = \prod_{w \neq v} V_w \) with \( V_w = \text{GL}_2(O_{F_w}) \) if neither \( D \) nor \( \pi \) ramifies at \( w \), and \( V_w \subseteq 1 + pM_2(O_{F_w}) \) if \( w \mid p \) (\( w \neq v \)). Then for \( \pi \) as in [1] we have \( \dim_{GL_2(F_v)}(\pi) = f \).

We also prove several variants and generalizations of Theorem 1.1. For instance, without the assumption \( V_w \subseteq 1 + pM_2(O_{F_w}) \) for \( w \mid p \), we still have \( \dim_{GL_2(F_v)}(\pi) \leq f \), see Remark 8.4.6. We can take \( V_w = \text{GL}_2(O_{F_w}) \) for \( w \) outside any finite set \( S \) containing the ramification places of \( D \) and \( \pi \) provided \( R_{\pi_w} \) is formally smooth for all \( w \in S \) prime to \( p \) (see loc. cit.). It is likely that other variants of Theorem 1.1 can be proven, e.g. by fixing types at some places \( w \) prime to \( p \) instead of assuming \( R_{\pi_w} \) formally smooth. For instance, we have \( \dim_{GL_2(F_v)}(\pi|_{D,v}(\pi)) = f \), where \( \pi|_{D,v}(\pi) \) is the “local factor” \( \pi|_{D,v}(\pi) \) of [BD14 (3.3)] and [EGS15 §6.5] (see Remark 8.4.4).

We now sketch the proof of Theorem 1.1. By dévissage, we can replace \( \pi \) by

\[
\pi = \lim_{V_v} \prod_{w \neq v} \Hom_{\text{GL}_2(O_{F_w})} \left( \otimes_{w \mid p} \Hom_{G_F}(\pi, H^1_{et}(X_{V^w} \times \overline{F}^w, F)) \right), \tag{2}
\]

where, for \( w \mid p \), \( w \neq v \), \( \sigma_w \) is any Serre weight in the set \( W(\pi^w) \) of [BDJ10 §3] and \( V_w \subseteq 1 + pM_2(O_{F_w}) \) is normal in \( \text{GL}_2(O_{F_w}) \) (and \( V_w \) is sufficiently small at a nice place \( w_1 \) where nothing ramifies). The representation \( \pi \) of \( \text{GL}_2(F_v) \) in (2) can be “patched” as in [CEG+16] or [DL] §6, and it follows from the arguments of Gee and Newton in [QN] Appendix A that we have \( \dim_{GL_2(F_v)}(\pi) \geq f \). It is therefore enough to prove the upper bound \( \dim_{GL_2(F_v)}(\pi) \leq f \) for \( \pi \) as in (2).
We let $k(\cong \mathbb{F}_p)$ be the residue field of $F_v$, and for each Serre weight $\sigma \in W(\tau'_v)$, we define $D_{0,\sigma}$ as the largest subrepresentation of the injective envelope $\text{Inj}_{\text{GL}_2(k)}(\sigma)$ such that $\sigma$ only appears in the socle of $D_{0,\sigma}$ and no other Serre weight of $W(\tau'_v)$ is a constituent of $D_{0,\sigma}$. We set $D_0(\tau'_v) := \bigoplus_{\sigma \in W(\tau'_v)} D_{0,\sigma}$ as in [BP12 §13]. We also denote by $m_{K_1/Z_1}$ the maximal ideal of $\mathbb{F}[[K_1/Z_1]]$. In order to get the above upper bound on $\text{dim}_{\text{GL}_2(F_v)}(\pi)$, we will apply the following theorem to $\pi$ in (2).

**Theorem 1.2 (Theorem 6.4.7).** Let $\pi$ be an admissible smooth representation of $\text{GL}_2(F_v)$ over $\mathbb{F}$ with a central character. Assume that

(i) we have an isomorphism $\pi^{K_1} = \pi[m_{K_1/Z_1}] \cong D_0(\tau'_v)^{\oplus r}$ of representations of $\text{GL}_2(k)$ for some $r \geq 1$;
(ii) we have $[\pi[m_{K_1/Z_1}^3] : \sigma] = [\pi[m_{K_1/Z_1}] : \sigma]$ for all $\sigma \in W(\tau'_v)$. 

Then $\text{dim}_{\text{GL}_2(F_v)}(\pi) \leq f$.

(In fact we prove in Theorem 6.4.7 a slightly stronger statement.) Condition (i) in Theorem 1.2 is already familiar, for instance it is satisfied with $r = 1$ by the representation $\pi_{D,v}(\tau)$ mentioned above (see [HW18] and [LMS], which build upon [BP12] and [ECST5]). Thus it is rather condition (ii) which is important. Though it is purely local, the proof of Theorem 1.2 is not at all trivial, and it took us a long time before finding a proof (or even convincing ourselves that the statement was true). The key idea is to look at the action on $\pi$ of the Iwahori subgroup $I$ of $K$ instead of $K$ itself. The proof of Theorem 1.2 is divided into two steps. The first step is the following result, where $I_1 \subseteq I$ is the pro-$p$-Iwahori subgroup and $m_{I_1/Z_1}$ is the maximal ideal of the Iwasawa algebra $\mathbb{F}[[I_1/Z_1]]$.

**Theorem 1.3 (Proposition 6.4.6).** Let $\pi$ be an admissible smooth representation of $\text{GL}_2(F_v)$ over $\mathbb{F}$ with a central character and assume $\pi$ satisfies (i) and (ii) of Theorem 1.2. Then we have for all continuous characters $\chi : I \to \mathbb{F}^\times$ that

$$[\pi[m_{I_1/Z_1}^3] : \chi] = [\pi[m_{I_1/Z_1}] : \chi].$$

Note that socle$(\pi|_I) = \pi[m_{I_1/Z_1}] = \pi^{I_1}$ since $p > 2$. The proof of Theorem 1.3 is given in §6. It is a bit long and technical, but is rather standard (to apply Proposition 6.4.6 to $\pi$ as in Theorem 1.2 one actually needs Corollary 6.3.13 and Lemma 6.4.3, see §6.4).

The second step is the following key result which gives the sought-after upper bound on the Gelfand–Kirillov dimension.

**Theorem 1.4 (Corollary 5.3.5).** Let $\pi$ be an admissible smooth representation of $\text{GL}_2(F_v)$ over $\mathbb{F}$ with a central character and assume $[\pi[m_{I_1/Z_1}^3] : \chi] = [\pi[m_{I_1/Z_1}] : \chi]$ for all $\chi : I \to \mathbb{F}^\times$. Then $\text{dim}_{\text{GL}_2(F_v)}(\pi) \leq f$.

Let us sketch the proof of Theorem 1.4. We view the algebraic dual $\pi^\vee$ as a (finitely generated) module over $\mathbb{F}[I_1/Z_1]$ and denote by $\text{gr}_m \pi^\vee$ the associated graded module over $\text{gr}_m \mathbb{F}[I_1/Z_1]$ for the $m_{I_1/Z_1}$-adic filtration. The graded ring $\text{gr}_m \mathbb{F}[I_1/Z_1]$ is not commutative, as the pro-$p$ group $I_1/Z_1$ is not uniform (see [Clo17] and §5.3). But the assumption $[\pi[m_{I_1/Z_1}^3] : \chi] = [\pi[m_{I_1/Z_1}] : \chi]$ implies that the action of $\text{gr}_m \mathbb{F}[I_1/Z_1]$ on $\pi^\vee$ factors through a commutative...
quotient \((\text{gr}_m \mathbb{F}[I_1/Z_1])/I_{1/Z_1}\), where \(I_{1/Z_1}\) is an explicit 2-sided ideal of \(\text{gr}_m \mathbb{F}[I_1/Z_1]\) generated by certain degree 2 elements (see Theorem 5.3.4). More precisely one has

\[
(\text{gr}_m \mathbb{F}[I_1/Z_1])/I_{1/Z_1} \cong \mathbb{F}[e_i, f_i; 0 \leq i \leq f - 1]/(e_i f_j; 0 \leq j \leq f - 1),
\]

where \((\text{gr}_m \mathbb{F}[I_1/Z_1])/I_{1/Z_1}\) by a regular sequence \((h_0, \ldots, h_{f-1})\) of central elements. By a general lemma (Lemma 5.1.3), \(\dim_{\text{GL}_2(F)}(\pi)\) is equal to the dimension of the support of \(\text{gr}_m \pi'\) in the polynomial algebra

\[
(\text{gr}_m \mathbb{F}[I_1/Z_1])/(h_0, \ldots, h_{f-1}) \cong \mathbb{F}[e_i, f_i; 0 \leq i \leq f - 1],
\]

which by (3) is smaller or equal than \(\dim(\text{gr}_m \mathbb{F}[I_1/Z_1]/I_{1/Z_1}) = 2f - f = f\). So we see that the fact that \(\text{gr}_m \pi'\) (for an admissible smooth representation of \(\text{GL}_2(F)\) over \(\mathbb{F}\)) is a module over \((\text{gr}_m \mathbb{F}[I_1/Z_1])/I_{1/Z_1}\), and not just over \(\text{gr}_m \mathbb{F}[I_1/Z_1]\), turns out to be an important condition. We hope to come back to other consequences of this condition in future work.

We now apply Theorem 1.2 to \(\pi\) in (2). For this, we need to prove that \(\pi\) satisfies conditions (i) and (ii) of Theorem 1.2. We first sketch the proof of (ii), which is the harder and more important one. We fix an arbitrary Serre weight \(\sigma\) in \(W(\tau')\). We need to prove

\[
\text{Hom}_K(\sigma, \pi) \simrightarrow \text{Hom}_K((\text{Proj}_{K/Z_1} \sigma)/m^2_{K/Z_1}, \pi),
\]

where \(\text{Proj}_{K/Z_1} \sigma\) is the algebraic dual of the injective envelope \(\text{Inj}_{K/Z_1} \sigma'\) of \(\sigma'\) in the category of smooth representations of \(K/Z_1\) over \(\mathbb{F}\). We do not know any other way to prove (4) than to “patch” (the dual of) both sides using the patching functors of \([EGS15]\). This strategy is not new: it is initially due to Emerton, Gee, Savitt in \([EGS15]\) (generalizing work of Diamond, of Fujiwara, “patch” (the dual of) both sides using the patching functors of \([EGS15]\). Here \(K_1\) is the unique projective \(\text{GL}_2(\mathbb{F})\)-representation.

Thus proving (4) is equivalent to proving

\[
M_\infty((\text{Proj}_{K/Z_1} \sigma)/m^2_{K_1/Z_1})/m_\infty \simrightarrow M_\infty(\sigma)/m_\infty,
\]

where \(m_\infty\) is the maximal ideal of \(R_\infty\). The strategy in the above references to prove (a “multiplicity one” variant of (5) with \(m^2_{K_1/Z_1}\) replaced by \(m_{K_1/Z_1}\) is to use the isomorphism

\[
M_\infty(\text{Proj}_{GL_2(k)} \sigma)/(p) \cong M_\infty(\text{Proj}_{GL_2(k)} \sigma) = M_\infty((\text{Proj}_{K/Z_1} \sigma)/m_{K_1/Z_1}),
\]

where \(\text{Proj}_{GL_2(k)} \sigma\) is the unique projective \(W(\mathbb{F})[\text{GL}_2(k)]\)-module lifting \(\text{Proj}_{GL_2(k)} \sigma \cong \text{Inj}_{\text{GL}_2(\mathbb{F})} \sigma\), and to determine the support of \(M_\infty(\text{Proj}_{GL_2(k)} \sigma)\) in \(R_\infty\).

We apply a similar strategy in our case, which means we first have to lift \((\text{Proj}_{K/Z_1} \sigma)/m^2_{K_1/Z_1}\) to a \(W(\mathbb{F})[K]\)-module. This is significantly more complicated than to lift \((\text{Proj}_{K/Z_1} \sigma)/m_{K_1/Z_1}\). It is easy to check that the \(K\)-representation \((\text{Proj}_{K/Z_1} \sigma)/m^2_{K_1/Z_1}\) is a nonsplit extension

\[
0 \rightarrow (m_{K_1/Z_1}/m^2_{K_1/Z_1}) \otimes_{F} \text{Proj}_{GL_2(k)} \sigma \rightarrow (\text{Proj}_{K/Z_1} \sigma)/m^2_{K_1/Z_1} \rightarrow \text{Proj}_{GL_2(k)} \sigma \rightarrow 0.
\]
For convenience, let us fix an embedding \( \sigma_0 : k \cong \mathbb{F}_{p^j} \hookrightarrow \mathbb{F} \) and write all others as \( \sigma_0 \circ \varphi^j \), \( j \in \{0, \ldots, f - 1\} \), where \( \varphi \) is the Frobenius \( x \mapsto x^p \) on \( k \). Then we have

\[
m_{K/Z_1}/m_{K/Z_1}^2 \cong \bigoplus_{j=0}^{f-1} (\text{Sym}^2(\mathbb{F}^2) \otimes \mathbb{F} \text{ det}^{-1})^{(j)},
\]

where \( (j) \) means that \( \text{GL}_2(k) \) acts via \( \sigma_0 \circ \varphi^j \). Moreover, for each \( j \), we fix a (non-canonical) \( \text{GL}_2(k) \)-equivariant embedding

\[
\iota_j : \text{Proj}_{\text{GL}_2(k)} \sigma \hookrightarrow (\text{Sym}^2(\mathbb{F}^2) \otimes \mathbb{F} \text{ det}^{-1})^{(j)} \otimes_{\mathbb{F}} \text{Proj}_{\text{GL}_2(k)} \sigma.
\]

We set

\[
L_{-1} \overset{\text{def}}{=} \text{Proj}_{\text{GL}_2(k)} \sigma
\]

and

\[
R_{2,j} \overset{\text{def}}{=} (\text{Sym}^2(W(\mathbb{F})) \otimes W(\mathbb{F}) \text{ det}^{-1})^{(j)} \otimes_{W(\mathbb{F})} L_{-1} \quad j \in \{0, \ldots, f - 1\},
\]

and we define a \( K \)-invariant lattice \( L_j \) in the locally algebraic representation

\[
L_{-1}[1/p] \oplus \left( \bigoplus_{j' = 0}^{j} R_{2,j'}[1/p] \right)
\]

as follows

\[
L_j = \{(x, (x_{j'})_{0 \leq j' \leq j}) \in L_{-1} \oplus (\bigoplus_{j' = 0}^{j} R_{2,j'}) : (x_{j'} \mod pR_{2,j'}) = (x \mod pL_{-1}) \text{ via } \iota_{j'} : L_{-1}/pL_{-1} \hookrightarrow R_{2,j'}/pR_{2,j'} \quad \forall j' \in \{0, \ldots, j\}\}.
\]

Equivalently, we have for \( j \in \{0, \ldots, f - 1\} \) that

\[
L_j \overset{\text{def}}{=} L_{j-1} \times_{\text{Proj}_{\text{GL}_2(k)} \sigma} R'_{2,j},
\]

where \( R'_{2,j} \overset{\text{def}}{=} \{ x \in R_{2,j} : (x \mod pR_{2,j}) \in (\iota_j(L_{-1}/pL_{-1})) \} \) (another \( K \)-invariant lattice in \( R_{2,j}[1/p] \)). By explicit computations carried out in §7 we first prove that the lattice \( L_{f-1} \) lifts \( (\text{Proj}_{K/Z_1} \sigma)/m_{K/Z_1}^2 \).

**Theorem 1.5** (Corollary 7.3.4). We have a \( K \)-equivariant isomorphism

\[
L_{f-1}/pL_{f-1} \cong (\text{Proj}_{K/Z_1} \sigma)/m_{K/Z_1}^2.
\]

We then prove the following theorem.

**Theorem 1.6** (Corollary 8.3.8). For \( j \in \{-1, \ldots, f - 1\} \) the \( R_\infty \)-module \( M_\infty(L_j) \) is free of finite rank over \( R_\infty/\text{Ann}_{R_\infty} (M_\infty(L_j)) \). Moreover this rank depends neither on \( j \) nor on the fixed Serre weight \( \sigma \) in \( W(\mathbb{F}) \).

Denote by \( r \geq 1 \) the rank in Theorem 1.6 Applying Theorem 1.6 to both \( j = -1 \) and \( j = f - 1 \), and using Theorem 1.5 when \( j = f - 1 \), we see that the two \( \mathbb{F} \)-vector spaces in (5) both have dimension \( r \). Since the natural map from left to right in (5) is surjective by exactness of \( M_\infty \), we obtain that (5) is an isomorphism, and hence that \( \pi \) satisfies condition (iii) of Theorem 1.2.

Let us sketch the proof of Theorem 1.6 which is by induction of \( j \). By exactness of \( M_\infty \), (7) implies

\[
M_\infty(L_j) \cong M_\infty(L_{j-1}) \times_{M_\infty(\text{Proj}_{\text{GL}_2(k)} \sigma)} M_\infty(R'_{2,j}),
\]

and \( M_\infty(L_{j-1}) \) is free of rank \( r \) over \( R_\infty/\text{Ann}_{R_\infty} (M_\infty(L_{j-1})) \) by induction, hence it is enough to prove the following three statements for \( j \in \{0, \ldots, f - 1\} \):
(i) $M_\infty(L_{-1})$ is free of rank $r$ over $R_\infty/\text{Ann}_{R_\infty}(M_\infty(L_{-1}))$;
(ii) $M_\infty(R'_{2,j})$ is free of rank $r$ over $R_\infty/\text{Ann}_{R_\infty}(M_\infty(R'_{2,j}))$;
(iii) $\text{Ann}_{R_\infty}(M_\infty(\text{Pro}_{\text{GL}_2}(\ell))) = \text{Ann}_{R_\infty}(M_\infty(L_{-1-1})) + \text{Ann}_{R_\infty}(M_\infty(R'_{2,j}))$.

Statement (i) is proven in §8.2 (see Proposition 8.2.5) by a refinement of the techniques in 
[EGLS15 §10] and [LMS §4] together with some commutative algebra. Statement (ii) is proven in

Theorem 8.3.6 using standard dévissage techniques and “elementary” properties of the functor $M_\infty$ (in particular [Le19 Lemma 4.5] instead of [EGS15 Lemma 10.1.13]) and some results of

§8.2.

Statement (iii) is the most subtle and the most technical part of the paper and is proven in

Theorem 8.3.8. Recall that $R_{\ell^\nu}$ is the local $W(\mathbb{F})$-algebra parametrizing framed deformations of $\ell^\nu$. We let $R_{\ell^\nu}(0,\tau)^{(j)}$, resp. $R_{\ell^\nu}(2,1)^{(j)}$, for $j \in \{0, \ldots, f-1\}$, be the reduced $p$-torsion free quotient of $R_{\ell^\nu}$ parametrizing those deformations which have inertial type $\tau$ and parallel Hodge–Tate weights $(1,0)$, resp. Hodge–Tate weights $(2,-1)$ in the embedding $F_\nu \hookrightarrow W(\mathbb{F})[1/p]$ induced by $\sigma_0 \circ \nu$ and $(1,0)$ elsewhere. An explicit computation that builds on the recent advances of Le–Le Hung–Levin–Morra [LLLH18, LLLH19] (see Proposition 4.2.1) shows that these rings are all domains. It follows (see Proposition 8.2.5) that $R_\infty/\text{Ann}_{R_\infty}(M_\infty(L_{-1}))$ is a power series ring over $R_{\ell^\nu}/\cap_{\tau}p_{\tau}^{(1,0)}$, where $p_{\tau}^{(1,0)}$ is the prime ideal $\ker(R_{\ell^\nu} \to R_{\ell^\nu}(1,0),\tau)$ and $\tau$ runs over the tame inertial types such that $\sigma$ is a Jordan–Hölder factor in the mod $p$ semisimplification of $\sigma(\tau)$ (here $\sigma(\tau)$ is the usual irreducible smooth representation of $K$ associated by Henniart to $\tau$ in the appendix to [BM02]). Likewise, $R_\infty/\text{Ann}_{R_\infty}(M_\infty(R'_{2,j}))$ is a power series ring over $R_{\ell^\nu}/\cap_{\tau}p_{\tau}^{(2,1)^j}$, where $p_{\tau}^{(2,1)^j} = \ker(R_{\ell^\nu} \to R_{\ell^\nu}(2,1),\tau)$ and $\tau$ runs over the same tame types (see Theorem 8.3.6). To prove (iii) it is enough to prove for $j \in \{0, \ldots, f-1\}$ that

(8) $p \in \text{Ann}_{R_\infty}(M_\infty(L_{-1-1})) + \text{Ann}_{R_\infty}(M_\infty(R'_{2,j}))$.

For instance, if $j = 0$ we have to prove that

(9) $p \in \cap_{\tau}p_{\tau}^{(1,0)} + \cap_{\tau}p_{\tau}^{(2,1)^0}$

(note that the fixed choice of embedding $\sigma_0$ of course plays no role here). Both (9) and (8)

follow from an explicit computation, which, though technical, can be done entirely “by hand”, see

Proposition 4.3.2. We have compiled in Tables 1 to 5 all the explicit computations of deformation

rings that we use in the proofs (everything was checked “by hand”).

To apply Theorem 1.2 to $\pi$ in (2), it remains to show that $\pi$ satisfies condition (i) of Theorem

1.2. But using (5) together with standard injectivity properties of localizations of Hecke modules

at non-Eisenstein maximal ideals and (a lot of) representation theory of $K$ (see Corollary 6.3.13),

we actually obtain the complete structure of $\pi[m^2_{K_1/Z_1}]$ as a representation of $K$.

**Theorem 1.7** (Theorem 8.4.2). Let $\pi$ as in (2), we have

(10) $\pi[m^2_{K_1/Z_1}] \cong \left( \bigoplus_{\sigma \in W(\ell^\nu)} \tilde{D}_{\sigma} \right)^{\oplus r},$

where $r$ is the rank in Theorem 1.6 and $\tilde{D}_{\sigma}$ is the largest subrepresentation of $[\text{Inj}_{K/Z_1} \sigma][m^2_{K_1/Z_1}]$ containing $\sigma$ with multiplicity 1 (≠ its socle) and no other Serre weights of $W(\ell^\nu)$. Moreover, each irreducible constituent of $\pi[m^2_{K_1/Z_1}]$ has multiplicity $r$. 

Condition (4) of Theorem 1.2 then immediately follows from the isomorphism (10) in Theorem 1.7 by taking $K_1$-invariants on both sides. In particular we finally obtain:

**Theorem 1.8 (Theorem 8.4.1).** Let $\pi$ be as in (2). Then $\dim_{\GL_2(F_v)}(\pi) = f$.

Let us give one application of Theorem 1.8 the existence of many admissible unitary Banach representations of $\GL_2(F_v)$ lifting $\pi$ (for $\pi$ as in (2)). As briefly mentioned after (2), there exists a “big” profinite $R_\infty$-module $M_\infty$ endowed with an $R_\infty$-linear continuous action of $\GL_2(F_v)$ such that $M_\infty/m_\infty \cong \pi'$. By Schikhof duality (see [ST02, §1]), it is enough to prove that, when $\pi$ is a group and $J_\infty(G)$ is a dimension vector space, we let $\kappa$ be sufficiently large arithmetic Frobenius on $k$. Then the set $J_\infty(G)$ is identified with $\{0, \ldots, f - 1\}$.

We let $\eps$ (resp. $\omega$) denote the $p$-adic (resp. mod $p$) cyclotomic character of the absolute Galois group $G_F$, where $F$ is any finite extension of $\Q$ or $\Q_p$. We normalize Hodge–Tate weights so that $\eps$ has Hodge–Tate weight 1 at every embedding.

Given a profinite group $G$, we write $F[G]$ for its completed group algebra with $F$-coefficients, with augmentation ideal denoted by $m_G$. We recall that Pontryagin duality $M \mapsto M'$ induces an exact anti-equivalence between the category of smooth $G$-representations over $F$, and the category of pseudocompact $F[G]$-modules. Recall that given a pseudocompact $F[G]$-module $M$, we have the radical $\rad G M \defeq m_G M$. Dually, given a smooth $G$-representation $M$ we write $\soc_G M$ for its socle.

If $G$ is a group and $V$ a representation of $G$ on a finite-dimensional $E$-vector space we denote by $\overline{V}$ the semisimplification of a $G$-stable $\O$-lattice in $V$. If $V$ a representation of $G$ on a finite-dimensional vector space, we let $JH(V)$ denote the set of Jordan–Hölder factors of $V$. Also, if $\sigma$ is an irreducible representation of $G$, we let $[V : \sigma]$ be the multiplicity of $\sigma$ in the semisimplification of $V$.

### 1.1. Notation

We only give some very general notation here, more specific notation will be given in each section. We fix an algebraic closure $\overline{\Q}_p$ of $\Q_p$. All finite extensions of $\Q_p$ will be considered as subfields of $\overline{\Q}_p$. We let $v_p$ denote the valuation of $\overline{\Q}_p$ such that $v_p(p) = 1$.

We let $E$ be a finite extension of $\Q_p$, with ring of integers $\O$, uniformizer $\varpi$ and residue field $\F$, and will always assume that $E$ is sufficiently large. We let $k$ be a finite extension of $\F_p$ of degree $f \defeq [k : \F_p]$. We fix an embedding $\sigma_0 : k \hookrightarrow \F$ and let $\sigma_j \defeq \sigma_0 \circ \varphi^j$, where $\varphi : x \mapsto x^p$ is the arithmetic Frobenius on $k$. Then the set $J \defeq \Hom(k, \F)$ is identified with $\{0, \ldots, f - 1\}$.

We let $\eps$ (resp. $\omega$) denote the $p$-adic (resp. mod $p$) cyclotomic character of the absolute Galois group $G_F$, where $F$ is any finite extension of $\Q$ or $\Q_p$. We normalize Hodge–Tate weights so that $\eps$ has Hodge–Tate weight 1 at every embedding.

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**GELFAND–KIRILLOV DIMENSION AND MOD $p$ COHOMOLOGY FOR $\GL_2$**  

9
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2. Preliminaries

Throughout this section $K$ denotes the unramified extension of $\mathbb{Q}_p$ of degree $f$ with ring of integers $\mathcal{O}_K$ and residue field $k$. Recall from §I.1 that we have fixed an embedding $\sigma_0 : k \hookrightarrow \mathbb{F}$, hence an embedding $K \hookrightarrow \mathbb{F}$ which we still denote by the same symbol $\sigma_0$. In particular we have compatible identifications of $\mathcal{J} = \text{Hom}(k, \mathbb{F})$ with $\text{Hom}_{\mathcal{O}_p}(K, \mathbb{F})$ and with $\{0, \ldots, f - 1\}$.

2.1. Group theoretic preliminaries. We consider the group scheme $\text{GL}_n$ defined over $\mathbb{Z}$, let $T \subseteq \text{GL}_n$ be the diagonal maximal torus and $Z$ its center. We write $R$ for the set of roots of $(\text{GL}_n, T)$, $W$ for its Weyl group, with longest element $w$ and let $B \subseteq \text{GL}_n$ denote the Borel of upper-triangular matrices. In particular, $B$ determines the subsets $R^+$ of positive roots. We identify the set of characters $X^*(T)$ with $\mathbb{Z}^n$ in the standard way. If $n = 2$, let $\alpha \in R^+$ correspond to $(1, -1) \in \mathbb{Z}^2$ so that $R^+ = \{\alpha\}$. If $A$ is any ring, we write $\text{GL}_{n/A}$ to denote the base change of $\text{GL}_n$ to $A$.

Let $G_0$ be the algebraic group $\text{Res}_{\mathcal{O}_K/\mathcal{O}_p} \text{GL}_n/\mathcal{O}_K$ with $T_0$ the diagonal maximal torus and center $Z_0$. Let $G$ be the base change $G_0 \times_{\mathbb{Z}_p} \mathcal{O}$, and similarly define $T$ and $Z$.

There is a natural isomorphism $G \cong \prod_{\mathcal{J}} \text{GL}_n/\mathcal{O}$ induced by the ring homomorphism $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O} \cong \mathcal{O}_\mathcal{J}$ defined by $x \otimes 1 \mapsto (\sigma_j(x))_{j \in \mathcal{J}}$. One has similar isomorphisms for $T$, $Z$, $X^*(T)$: $R$, $R^\vee$, where $R$ (resp. $R^\vee$) denotes the set of roots (resp. coroots) of $(G, T)$. If $\mu \in X^*(T)$, then we correspondingly write $\mu = (\mu_j)_{j \in \mathcal{J}}$. We have an automorphism $\pi$ on $X^*(T)$, coming from the descent data of $T$ induced by $\mathcal{O}$ and corresponding to the arithmetic Frobenius, characterized by $\pi(\mu) = \mu_{j-1}$.

We identify $X^*(T) = \oplus_{\mathcal{J}} X^*(T)$ with $(\mathbb{Z}^n)^\mathcal{J}$ as above. Moreover, if $(a_1, \ldots, a_n) \in \mathbb{Z}^n$ we write $(a_1, \ldots, a_n)$ to denote the element of $X^*(T)$ whose corresponding tuple equals $(a_1, \ldots, a_n)$ at each embedding $j \in \mathcal{J}$. We let $\eta_j = (n - 1, \ldots, 1, 0)$ in the $j$-th coordinate and 0 otherwise. We let $\eta = \sum_{j \in \mathcal{J}} \eta_j = (n - 1, \ldots, 1, 0)$.

Given $\lambda \in X^*(T)$ (resp. $\lambda \in X^*(T)$), we let $V(\lambda)/\mathcal{O}$ denote the algebraic Weyl module of $\text{GL}_n/\mathcal{O}$ (resp. $G$) with highest weight $\lambda$ as defined in [Jan03 II.8.3]. If $A$ is an $\mathcal{O}$-algebra, we write $V_A(\lambda)$ to denote the restriction of $V(\lambda)/\mathcal{O}(A)$ to $\text{GL}_n(\mathcal{O}_K)$ via the injection $\text{GL}_n(\mathcal{O}_K) \hookrightarrow \text{GL}_n(A)$ induced by the ring homomorphism $\sigma_0$. If $j \in \mathcal{J}$ and $\lambda \in X^*(T)$, we write $V(\lambda)_{/\mathcal{O}}^{(j)}$ to denote the algebraic representation of $G$ obtained, by inflation from the $j$-th projection $G \cong \prod_{\mathcal{J}} \text{GL}_n/\mathcal{O} \rightarrow \text{GL}_{n/\mathcal{O}}$, from the algebraic Weyl module $V(\lambda)/\mathcal{O}$ of $\text{GL}_n/\mathcal{O}$.

Let $R^+ \subseteq R$ (resp. $R^+/\mathcal{O} \subseteq R^\vee$) be the subset of positive roots (resp. coroots) of $G$ with respect to the upper-triangular Borel in each embedding. If $n = 2$, let $\alpha_j \in R$ be $(1, -1)$ in the $j$-th coordinate and 0 otherwise, so that $R^+ = \{\alpha_j : j = 0, \ldots, f - 1\}$.

Let $X_{\text{reg}}^+(T)$ be the set of dominant weights, i.e. the set of weights $\lambda \in X^*(T)$ satisfying $0 \leq (\lambda, \alpha^\vee)$ for all $\alpha \in R^+$. We denote by $X_1(T) \subseteq X_{\text{reg}}^+(T)$ the subset of $p$-restricted weights $\lambda \in X^*(T)$ satisfying $0 \leq (\lambda, \alpha^\vee) \leq p - 1$ for all simple roots $\alpha \in R^+$. Let $X_{\text{reg}}(T) \subseteq X_{\text{reg}}^+(T)$ be the subset of weights $\lambda \in X^*_+(T)$ satisfying $0 \leq (\lambda, \alpha^\vee) < p - 1$ for all simple roots $\alpha \in R^+$. Finally, we let $X^0(T) \subseteq X_{\text{reg}}^+(T)$ be the subset of weights $\lambda \in X^*(T)$ satisfying $(\lambda, \alpha^\vee) = 0$ for all simple roots $\alpha \in R^+$. 

The lowest alcove is defined as
\[ C_0 \overset{\text{def}}{=} \{ \lambda \in X^*(T) \otimes \mathbb{R} : 0 < \langle \lambda + \eta, \alpha^\vee \rangle < p \ \forall \alpha \in \mathbb{R}^+ \}. \]

Given \( N \geq 0 \) and \( \mu \in C_0 \) we say that \( \mu \) is \( N \)-deep in \( C_0 \) if \( N < \langle \mu + \eta, \alpha^\vee \rangle < p - N \) for all \( \alpha \in \mathbb{R}^+ \).

(Thus the existence of an \( N \)-deep weight in \( C_0 \) implies \( p \geq 2N + 2 \).)

In particular, when \( n = 2 \), via the identifications above
\[ X_1(T) = \{ \lambda \in (\mathbb{Z}^2)^f : 0 \leq \lambda_{j,1} - \lambda_{j,2} \leq p - 1 \ \forall j = 0, \ldots, f - 1 \}, \]
\[ X_{\text{reg}}(T) = \{ \lambda \in (\mathbb{Z}^2)^f : 0 \leq \lambda_{j,1} - \lambda_{j,2} < p - 1 \ \forall j = 0, \ldots, f - 1 \}, \]

and \( C_0 \cap X^*(T) = X_{\text{reg}}(T) \).

Let \( W \) be the Weyl group of \( (G, T) \), with longest element \( w_0 \). It acts on \( X^*(T) \) and we have a compatible identification of \( W \) with \( \prod_{j \in J} W \). Given \( w \in W \), we write \( w_j \) to denote its \( j \)-th component via the identification above.

Let \( W_a \) and \( \tilde{W} \) be the affine Weyl group and extended affine Weyl group, respectively, of \( G \). Concretely, \( \tilde{W}_a \cong \varLambda_R \rtimes W \) and \( \tilde{W} \cong X^*(T) \rtimes W \), where \( \varLambda_R \subseteq X^*(T) \) is the root lattice of \( G \).

The image of \( \lambda \in X^*(T) \) in \( \tilde{W} \) is denoted by \( \tilde{\lambda} \). Note that \( \tilde{W} \cong (\mathbb{Z}^n \rtimes S_n) \) and we will also write \( t_\varrho \) for the image of \( \varrho \in \mathbb{Z}^n \) in \( \mathbb{Z}^n \rtimes S_n \). We have the \( p \)-dot action of \( W \) on \( X^*(T) \), defined as follows: if \( \tilde{w} = wt_\nu \in \tilde{W} \) and \( \mu \in X^*(T) \) then \( \tilde{w} \cdot \mu \overset{\text{def}}{=} w(\mu + \eta + p\nu - \eta) \).

Let \( \Omega \) be the stabilizer of the lowest alcove \( C_0 \) in \( \tilde{W} \), so \( \tilde{W} = W_a \rtimes \Omega \). Concretely, when \( n = 2 \), it is the subgroup of \( \tilde{W} \) generated by \( X^0(T) \) and \( \{ 1, wt_{(1,0)} \} \).

Recall that the choice of \( C_0 \) endows \( W_a \) with a Bruhat order, which is denoted by \( \leq \). This induces a partial order \( \preceq \) on \( \tilde{W} \), namely \( \tilde{w}_a \omega \preceq \tilde{w}_a' \omega' \) in \( W_a \rtimes \Omega = \tilde{W} \) if and only if \( \tilde{w}_a \leq \tilde{w}_a' \) in \( W_a \) and \( \omega = \omega' \) in \( \Omega \). We denote \( \tilde{W}^\vee \) the group \( \tilde{W} \), endowed with the Bruhat order induced by the choice of the antidominant base alcove, i.e.
\[ C_0^\vee \overset{\text{def}}{=} \{ \lambda \in X^*(T) \otimes \mathbb{R} : -p < \langle \lambda + \eta, \alpha^\vee \rangle < 0 \ \forall \alpha \in \mathbb{R}^+ \}. \]

We have an anti-isomorphism
\[ \tilde{W}^\vee \overset{\sim}{\to} \tilde{W}, \quad \tilde{w} \mapsto \tilde{w}^\vee \]
defined by \((st\mu)^\vee \rangle = t_{\mu_{f - 1}} s_{f - 1}^{-1} \) such that \( \tilde{w}_1 \leq \tilde{w}_2 \) if and only if \( \tilde{w}_1^\vee \leq \tilde{w}_2^\vee \) \[ \text{[LLHL19] Lemma 2.1.3}. \) Given \( \lambda \in X^*(T) \) we let \( \text{Adm}^\vee(t_\lambda) \) denote the \( \lambda \)-admissible set in the sense of \[ \text{[KR00]} \] relative to the Bruhat order defined above on \( \tilde{W}^\vee \).

Let \( R \) be a commutative ring. If \( (x_1, \ldots, x_n) \in R^n \) we write \( \text{Diag}(x_1, \ldots, x_n) \) for the diagonal matrix of \( M_n(R) \) whose \( i \)-th diagonal entry is \( x_i \). If \( \mu \in \mathbb{Z}^n \) and \( x \in R \) then we write \( x^\mu \) for the diagonal matrix \( \text{Diag}(x^{\mu_1}, \ldots, x^{\mu_n}) \in M_n(R) \).

Sometimes it will be convenient to consider \( \tilde{W}^\vee \) as subgroup of \( \text{GL}_n(F((x))) \) by the injective homomorphism sending \( st_\mu \) to \( \tilde{s}_j \tilde{t}_\nu \rangle \), where \( \tilde{s}_j \) is the permutation matrix associated to \( s_j \in S_n \).

If \( w \in S_n \) we let \( \text{sgn}(w) \in \{ \pm 1 \} \) denotes its sign.
2.2. The inertial local Langlands correspondence and Serre weights. An inertial type is a representation \( \tau : I_K \to \GL_2(\mathbb{Q}_p) \) with open kernel which can be extended to \( G_K \).

By a result of Henniart (see the appendix to \([BM02]\)), given an inertial type \( \tau \), there is an irreducible smooth \( \GL_2(O_K) \)-representation \( \sigma(\tau) \) over \( \mathbb{Q}_p \) associated to it, normalized as in \([BM02] \S 2.1.1\). (This is often referred as the inertial local Langlands correspondence; the representation \( \sigma(\tau) \) above is the same as the representation \( \sigma(\tau) \) appearing in \([CEG^+16] \) Thm. 3.7] when, in the notation of loc. cit. \( G = \GL_2(K) \)). We remark that for any inertial type \( \tau \), the representation \( \sigma(\tau) \) can be realized over \( E \), up to enlarging \( E \) if necessary.

A Serre weight of \( G_0 \times_{Z_f} \mathbb{F}_p \) is an isomorphism class of an (absolutely) irreducible representations of \( G_0(\mathbb{F}_p) = GL_n(k) \) over \( \mathbb{F} \). If \( \lambda \in X^*_T(\mathbb{F}) \), we write \( L(\lambda)/\mathbb{F} \) (or sometimes just \( L(\lambda) \)) for the irreducible algebraic representation of \( G \times_{\mathcal{O}} \mathbb{F} \) of highest weight \( \lambda \), and \( F(\lambda) \) for the restriction of \( L(\lambda)/\mathbb{F} \) to the group \( G_0(\mathbb{F}_p) \). The map \( \lambda \mapsto F(\lambda) \) induces a bijection between \( X^*_T(\mathbb{F})/(p-\pi)X^0(\mathbb{T}) \) and the set of Serre weights of \( G_0 \times_{Z_f} \mathbb{F}_p \) (cf. \([GHS18, Lemma 9.2.4]\)). A Serre weight \( \sigma \) is regular if \( \sigma \cong F(\lambda) \) with \( \lambda \in X_{reg}(\mathbb{T}) \), cf. \([Her03, Def. 6.1]\).

If \( n = 2 \) and \( \mathfrak{p} : G_K \to GL_2(\mathbb{F}) \) is a tame Galois representation then we have a set \( W(\mathfrak{p}) \) of Serre weights, defined by Buzzard–Diamond–Jarvis in \([BDJ10]\). We emphasize that \( W(\mathfrak{p}) \) depends only on \( \mathfrak{p}|_{I_K} \).

2.3. Tame inertial types. Fix a pair \( (s, \mu) \in W \times X^*(\mathbb{T}) \), which we will use to define a tame inertial type.

Writing \( s = (s_0, \ldots, s_f-1) \in W \) we set \( s_r \defeq s_0 s_f s_{f-2} \cdots s_1 \in S_n \) and let \( r \) denote the order of \( s_r \). Let \( f' = r f, e' \defeq p^{f' - 1} \). Let \( K'/K \) be the unramified extension of \( K \) of degree \( r \) with residue field \( k' \). We fix an embedding \( \sigma_0 : k' \to \mathbb{F} \) extending \( \sigma_0 \), so we can identify \( J' \defeq \text{Hom}(k', \mathbb{F}) \) with the set \( \{0, \ldots, f' - 1\} \) via \( \sigma'_j \defeq \sigma_0 \circ \phi' \to j' \). We define the tame fundamental character \( \omega' \defeq I_K = I_{K'} \to O_{K'}^\times \to k'^\times \to \mathbb{F}^\times \), where the first map is the local Artin map, normalized so that uniformizers correspond to geometric Frobenius elements, and the last map is given by \( \sigma'_0 \). We also let \( \tilde{\omega}' : I_K \to O_K^\times \) denote the Teichmüller lift of \( \omega' \).

Define \( \alpha'_{(s, \mu)} \in (\mathbb{Z})^{\text{Hom}(k', \mathbb{F})} \cong X^*(\mathbb{T})^r \) by

\[
\alpha'_{(s, \mu), j} \defeq s_1^{-1} s_2^{-1} \cdots s_j^{-1} (\mu_j + \eta_j),
\]

where the indices on the right-hand side are considered modulo \( f \). In particular, \( \alpha'_{(s, \mu), j+kf} = s_r^{-k} \alpha'_{(s, \mu), j} \), showing that \( \alpha'_{(s, \mu), j} \) only depends on \( j \) modulo \( f' \). Also define

\[
a'_j(\mu) \defeq \sum_{i=0}^{f'-1} \alpha'_{(s, \mu), -j+i} p^{f'_i} \in \mathbb{Z}^n.
\]

Definition 2.3.1. Given \( (s, \mu) \in W \times X^*(\mathbb{T}) \) define

\[
\tau(s, \mu + \eta) \defeq \bigoplus_{1 \leq i \leq n} \omega_{f'}^{a'_i(\mu)}(s, \mu + \eta) : I_K \to GL_n(\mathcal{O}).
\]
Setting $a^{(0)} \overset{\text{def}}{=} \sum_{j=0}^{f-1} \alpha'_{(s,\mu),j}p^j$ we can also write it as

$$\tau(s, \mu + \eta) = \bigoplus_{1 \leq i \leq n} \omega^{i-1} a^{(0)}_{j(i)} p/k.$$  

From (11) we see that $\tau(s, \mu + \eta)$ is a tame inertial type, i.e. can be extended to $G_K$. Given a tame inertial type $\tau(s, \mu + \eta)$, we write $\tau(s, \mu + \eta)$ for its reduction mod $\varpi$.

**Remark 2.3.2.** Due to our choice of labeling of the embeddings of $k$ in $F$, namely $\sigma_j = \sigma_0 \circ \varphi^j$, our definition of $\tau(s, \mu + \eta)$ is not compatible with [LLHL19, Def. 2.2.1]. This choice is motivated by the fact that we do not think that the definition in loc. cit. is compatible with [Her09] and [GHS18]. However we checked that it does not affect our further references to [LLHL19].

**Definition 2.3.3.** Let $\tau$ be a tame inertial type.

(i) We say that $\tau$ is $N$-generic if there is an isomorphism $\tau \cong \tau(s, \lambda + \eta)$ for some $s \in W$ and $\lambda \in X^*(\mathcal{T})$ which is $N$-deep in alcove $C_0$.

(ii) A lowest alcove presentation of $\tau$ is a pair $(s, \mu) \in W \times C_0$ such that $\tau \cong \tau(s, \mu + \eta)$ (which by definition exists exactly when $\tau$ is (0-generic).

We also recall the following definition.

**Definition 2.3.4.** Let $\overline{\rho} : G_K \to \GL_2(F)$ be a Galois representation and let $N \in \mathbb{N}$. Let $\overline{\rho}^{\text{ss}}|_{I_K}$ denote the restriction to $I_K$ of the semisimplification of $\overline{\rho}$. We say that $\overline{\rho}$ is $N$-generic if $\overline{\rho}^{\text{ss}}|_{I_K} \cong \tau(s, \mu)$ for some $s \in W$ and $\mu - \eta \in X^*(\mathcal{T})$ which is $N$-deep in alcove $C_0$.

**Remark 2.3.5.** Note that if a type $\tau$ is $N$-generic and $(s, \lambda)$ is a lowest alcove presentation of $\tau$, the weight $\lambda$ is not necessarily $N$-deep in $C_0$. However by [LLHL19, Prop. 2.2.15], we know that $\lambda$ is $(N - 1)$-deep in $C_0$. (Similar comments apply to genericity of $\overline{\rho}$.)

Below we will need the “orientation” $s^{(i)}_{or,j} \in (S_n)^{\text{Hom}(k', F)} \cong W^r$ of $\alpha'_{(s,\mu)}$, which is defined by

$$s^{(i)}_{or,j} \overset{\text{def}}{=} s^{(i)}_{1} s^{(i)}_{2} \cdots s^{(i)}_{f'-1-j},$$

where the indices on the right-hand side are considered modulo $f$. Hence $s^{(i)}_{or,j+kf} = s^{(i)}_{or,j}$, showing that $s^{(i)}_{or,j}$ only depends on $j$ modulo $f'$. (We remark that $s^{(i)}_{or,j}$ is chosen so that $(s^{(i)}_{or,j})^{-1} \alpha'_{(s,\mu)} \in X^*(\mathcal{T})$ is dominant.)

### 2.4. Combinatorics of types and Serre weights.

Let $n = 2$. We collect results on Serre weights for mod $p$ Galois representations and Jordan–Hölder constituents of reductions of generic Deligne–Lusztig representations, expressed in terms of the extension graph of [LMS, §2]. We caution the reader that we modify slightly the definition of the extension graph and translation map appearing in loc. cit.

Let $\Lambda_W \overset{\text{def}}{=} X^*(\mathcal{T})/X^0(\mathcal{T})$ denote the weight lattice of $\Res_{k/F_p} \SL_2$. We identify $\Lambda_W$ with $\mathbb{Z}^f$ in the usual way. For $\mu \in X^*(\mathcal{T})$ we define

$$\Lambda_W^{\mu} \overset{\text{def}}{=} \{ \omega \in \Lambda_W : 0 \leq \langle \overline{\mu} + \omega, \alpha^\vee \rangle < p - 1 \forall \alpha \in \mathbb{F}_p^+ \},$$

where $\overline{\mu}$ denotes the image of $\mu$ in $\Lambda_W$. The set $\Lambda_W^{\mu}$ is called the **extension graph associated to $\mu$**.
We have an injective map
\[ t_\mu : \Lambda^\mu_W \to X_{\text{reg}}(\mathcal{T})/(p-\pi)X^0(\mathcal{T}) \]
whose image consists of the weights \( \lambda \in X_{\text{reg}}(\mathcal{T}) \) such that \( \lambda|_\mathcal{Z} = \mu|_\mathcal{Z} \) modulo \( (p-\pi)X^*(\mathcal{Z}) \). (In other words, the map \( \omega \mapsto F(t_\mu(\omega)) \) defines a bijection between \( \Lambda^\mu_W \) and regular Serre weights with central character \( \mu|_\mathcal{Z} \).

The map \( t_\mu \) is constructed as follows. Given \( \omega' \in X^*(\mathcal{T}) \) there is a unique \( \tilde{\omega}' \in \Omega \cap t_{-n-1}(\omega)W_\mu \). Setting
\[ t'_\mu(\omega') \overset{\text{def}}{=} \tilde{\omega}' \cdot (\mu + \omega') \mod (p-\pi)X^0(\mathcal{T}) \]
we thus obtain a map \( t'_\mu : X^*(\mathcal{T}) \to X^*(\mathcal{T})/(p-\pi)X^0(\mathcal{T}) \), which further factors through \( X^*(\mathcal{T}) \to X^*(\mathcal{T})/X^0(\mathcal{T}) = \Lambda^\mu_W \), by the definition of \( \tilde{\omega}' \) and since \( \cdot \) is the \( p \)-dot action. We write \( t_\mu \) for the restriction of such a map to \( \Lambda^\mu_W \), and note that \( t_\mu \) has image in \( X_{\text{reg}}(\mathcal{T})/(p-\pi)X^0(\mathcal{T}) \) by definition of \( \Lambda^\mu_W \).

**Remark 2.4.1.** In the notation of [LMS, §2.2] the set \( \Lambda^\mu_W \) above would be denoted by \( \Lambda^\mu_W^t \), and the map \( t_\mu \) above by \( t_{\mu+\eta} \).

In terms of the identification \( \Lambda_W \cong \mathbb{Z}^J \) the map \( t_\mu \) is described as follows: if \( \mu = (a_j, b_j)_j \in X^*(\mathcal{T}) \) and \( \omega = (2n_j + \delta_j)_j \in \Lambda^\mu_W \) with \( n_j \in \mathbb{Z} \), \( \delta_j \in \{0,1\} \), then a representative of \( t_\mu(\omega) \) is given by
\[
(t_\mu(\omega))_j = \begin{cases} 
(a_j + n_j + \delta_j, b_j - n_j) & \text{if } \delta_j + 1 = 0,
(b_j - 1 - n_j, a_j + n_j + \delta_j - p + 1) & \text{if } \delta_j + 1 = 1.
\end{cases}
\]

We now recall and slightly improve on a few results about \( t_\mu \) which will be important in §4 (for the combinatorics of tame inertial types and Serre weights) and in §6.2 (for the structure of certain \( GL_2(O_K) \)-representations with \( \mathbb{F} \)-coefficients).

Given \( J \subseteq \mathcal{J} \) we define \( \eta_J \overset{\text{def}}{=} \sum_{j \in J} \eta_j \in X^*(\mathcal{T}) \) and write \( \eta_J \) for the image of \( \eta_J \) in \( \Lambda_W = X^*(\mathcal{T})/X^0(\mathcal{T}) \). Define \( \Sigma \subseteq \Lambda_W \) to be the set \( \{ \eta_J : J \subseteq \mathcal{J} \} \).

**Proposition 2.4.2.** Suppose that \( \mathcal{P} : G_K \to GL_2(\mathbb{F}) \) is a tame Galois representation such that \( \mathcal{P}|_{I_K} \cong \mathcal{P}(s, \mu) \) for some \( (s, \mu) \in W \times X^*(\mathcal{T}) \) with \( \mu - \eta \) lying \( 1 \)-deep in alcove \( C_0 \). Then
\[
W(\mathcal{P}) = \{ F(t_{\mu-\eta}(sw)) : \omega \in \Sigma \}.
\]

**Proof.** From the proof of [LMS, Prop. 2.11] we see that the right-hand side of (13) is \( W_{\text{obs}}(\mathcal{P}) \), which is the set of weights defined in [GHS18, Def. 7.1.3]. By [GHS18, Ex. 7.1.7] we have \( W_{\text{obs}}(\mathcal{P}) = W(\mathcal{P}) \).

**Proposition 2.4.3.** Suppose \( \tau \overset{\text{def}}{=} \tau(sw^{-1}, \mu - sw^{-1}(\nu)) \) for some \( (s, \mu), (w, \nu) \in W \times X^*(\mathcal{T}) \) such that \( \mu - sw^{-1}(\nu) - \eta \) is \( 1 \)-deep in alcove \( C_0 \). If \( \nu \in \eta + \Lambda_R \), then
\[
JH(\sigma(\tau)) = \{ F(t_{\mu-\eta}(sw^{-1}(\omega - \mathcal{P})) : \omega \in \Sigma \}.
\]

**Proof.** Recall that, in the notation of [DL, LLHL19], we have \( \sigma(\tau) \cong R_{sw^{-1}}(\mu - sw^{-1}(\nu)) \) by [LLHL19, Cor. 2.3.5] (the deepness assumption on \( \mu - sw^{-1}(\nu) - \eta \) ensures that \( \tau \) is \( 1 \)-generic in the terminology of loc. cit., hence regular, see [LLHL19, Def. 2.2.9] and the comment after it; thus [LLHL19, Cor. 2.3.5] applies). Moreover, the deepness assumption on \( \mu - sw^{-1}(\nu) - \eta \) reads
1 < \langle \mu - sw^{-1}(\nu), \alpha^\vee \rangle < p - 1 \text{ for } \alpha \in R^+ \text{ and since } \langle sw^{-1}(\Sigma), \alpha^\vee \rangle \in \{-1, 0, 1\} \text{ we conclude that}
0 < \langle p + sw^{-1}(\Sigma - \nu), \alpha^\vee \rangle < p \text{ for } \alpha \in R^+. \text{ This is exactly the condition that } sw^{-1}(\Sigma - \nu) \subseteq \Lambda_W^{\mu - \eta}\text{ and the statement is thus immediate from [DL] Prop. 2.15} (keeping in mind that the translation map in loc. cit. is an } \eta \text{-shift of ours).}

We recall the following “change of origin” formula for the map \( t_\lambda \), obtained from [LMS] Prop. 2.5. For \( \omega \in \Lambda_W^\mu \) let \( \omega' = X^*(T) \) denote a lift of \( \omega \) and define \( w_\omega \) as the image of the unique element \( \bar{w}' \in \Omega \cap t_{-\pi^{-1}(\omega')}W_\alpha \) (as above) in \( W \). By definition, \( w_\omega \) does not depend on the choice of lift \( \omega' \) of \( \omega \) and in fact only depends on the image of \( \omega \) in \( \Lambda_W / \Lambda_R \).

**Lemma 2.4.4.** Let \( \omega \in \Lambda_W^\mu \) and let \( \lambda \in X^*(T) \) be such that \( t_\mu(\omega) \equiv \lambda \mod (p - \pi)X^0(T) \). Then \( t_\lambda(\omega') = t_\mu(w_\omega^{-1}(\omega') + \omega) \) for all \( \omega' \in \Lambda_W^\mu \). Equivalently \( t_\mu(\omega') = t_\lambda(w_\omega(\omega' - \omega)) \).

**Remark 2.4.5.** By [DL, Prop. 2.15] we have
\[ \dim_F \left( \text{Ext}_{GL_{2(k)}}(F(t_\mu(\omega)), F(t_\mu(\omega')) \right) = \begin{cases} 1 & \text{if } \omega, \omega' \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases} \]

**Proof.** Let \( \lambda \equiv t_\mu(\omega) \). By Lemma 2.4.4 we have \( t_\mu(\omega') = t_\lambda(\omega'') \) with \( \omega'' = w_\omega(\omega' - \omega) \). As \( \omega'' \) and 0 are adjacent if and only if \( \omega \) and \( \omega' \) are adjacent, we may assume that \( \omega = 0 \). By letting \( \bar{\eta}_i \) be \( \eta_i \mod X^0(T) \) we compute
\[ t_\mu(\bar{\eta}_i) \equiv w_{i-1} t_{-\eta_i} \cdot (\mu + \eta_i) \mod (p - \pi)X^0(T), \]
\[ t_\mu(-\eta_i) \equiv t_{\eta_i-1} w_{i-1} \cdot (\mu - \eta_i) \mod (p - \pi)X^0(T). \]

These are precisely the Serre weights that extend with \( F(\mu) \) by [BPT2, Cor. 5.6]. (Note that by assumption all Serre weights in this lemma are regular.)

**Remark 2.4.7.** The “change of origin” map \( \Lambda_W^\mu \to \Lambda_W^\mu \) sending \( \omega' \) to \( w_\omega^{-1}(\omega') + \omega \) (see Lemma 2.4.4) clearly preserves adjacency, i.e. is a graph automorphism. Under the identification \( \Lambda_W \cong Z^J \) it is of the form \((a_0, \ldots, a_{f-1}) \mapsto (\varepsilon_0 a_0 + n_0, \ldots, \varepsilon_{f-1} a_{f-1} + n_{f-1}) \) for some \( \varepsilon_i \in \{\pm 1\} \) and \( n_i \in Z \).
3. Galois deformations: background and lemmas

3.1. Kisin modules with descent data and the monodromy condition. We keep the setup of \(\S 2\) in particular \(K\) denotes the unramified extension of \(\mathbb{Q}_p\) of degree \(f\), with residue field \(k\). For this section we will recall and slightly extend some relevant background and notation from [LLHLM18], [LLHL20], and [LLHL19].

3.1.1. Kisin modules. From now on we fix a tame inertial type \(\tau\) together with a lowest alcove presentation \((s, \mu)\) for \(\tau\). (The lowest alcove presentation fixes an ordering of the characters in \(\tau\). This will be important in defining many of the concepts below, see Remark \(3.1.3\).) Recall that \(s_\tau = s_0s_{f-1}s_{f-2} \cdots s_1 \in S_n\) and that \(\tau\) denotes the order of \(s_\tau\). As in \(\S 2.3\) we let \(L'/K\) be the unramified extension of \(K\) of degree \(r\) with residue field \(k'\). Fix an \(e'\)-th root \((-p)^{1/e'}\) of \(-p\), let \(E(u') = (u')^{e'} + p = v + p\) denote the minimal polynomial of \((-p)^{1/e'}\) over \(K'\), and let \(L' = L'((-p)^{1/e'})\).

Let \(\Delta' \overset{\text{def}}{=} \text{Gal}(L'/K) \subseteq \Delta \overset{\text{def}}{=} \text{Gal}(L'/K)\). If \(R\) is a complete noetherian local \(\mathcal{O}\)-algebra with finite residue field define \(\mathcal{S}_{L', R} \overset{\text{def}}{=} \left(\mathcal{O}(k') \otimes_{\mathbb{Z}_p} R\right)[u']\). Given a \((\mathcal{O}(k') \otimes_{\mathbb{Z}_p} R)[u']\)-module \(\mathcal{M}\) we define \(\mathcal{M}(\mathcal{J}') \overset{\text{def}}{=} \mathcal{M} \otimes_{\mathcal{S}_{L', R}} \mathcal{J}'\), and we thus have an \(R\)-linear isomorphism \(\mathcal{M} \xrightarrow{\sim} \oplus_{\mathcal{J}' \in \mathcal{J}} \mathcal{M}(\mathcal{J}')\).

(We warn the reader that, due to our choice of normalization \(\sigma'_{\mathcal{J}'} \overset{\text{def}}{=} \sigma_0 \circ \mathcal{J}'\), we need to use the minus sign in the definition \(\mathcal{M}(\mathcal{J}') \overset{\text{def}}{=} \mathcal{M} \otimes_{\mathcal{S}_{L', R}} \mathcal{J}'\) in order to be compatible with the convention of [LLHL19] on Kisin modules, see Remark \(2.3.2\) above.)

Recall from [LLHL20, \S 3.1] that \(\mathcal{S}_{L', R}\) is endowed with an action of \(\Delta\) and by letting \(v \overset{\text{def}}{=} (u')^{e'}\) we have
\[(\mathcal{S}_{L', R})^{\Delta = 1} = \left(\mathcal{O}(k) \otimes_{\mathbb{Z}_p} R\right)[v]\]

Let \(h \geq 0\) be an integer. We define the category of Kisin modules over \(R\) of \(E(u')\)-height \(\leq h\) and descent data of type \(\tau\) as in [LLHL20, Def. 3.1.3] (with the caveat that we consider modules of rank \(n\) as opposed to 3 in loc. cit.), and denote it by \(Y^{[0, h], \tau}(R)\). Given an object \(\mathfrak{M} \in Y^{[0, h], \tau}(R)\) we have the notion of eigenbasis \(\beta = (\beta(\mathcal{J}'))\) for \(\mathfrak{M}\), as defined in [LLHL20, Def. 3.1.6], [LLHL19, Def. 3.2.8].

In particular, given a Kisin module \(\mathfrak{M} \in Y^{[0, h], \tau}(R)\) and an eigenbasis \(\beta\) of \(\mathfrak{M}\) we can consider the matrix of the Frobenius morphism \(\varphi\).

Definition 3.1.1. We let \(C_{\mathcal{M}, \beta}(\mathcal{J}') \in M_n(R[[u']])\) denote the matrix of \(\varphi^*(\mathcal{M}(\mathcal{J}')) \rightarrow \mathcal{M}(\mathcal{J}'+1)\) with respect to the bases \(\varphi^*(\beta(\mathcal{J}'))\) and \(\beta(\mathcal{J}'+1)\), i.e. \(\beta(\mathcal{J}'+1)C_{\mathcal{M}, \beta}(\mathcal{J}') = \varphi(\mathcal{J}') \cdot (\varphi^*(\beta(\mathcal{J}'))).\) We denote by \(A_{\mathcal{M}, \beta}(\mathcal{J}') \in M_n(R[[v]])\) the matrix
\[A_{\mathcal{M}, \beta}(\mathcal{J}') \overset{\text{def}}{=} \text{Ad} \left(\left(s_{\mathfrak{m}, \mathcal{J}'+1}\right)^{-1}(u')^{-\mathcal{a}(\mathcal{J}', \mathfrak{m})}(C_{\mathcal{M}, \beta}(\mathcal{J}'))\right)\]
(see also [LLHM] equation (5.4)], where \(C_{\mathcal{M}, \beta}(\mathcal{J}')\) in loc. cit. denotes the matrix of \(\varphi^*(\mathcal{M}(\mathcal{J}'-1)) \rightarrow \mathcal{M}(\mathcal{J}')\).
Remark 3.1.2. We caution that $\text{Ad}(s(u')^\mu)$ denotes $\text{Ad}(\hat{s}) \text{Ad}((u')^\mu)$ and not $\text{Ad}((u')^{s(\mu)})$, and we remind the reader that $\hat{s}$ is the permutation matrix representing $s$ and that we have $(u')^\mu = \text{Diag}((u')^{\mu_1}, \ldots, (u')^{\mu_n})$ for $\mu \in \mathbb{Z}^n$.

Remark 3.1.3. We stress that the notion of eigenbasis and the definition of $A_{\mathfrak{m}, \beta}^{(j')}$ depends on the choice of the lowest alcove presentation $(s, \mu)$ for $\tau$. Moreover, when $\mu$ is 1-deep in alcove $C_0$, the matrix $A_{\mathfrak{m}, \beta}^{(j')}$ only depends on $j'$ modulo $f$ and is upper-triangular modulo $v$ (see the discussion after [LLHLM, Rk. 5.1.7]).

If $\lambda = (\lambda_{j,1}, \ldots, \lambda_{j,n}) \in X^*(T)$ is a dominant character such that $\lambda_{j,i} \in \{0, \ldots, h\}$ for all $j, i$, we have a closed $p$-adic formal substack $Y^{\leq h, \tau}$ of $Y^{[0, h], \tau}$ defined in [CL18, Theorem 5.3], which is flat over $\mathcal{O}$ and has reduced versal rings. It is characterized by the property that for any flat $p$-adically complete noetherian local $O$-algebra $R$, a Kisin module $\mathfrak{M} \in Y^{[0, h], \tau}(R)$ belongs to $Y^{\leq h, \tau}(R)$ if and only if all $i$ by $i$ minors of $A_{\mathfrak{m}, \beta}^{(j)}$ are divisible by $(v + p)^{\sum_{k=1}^{i} \lambda_{j,n+1-k}}$, for $i \in \{1, 2, \ldots, n\}$ (cf. [LLHLM] the discussion after Warning 5.3.2, see also [LLHLM18, Prop. 4.18]). This definition does not depend on the choice of the eigenbasis for $\mathfrak{M}$. We caution that Remark 3.1.2.

Definition 3.1.4. Let $\tilde{\mathfrak{M}} \in Y^{[0, h], \tau}(\mathbb{F})$. Write $\mathcal{I}(\mathbb{F})$ for the Iwahori subgroup of $\text{GL}_n(\mathbb{F}[v])$ consisting of matrices which are upper triangular modulo $v$. We say that $\tilde{\mathfrak{M}}$ has shape $\tilde{\omega} \in \tilde{W}$ with respect to $\tau$ if for any choice of eigenbasis $\tilde{\beta}$ the equality

$$\mathcal{I}(\mathbb{F}) A_{\tilde{\mathfrak{m}}, \tilde{\beta}}^{(j)} \mathcal{I}(\mathbb{F}) = \mathcal{I}(\mathbb{F}) \tilde{\omega}_j \mathcal{I}(\mathbb{F})$$

holds in $\text{GL}_n(\mathbb{F}(v))$ for all $j = 0, \ldots, f - 1$. This notion is independent of $\tilde{\beta}$ by [LLHLM18, Prop. 2.15, 2.16], but again depends on the choice of lowest alcove presentation of $\tau$.

Fix $\tilde{\mathfrak{M}} \in Y^{[0, h], \tau}(\mathbb{F})$ we recall that an eigenbasis $\tilde{\beta}$ is a gauge basis if $A_{\tilde{\mathfrak{m}}, \beta}^{(j)}$ has a particularly simple form [LLHL19, Def. 3.2.23]. A gauge basis always exists and is unique up to scaling by $\{(t_j) \in \mathcal{I}(\mathbb{F})' : t_j \equiv t_k \pmod{f}\}$ (this is [LLHL19, Prop. 3.2.22] in the particular case $h = n - 1$, and the general case follows from [LLHLM] Prop. 5.1.8, Lemma 5.2.2).

We now fix $\tilde{\mathfrak{M}} \in Y^{[0, h], \tau}(\mathbb{F})$ together with a gauge basis $\tilde{\beta}$. Write $\tilde{\omega} = (w_j t_{w_j})_j \in \tilde{W}'$ for its shape with respect to $\tau$.

The following result, generalizing [LLHLM18, Thm. 4.1, Thm. 4.16], [LLHL19, Prop. 3.4.3], is a particular case of [LLHLM Prop. 5.2.7].

Proposition 3.1.5. Let $R$ be a complete noetherian local $O$-algebra with residue field $\mathbb{F}$, and let $\tau$ be an $(h + 1)$-generic tame inertial type. Let $\tilde{\mathfrak{m}} \in Y^{[0, h], \tau}(R)$ together with an isomorphism $\mathfrak{M} \cong \mathfrak{M} \cong \tilde{\mathfrak{M}}$.

Then there exists an eigenbasis $\beta$ for $\mathfrak{M}$ lifting $\beta$ such that for all $1 \leq i, k \leq n$ and all $j = 0, \ldots, f - 1$ we have

1. $A_{\tilde{\mathfrak{m}}, \beta}^{(j)} \in v^{\delta_{j,k}} R[v + p]$,
2. $\deg_v A_{\tilde{\mathfrak{m}}, \beta}^{(j)} \leq v_{j,k} - \delta_{j,w_j(k)}$ with equality if $(i, k) = (w_j(k), k)$,

where $A^{(j)} := A_{\tilde{\mathfrak{m}}, \beta}^{(j)}$. Furthermore, such a $\beta$ is uniquely determined up to scaling by the group $\{(t_j) \in (\ker(T(R) \to T(\mathbb{F})))' : t_j = t_k \pmod{f}\}$.
Definition 3.1.6. Let $R$ be a complete noetherian local $O$-algebra with residue field $F$, and let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ together with an isomorphism $\mathfrak{M} \otimes_R F \cong \mathfrak{M}$. A gauge basis of $\mathfrak{M}$ is an eigenbasis $\beta$ lifting $\beta$ that satisfies conditions (i) and (ii) of Proposition 3.1.5.

3.1.2. Monodromy condition. Let $R$ be a $p$-adically complete flat $O$-algebra that is topologically of finite type. Define $\mathcal{O}^\text{rig}_R$ as the inverse limit over $n$ of $R[[u', u''_n]][1/p]$, the transition maps being the natural inclusions. The Frobenius $\varphi: u' \mapsto (u')^p$ on $R[[u']]$ extends naturally to $\mathcal{O}^\text{rig}_R$. By letting

$$
\lambda \overset{\text{def}}{=} \prod_{n=0}^{\infty} \varphi^n \left( \frac{E(u')}{{p}} \right) = \prod_{n=0}^{\infty} \left( 1 + \frac{\varphi^n}{{p}} \right) \in \mathcal{O}^\text{rig}_R \subseteq \mathcal{O}^\text{rig}_R.
$$

we have the derivation $N_{\mathfrak{M}} \overset{\text{def}}{=} -u'\lambda \frac{d}{du'}$ of $\mathcal{O}^\text{rig}_R$.

Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ and write $\mathfrak{M}^\text{rig}$ for the base change $\mathfrak{M} \otimes_R u'[1/p] \mathcal{O}^\text{rig}_R$, which decomposes as $\mathfrak{M}^\text{rig} = \bigoplus_{j'} \mathfrak{M}^\text{rig}(j')$.

The following result builds on [Kis06, Cor. 1.3.15] and is stated in [LLHLM, Prop. 7.1.3].

Proposition 3.1.7. Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ for $R$ a $p$-adically complete flat $O$-algebra that is topologically of finite type. Then, $\mathfrak{M}^\text{rig}[1/\lambda]$ is equipped with a unique derivation $N_{\mathfrak{M}^\text{rig}}$ over $\mathfrak{M}$ such that

$$
N_{\mathfrak{M}^\text{rig}} \varphi_{\mathfrak{M}^\text{rig}} = E(u') \varphi_{\mathfrak{M}^\text{rig}} N_{\mathfrak{M}^\text{rig}}
$$

and $N_{\mathfrak{M}^\text{rig}} \mod u' = 0$.

We have a decomposition of $N_{\mathfrak{M}^\text{rig}}$ into $N^{(j')} : \mathfrak{M}^\text{rig}(j') \to \mathfrak{M}^\text{rig}(j')$ and we write $N^{(j')}_{\mathfrak{M}^\text{rig}, \beta}$ to denote the matrix of the endomorphism $N^{(j')}_{\mathfrak{M}^\text{rig}}$ with respect to the basis $\beta(j')$, i.e. $\beta(j') N^{(j')}_{\mathfrak{M}^\text{rig}, \beta} = N^{(j')}_{\mathfrak{M}^\text{rig}}(\beta(j'))$.

Definition 3.1.8. Let $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ with eigenbasis $\beta$. The monodromy condition is the condition that $\lambda^{h-1} N^{(j')}_{\mathfrak{M}^\text{rig}, \beta}$ vanishes to order $h - 1$ at $u' = (-p)^{1/\epsilon'}$ for all $j'$. We see as in [LLHLM, Prop. 5.3] that the condition above is equivalent to $N_{\mathfrak{M}^\text{rig}}(\mathfrak{M}^\text{rig}) \subseteq \mathfrak{M}^\text{rig}$. As in the proof of Thm. 6.14 in [LLHLM], the monodromy condition only depends on $j'$ modulo $f$.

As in [LLHLM, Thm. 5.6], [LLHL, Prop. 3.4.12], given $\mathfrak{M} \in Y^{[0,h],\tau}(R)$ with eigenbasis $\beta$, the matrix $N^{(j')}_{\mathfrak{M}^\text{rig}, \beta}$ can be expressed as

$$
N^{(j')}_{\mathfrak{M}^\text{rig}, \beta} = N^1(j') + \sum_{i=1}^{\infty} \left( \prod_{k=0}^{i-1} \varphi^k(C^{(j'-k+1)}_{\mathfrak{M}^\text{rig}, \beta}) \right) \varphi^i(N_1(j'-i)) \left( \prod_{k=i-1}^{0} \varphi^k(E(u')(C^{(j'-k+1)}_{\mathfrak{M}^\text{rig}, \beta})^{-1}) \right),
$$

where $N^1(j')$ satisfies

$$
\text{Ad} \left( (s_{\alpha,j'})^{-1}(u')^{-a_{(s',\alpha)}} \right) (\lambda^{h-1} N^1(j')) =
$$

$$
= -\left( \frac{\varphi(\lambda)}{p} \right)^h \left( -\epsilon' \frac{d}{du} A^{(j'-1)}_{\mathfrak{M}^\text{rig}, \beta} - \left[ \text{Diag}(s_{\alpha,j'})^{-1}(a_{(s',\alpha)}) \right] (v + p)^h (A^{(j'+1)}_{\mathfrak{M}^\text{rig}, \beta})^{-1} \right).$$
In what follows, define the *leading term of the monodromy condition*

\[
P_N(A^{(j-1)}_{2\mathfrak{m},\beta}) \triangleq \left( -e^t \frac{d}{dv} A^{(j-1)}_{2\mathfrak{m},\beta} - \left[ \text{Diag}((s'_{\alpha,r,j})^{-1}(a'_{\tau,\nu,j})), A^{(j-1)}_{2\mathfrak{m},\beta} \right] \right)(v + p)h (A^{(j-1)}_{2\mathfrak{m},\beta})^{-1}
\]

(where \([M, N] \triangleq MN - NM\), which again only depends on \(j\) modulo \(f\).

**Proposition 3.1.9 (LLHLM18).** Let \(\mathfrak{m} \in Y^{[0,h]}(R)\) with eigenbasis \(\beta\). The monodromy condition is equivalent to the condition that

\[
\left( \frac{d}{dv} \right)^t \bigg|_{v'=-p^{1/c'}} \left[ \text{Ad} \left( (s'_{\alpha,r,j})^{-1}(a'_{\tau,\nu,j}) \right)(\lambda^{h-1}N^{(j')}_{2\mathfrak{m},\beta}) \right] = 0
\]

for all \(t = 0, \ldots, h - 2, j' = 0, \ldots, f' - 1\) and only depends on \(j'\) modulo \(f\).

Assume that \(\tau\) is \(N\)-generic, where \(N \geq 2h - 3\) and \((N - 1)(p - 1) \geq h\). Then the monodromy condition has the form

\[
P_N(A^{(j-1)}_{2\mathfrak{m},\beta}) \in (v + p)h^{-1} M_n(R[v]) + p^{N-2h+3} M_n(R[v])
\]

for all \(j = 0, \ldots, f - 1\).

**Proof.** The proof is along the lines of LLHLM18 Thm. 5.6] and LLHL19 Prop. 3.4.12. Recall from the proof of LLHL19 Prop. 3.4.12 that the monodromy condition is equivalent to \(\lambda^{h-1}N^{(j')}_{2\mathfrak{m},\beta}\) vanishing to order \(h - 1\) at \(v' = (-p)^{1/c'}\) for all \(j'\), i.e. equivalent to the vanishing of \(\left( \frac{d}{dv} \right)^t \bigg|_{v'=-p^{1/c'}} (\lambda^{h-1}N^{(j')}_{2\mathfrak{m},\beta})\) for all \(t = 0, \ldots, h - 2\) and all \(j'\). As \(u'\) is invertible in \(R[u'/E(u')]\)[loc. cit.], the vanishing condition above is equivalent to condition (16) for all \(t = 0, \ldots, h - 2\) and all \(j'\).

As in the proof of LLHL19 Prop. 3.4.12, we see that

\[
\text{Ad} \left( (s'_{\alpha,r,j})^{-1}(a'_{\tau,\nu,j}) \right)(\lambda^{h-1}N^{(j')}_{2\mathfrak{m},\beta}) = \left( \frac{\varphi(\lambda)}{p} \right)^h \sum_{i=0}^{\infty} (\varphi^{i+1}(\lambda))^{h} Z^{(j')}_{i}
\]

where

\[
Z^{(j')}_{i} \triangleq A^{(j' - 1)}_{2\mathfrak{m},\beta} \text{Ad} \left( s_{j'-1,j'}^t u^t v^{\epsilon_{j'-1}} \right)(\varphi^{i+1}(\lambda))^{h} Z^{(j')}_{i}
\]

for \(i > 0\) and

\[
Z^{(j')}_{0} \triangleq -P_N(A^{(j'-1)}_{2\mathfrak{m},\beta}).
\]

(These are analogous to the formulas in LLHLM18 Thm. 5.6). We point out the incorrect definition of \(Z^{(j'-1)}_{0}\) in loc. cit.)

As in LLHL19 Prop. 3.4.12 we see that \(Z^{(j')}_{i} \in \frac{p^{(N-1)i}}{p^{(h-1)}} M_n(R[v])\) for \(i > 1\) and \(Z^{(j')}_{1} \in \frac{p^{N}}{p^{h-1}} M_n(R[v])\), as \(\tau\) is \(N\)-generic. Hence, by letting \(M^{(j')} \triangleq \frac{1}{\varphi(\lambda)^h} \sum_{i=1}^{\infty} (\varphi^{i+1}(\lambda))^{h} Z^{(j')}_{i}\) we conclude that \(\left( \frac{1}{d\epsilon_{j'-1}} \right)^t \bigg|_{v=-p} M^{(j')} \in p^{N-(h-1)-t} M_n(R)\) for \(t = 0, \ldots, h - 2\). (Note that

\[
\left( \frac{d}{dv} \right)^t \bigg|_{v=-p} (\varphi^{i+1}(\lambda)/\varphi(\lambda))^{h} Z^{(j')}_{i}
\]
Remark 3.2.2. \( \text{modulo } f \) In this case, any extension of Definition 3.2.4. (\cite{LLHL19, Def. 3.1.1}) For \( \tau \) (dealing with the factor \( (\varphi(\lambda)/p)^h \) in (17)), that the monodromy condition is equivalent to

\[
\left( \frac{d}{dv} \right)^t |_{v = -p} \left[ -P_N(A_{j_0, j_1}^{(j)} - 1) + M^{(j)} \right] = 0
\]

for all \( j' \) and all \( t = 0, \ldots, h - 2 \), i.e., by the previous paragraph, equivalent to

\[
\left( \frac{d}{dv} \right)^t |_{v = -p} \left( P_N(A_{j_0, j_1}^{(j)} - 1) \right) + O(p^{N-(h-1)-t}) = 0
\]

for all \( j' \) and all \( t = 0, \ldots, h - 2 \). But the condition above can be rewritten as

\[
P_N(A_{j_0, j_1}^{(j)} - 1) \in (v + p)^{h-1} M_n(R[v]) + p^{N-2h+3} M_n(R[v])
\]

\( \square \)

3.2. Lemmas on mod \( p \) Galois representations. Given \( (s, \mu) \in W \times X^*(T) \), consider the reduction \( \tau(s, \mu) : I_K \to \text{GL}_n(F) \) of the same inertial type \( \tau(s, \mu) \). Typically, the length of \( \tau(s, \mu) \) as representation of \( I_K \) equals the number of orbits of \( s_\tau = s t s t_1 \cdots s_1 \in S_n \). The following definition gives the precise condition for this to be true.

Definition 3.2.1. We say that \( (s, \mu) \in W \times X^*(T) \) is good if

\[
\sum_{j=0}^{fd(i)-1} p^j(s_1^{-1} \cdots s_1^{-1}(\mu_j)) \not\equiv 0 \pmod{q^{d(i)} - 1} \quad \forall 1 \leq i \leq n \quad \forall d | d(i), 1 \leq d < d(i),
\]

where \( d(i) \geq 1 \) is minimal such that \( s_1^{-1} s_2^{-1} \cdots s_1^{-1}(i) = i \) (and where the indices are considered modulo \( f \)).

Remark 3.2.2. Definition 3.2.1 generalizes \cite{Her09} Def. 6.19. We see that \( \tau(s, \mu) \) is the restriction to \( I_K \) of an irreducible representation of \( G_K \) if and only if \( s_\tau \) has order \( n \) and \( (s, \mu) \) is good. Just note from Definition 3.2.1 that

\[
\tau(s, \mu) \cong \bigoplus_{i=1}^n \omega^{ \sum_{j=0}^{fd(i)-1} p^j(s_1^{-1} \cdots s_1^{-1}(\mu_j)) }.
\]

In this case, any extension of \( \tau(s, \mu) \) to a \( G_K \)-representation is irreducible.

Lemma 3.2.3. If \( \mu - \eta \in C_0 \), then \( (s, \mu) \) is good for any \( s \in W \).

Proof. Fix \( i \in \{1, \ldots, n\} \). Let \( \nu \defeq \sum_{j=0}^{f(i)-1} p^j s_1^{-1} \cdots s_1^{-1}(\mu_j) \in \mathbb{Z}^n \) and let \( c_k \defeq (s_\tau^{-k} \nu)_i \). By assumption, \( 0 < \langle \mu_j, \alpha_i^\vee \rangle < p \) for all \( i \), which implies that \( 0 < |c_k - c_\ell| \) for all \( k \neq \ell \) (mod \( d(i) \)). It suffices to show that \( \sum_{k=0}^{d(i)-1} k c_k \not\equiv 0 \pmod{q^{d(i)} - 1} \) for all \( d | d(i), 1 \leq d < d(i) \). This follows exactly as in the proof of \cite{Her09} Lemma 6.24. (Alternatively one can check that Definition 3.2.1 is equivalent to the definition given in \cite{LLHL19} §2.2 and invoke \cite{LLHL19} Lemma 2.2.3.) \( \square \)

Definition 3.2.4. (\cite{LLHL19} Def. 3.1.1) For \( w \in \hat{W} \) and \( D \in T^*(F) \), let \( \mathcal{M}(\hat{w}, D) \) denote the \( \text{étale } \varphi \)-module of rank \( n \) over \( k((v)) \otimes_{F_p} F \) such that \( \text{Mat}(\varphi^{(j)}) = D_j \hat{w}_j \) with respect to the standard basis.
**Definition 3.2.5.** For $\tilde{w} \in \overline{W}^\vee$ and $D \in \mathcal{T}(\mathbb{F})$, let $V(\tilde{w}, D)$ be the unique tame representation of $G_K$ over $\mathbb{F}$ of dimension $n$ such that

$$V(\tilde{w}, D)|_{G_{K_{\infty}}} \cong \mathcal{V}^*_K(M(\tilde{w}, D)),$$

where $\mathcal{V}^*_K$ denotes the contravariant functor of $\text{Fon90}$ from étale $\varphi$-modules to representations of $G_{K_{\infty}}$ (see also [LLHL19, §3.1], where it is denoted by $\mathcal{V}^*$). Its existence and uniqueness is guaranteed by [LLHL19, Prop. 3.1.2] and the equivalence for tame representations in [LLHL19 §3.1].

**Lemma 3.2.6.** For $\lambda \in (\mathbb{F}^\times)^f$ we have

$$V(\tilde{w}, \lambda D) \cong V(\tilde{w}, D) \otimes_{\mathbb{F}} \text{nr}(\prod_{j=0}^{f-1} \lambda_j),$$

where $\text{nr}(\alpha)$ denotes the unramified character of $G_K$ sending an arithmetic Frobenius to $\alpha \in \mathbb{F}^\times$.

**Proof.** As $\mathcal{M}(\tilde{w}, \lambda D)$ is the tensor product of $\mathcal{M}(\tilde{w}, D)$ and $\mathcal{M}(1, \lambda)$ over $k((v)) \otimes_{\mathbb{F}_p} \mathbb{F}$ and $\mathcal{V}^*_K$ is a tensor functor, it suffices to show that

$$V(1, \lambda) \cong \text{nr}(\prod_{j=0}^{f-1} \lambda_j).$$

Note that $\mathcal{M}(1, \lambda)$ is isomorphic to the rank one étale $\varphi$-module with

$$\varphi(j) = \begin{cases} 1 & \text{if } 0 \leq j < f - 1, \\ \prod_{j'=0}^{f-1} \lambda_{j'} & \text{if } j = f - 1 \end{cases}$$

in the standard basis. By the proof of [GLS14, Lemma 6.3], $\mathcal{V}^*_K(\mathcal{M}(1, \lambda)) \cong \text{nr}(\prod_{j=0}^{f-1} \lambda_j)|_{G_{K_{\infty}}}$. □

**Proposition 3.2.7.** Suppose $\tilde{w} \in \overline{W}^\vee$, $\tilde{w}^s = t \mu s'$ with $(s', \mu') \in \overline{W} \times X^s(\mathcal{T})$ good. Then

$$\{ \rho : G_K \to \text{GL}_n(\mathbb{F}) : \rho|_{I_K} \cong \tau(s', \mu') \}_{/\cong} = \{ V(\tilde{w}, D) : D \in \mathcal{T}(\mathbb{F}) \}_{/\cong}.$$

**Proof.** By [LLHL19, Prop. 3.1.2] we know that the right-hand side is contained in the left-hand side. As in line 1 of the proof of [LLHL19, Prop. 3.1.2] we may assume that $(\tilde{w}^s)_j = 1$ for all $0 \leq j < f - 1$. Then we can split $\mathcal{M}(\tilde{w}, D)$ into a direct sum of $\varphi$-modules according to the orbits of $(s')_{f-1} \in S_n$, so without loss of generality $s'_0$ has only one orbit. (Note that the goodness of $(s', \mu')$ is compatible with this decomposition.) As $(s', \mu')$ is good and $s'_0$ has only one orbit, we deduce by Remark 3.2.2 that $V(\tilde{w}, D)$ is irreducible. By Lemma 3.2.6 it follows that the left-hand side is contained in the right-hand side. □

Recall that $\overline{\rho} : G_K \to \text{GL}_n(\mathbb{F})$ is cyclotomic free if $\overline{\rho}$ becomes upper triangular over an unramified extension $K'/K$ of degree prime to $p$ such that $H^0(G_{K'}, (\overline{\rho}|_{G_{K'}})^{18} \otimes_{\mathbb{F}} \omega^{-1}) = 0$ [LLHLM18, Def. 3.8].

**Lemma 3.2.8.** If $\overline{\rho}_1, \overline{\rho}_2$ are finite-dimensional representations of $G_K$ over $\mathbb{F}$ such that $\overline{\rho}_1 \otimes_{\mathbb{F}} \overline{\rho}_2$ is cyclotomic free, then the natural map

$$\text{Hom}_{G_K}(\overline{\rho}_1, \overline{\rho}_2) \to \text{Hom}_{G_{K_{\infty}}}(\overline{\rho}_1|_{G_{K_{\infty}}}, \overline{\rho}_2|_{G_{K_{\infty}}})$$

is an isomorphism.
Proof. The statement is true if \( \mathfrak{p}_1, \mathfrak{p}_2 \) are tame, or equivalently semisimple, cf. the beginning of [LLHL19] \S3.1. In particular, the lemma holds if \( \mathfrak{p}_1, \mathfrak{p}_2 \) are irreducible. For the general case, we first notice that for any subquotients \( \mathfrak{p}_i' \) of \( \mathfrak{p}_i \) (\( i = 1, 2 \)), the tensor product \( \mathfrak{p}_1' \otimes \mathfrak{p}_2' \) is cyclotomic free (as it is a subquotient of \( \mathfrak{p}_1, \mathfrak{p}_2 \)), so that \( \text{Ext}^1_{G_K}(\mathfrak{p}_1, \mathfrak{p}_2) \to \text{Ext}^1_{G_K}(\mathfrak{p}_1', \mathfrak{p}_2') \) is injective by [LLHL19] Lemma 3.10]. We can therefore argue by dévissage on \( \mathfrak{p}_1 \) and \( \mathfrak{p}_2 \).

**Corollary 3.2.9.** If \( \mathfrak{p}_1, \mathfrak{p}_2 \) are finite-dimensional representations of \( G_K \) over \( \mathbb{F} \) such that \( \mathfrak{p}_1 \) is 2-generic (defined analogously to [LLHL19] Def. 3.7), then the natural injective map

\[
\text{Isom}_{G_K}(\mathfrak{p}_1, \mathfrak{p}_2) \to \text{Isom}_{G_{K\infty}}(\mathfrak{p}_1|_{G_{K\infty}}, \mathfrak{p}_2|_{G_{K\infty}})
\]

is a bijection.

**Proof.** We first claim that \( \mathfrak{p}^{ss}|_{G_{K\infty}} \cong (\mathfrak{p}|_{G_{K\infty}})^{ss} \) for any finite-dimensional representation \( \mathfrak{p} \) of \( G_K \) over \( \mathbb{F} \), i.e. that \( \mathfrak{p}^{ss}|_{G_{K\infty}} \) is already semisimple. This follows as in [LLHL19] \S3.1]: \( \mathfrak{p}^{ss} \) is a representation of \( G_K/I_{K'}^e \), where \( I_{K'}^e \) is the wild inertia group and \( G_{K\infty}/(G_{K\infty} \cap I_{K'}^e) \cong G_K/I_{K}^e \), as \( K_{\infty}/K \) is a totally ramified \( p \)-extension.

Assume \( \text{Isom}_{G_{K\infty}}(\mathfrak{p}_1|_{G_{K\infty}}, \mathfrak{p}_2|_{G_{K\infty}}) \neq 0 \). By the previous paragraph and again by the beginning of [LLHL19] \S3.1] we thus have \( \mathfrak{p}_1^{ss} \cong \mathfrak{p}_2^{ss} \), hence \( (\mathfrak{p}_1 \otimes \mathfrak{p}_2)^{ss} \cong \text{ad}(\mathfrak{p}_1)^{ss} \). As \( \text{ad}(\mathfrak{p}_1) \) is cyclotomic free by the analog of [LLHL19] Prop. 3.9], we obtain \( \mathfrak{p}_1 \otimes \mathfrak{p}_2 \) cyclotomic free, and we can then conclude by Lemma 3.2.8. \( \square \)

### 3.3. Some commutative algebra lemmas.

**Lemma 3.3.1.** Let \( A \overset{\text{def}}{=} \mathcal{O}[[x_1, \ldots, x_n]] \), where \( \mathcal{O} \) is a complete DVR with uniformizer \( \varpi \) and \( n \geq 2 \). If \( f \in A^\times \) and \( d > 0 \), then \( x_1 x_2 + \varpi^d f \) is irreducible in \( A \). Moreover the ideals \((x_1 x_2 + \varpi^d f)\) and \((x_1)\) are distinct, and the ideals \((x_1 x_2 + \varpi^d f)\) and \((x_1 x_2 + \varpi^d g)\) are distinct if \( f \neq g \mod \mathfrak{m}_A \).

**Proof.** By the \( \mathcal{O} \)-automorphism of \( A \) sending \( x_2 \) to \( x_1 + x_2 \) and fixing \( x_1 \) (\( i \neq 2 \)), we may instead consider \( x_1^2 + x_1 x_2 + \varpi^d g \) \((g \in A^\times)\). By the Weierstrass preparation theorem, if \( x_1^2 + x_1 x_2 + \varpi^d g \) is reducible then it has a factor of the form \( x_1 - b \) for some \( b \in \mathcal{O}[x_2, \ldots, x_n] \). Evaluating at \( x_1 = b \) we see that \( b^2 + bx_2 + \varpi^d g(b, x_2, \ldots, x_n) = 0 \), so \( \varpi^d \mid b(x_2 + x_n) \). Hence \( \varpi^d \mid b \) or \( \varpi^d \mid (b + x_2) \). In the first case, \( b = \varpi^d c \) and \( \varpi^d c^2 + cx_2 + g(\varpi^d c, x_2, \ldots, x_n) = 0 \), so \( g \in \mathfrak{m}_A \), contradiction. The second case is similar, and the last part is straightforward. \( \square \)

**Lemma 3.3.2.** Suppose \( R, S \) are complete noetherian local \( \mathcal{O} \)-algebras and \( I_i \) for \( 1 \leq i \leq n \) (resp. \( J_j \) for \( 1 \leq j \leq m \)) are (closed) ideals of \( R \) (resp. \( S \)). If \( R/I_i \) and \( S/J_j \) are \( \mathcal{O} \)-flat for all \( i, j \) then

\[
\bigcap_{i,j} (I_i, J_j) = \left( \bigcap_i I_i, \bigcap_j J_j \right)
\]

as ideals of \( R \otimes \mathcal{O} S \).

**Proof.** By induction we may assume that \( n = 1 \) and \( m = 2 \). Let \( I \overset{\text{def}}{=} I_1 \). We recall that \( R \) and \( S \) with their natural topologies are pseudocompact \( \mathcal{O} \)-modules. Hence if \( M \) is any pseudocompact \( \mathcal{O} \)-module, the functor \( M \otimes \mathcal{O} (-) \) is right exact and it is exact if and only if \( M \) is \( \mathcal{O} \)-flat [ABD+65 Exposé VIIb, 0.3.7, 0.3.8]. Consider the homomorphisms

\[
R \otimes \mathcal{O} S \to R/I \otimes \mathcal{O} S/(J_1 \cap J_2) \to (R/I \otimes \mathcal{O} S/J_1) \oplus (R/I \otimes \mathcal{O} S/J_2).
\]
The second map is injective by flatness, and the kernel of the first map is \((I, J_1 \cap J_2)\) by right exactness. By considering the kernel of the composite, we deduce that \((I, J_1 \cap J_2) = (I, J_1) \cap (I, J_2)\). \(\square\)
4. Galois deformation rings

4.1. Setup. From now on we consider the situation where \( n = 2 \).

Throughout this section we fix a semisimple Galois representation \( \bar{\rho} : G_K \to \text{GL}_2(\mathbb{F}) \) such that \( \bar{\rho}|_{I_K} \cong \tau(s, \mu) \), where

\[
\begin{align*}
(\text{i}) &\quad s_j \neq 1 \text{ (hence, } s_j = w) \text{ precisely when } j = 0 \text{ and } \bar{\rho} \text{ is irreducible;} \\
(\text{ii}) &\quad \mu - \eta \text{ is } N\text{-deep in } \mathcal{C}_0 \text{ with } N \geq 9.
\end{align*}
\]

(This specific form of the lowest alcove presentation for \( \bar{\rho} \) depends on the choice of the embedding \( \sigma_0 \); however, we see from Remark 2.3.5 that when \( \bar{\rho} \) is 10-generic the conditions (i)(ii) above can always be arranged by an appropriate choice of \( s \).) Up to a twist by a power of \( \omega_f \) we can furthermore assume that \( \mu_j = (r_j + 2, 1) \in \mathbb{Z}^2 \) with \( N < r_j + 1 < p - N \) for all \( j \), and hence

\[
\bar{\rho}|_{I_K} \cong \left\{ \begin{array}{ll}
(\sum_{j=0}^{f-1}(r_j+1)p^j \oplus \mathbb{1}) \otimes \omega & \text{if } \bar{\rho} \text{ is reducible}, \\
(\sum_{j=0}^{f-1}(r_j+1)p^j \oplus \sum_{j=0}^{f-1}(r_j+1)p^j+1) \otimes \omega & \text{if } \bar{\rho} \text{ is irreducible}.
\end{array} \right.
\]

In this section we will study various framed Galois deformation rings of \( \bar{\rho} \), for which 3\( f \) tame inertial types play a role, and we now introduce them. Given

\[ \tilde{w} \in \text{Adm}^\vee(t_{(2,1)}) = \{ t_{(2,1)}, w t_{(2,1)}, t_{(1,2)} \} \]

arbitrary, write \( \tilde{w}^\ast = t_{w^\ast}w \) for \( (w, \nu) \in \mathsf{W} \times X^\ast(T) \). Define the type

\[ \tau_{\tilde{w}} \overset{\text{def}}{=} \tau(sw^{-1}, \mu - sw^{-1}(\nu)), \]

which we always consider together with its lowest alcove presentation \( (s(\tau), \mu(\tau)) = (sw^{-1}, \mu - sw^{-1}(\nu) - \eta) \).

Concretely, \( s(\tau)_j = w_j^{-1} \) except when \( j = 0 \) and \( \bar{\rho} \) is irreducible, in which case we have \( s(\tau)_0 = w_0^{-1}, \) and

\[ \mu(\tau)_j + \eta_j = \begin{cases} (r_j, 0) & \text{if } (t_{w^\ast}w_j, s_j) \in \{ (t_{(2,1)}, 1), (t_{(2,1)}w, w), (t_{(1,2)}, w) \}, \\
(r_j + 1, -1) & \text{if } (t_{w^\ast}w_j, s_j) \in \{ (t_{(2,1)}, w), (t_{(2,1)}w, 1), (t_{(1,2)}, 1) \}.
\end{cases} \]

Then

\begin{equation}
\tau_{\tilde{w}} \cong \left\{ \begin{array}{ll}
\tilde{\omega}_f^{(0)} \oplus \tilde{\omega}_f^{(0)} & \text{if } \prod_{j=0}^{f-1} s(\tau)_j = 1, \\
\tilde{\omega}_2^{(0)} + p \tilde{\omega}_2^{(0)} & \oplus \tilde{\omega}_2^{(0)} + p \tilde{\omega}_2^{(0)} & \text{otherwise},
\end{array} \right.
\end{equation}

where \( a^{(0)} = (a_1^{(0)}, a_2^{(0)}) \in \mathbb{Z}^2 \) is defined to be \( a^{(0)} \overset{\text{def}}{=} \sum_{j=0}^{f-1} p^j(\prod_{i=1}^{j} w_j)(\mu(\tau)_j + \eta_j) \).

**Lemma 4.1.1.** Up to isomorphism there exists a unique (semisimple) Kisin module \( \overline{\mathfrak{M}} \) in \( Y(3,0), \tau_{\tilde{w}}(\mathbb{F}) \) of shape \( \tilde{w} \) such that \( T_{\text{def}}(\overline{\mathfrak{M}}) \cong \bar{\rho}|_{G_{K_{\infty}}} \).

**Proof.** Define a Kisin module \( \overline{\mathfrak{M}} \) of type \( \tau_{\tilde{w}} \) by \( A^{(j)} = D_j \tilde{w}_j \) (keeping the notation of Definition 3.1.1) for some \( D = (D_j) \in \mathcal{T}(\mathbb{F}) \). By definition it has shape \( \tilde{w} \). As \( \tilde{w} \in \text{Adm}^\vee(t_{(2,1)}) \subseteq \text{Adm}^\vee(t_{(2,1)}), \)
Adm(\(t_{(3,0)}\)) we know that \(\\mathcal{M}\in Y^{(3,0),\tau_\nu}(F)\) (\cite{LLHLM19} §3.2). By \cite{LLHLM20} Prop. 3.2.1] the associated étale \(\varphi\)-module is given by

\[
\text{Mat}(\varphi(j)) = (D\tilde{w}(sw^{-1})^\ast t_{(\mu-sw^{-1}(\nu))})_j = (Ds^\ast t_{\mu})_j
\]

in some suitable basis. As \(\mu-\eta\in C_0\) we know by Lemma \ref{lem:to} that \((s,\mu)\) is good, hence by Proposition \ref{prop:1} we can choose \(D\in T(F)\) such that \(T_{\text{def}}(\mathcal{M}) \cong \mathcal{P}_{\tau_\nu}.\) The uniqueness of \(\mathcal{M}\) follows as in \cite{LLHLM18} Thm. 3.2, \cite{LLHLM19} Prop. 3.2.18 (this uses that \(3<\langle \mu(\tau) + \eta, \alpha_j^\vee \rangle < p-4\) for all \(j\)).

\[\text{Lemma 4.1.2.}\]

There is a unique bijection \(\theta : W(\mathfrak{p}) \rightarrow \{t_{(2,1)}, t_{(1,2)}\}^f\) such that for \(\sigma \in W(\mathfrak{p})\) and \(\tilde{w} \in \text{Adm}(\mathfrak{t}_{(2,1)})\) we have

\[
\sigma \in \text{JH} (\sigma(\tau_\omega) \otimes_F (N_{k/\mathbb{F}_p} \circ \text{det})) \iff (\tilde{w}_j \neq \theta(\sigma)j \forall j).
\]

\[\text{Proof.}\] Recall that \(\mathfrak{p}|_{I_k} \cong \tau(s,\mu),\) where \(\mu-\eta\) is \(N\)-deep in alcove \(C_0\) and that, for \(\tilde{w} \in \text{Adm}(\mathfrak{t}_{(2,1)})\), we write \(\tilde{w}^* = t_{\nu}w\) for \((w,\nu) \in W \times X^*(T)\) and \(\tau_\omega = \tau(sw^{-1},\mu-sw^{-1}(\nu))\).

We note that \(\sigma(\tau_\omega) \otimes_F (N_{k/\mathbb{F}_p} \circ \text{det}) \cong \sigma(sw^{-1},\mu-sw^{-1}(\nu) + (1,1)),\) and as \(\tilde{w} \in \text{Adm}(\mathfrak{t}_{(2,1)})\) we see that \(\nu - (1,1) \in \eta + \Lambda_R.\)

Recall from §2.4 that the map \(\omega \mapsto F(t_{\mu-\eta}(s(\omega)))\) induces a bijection between \(\Lambda_{W}^{w^{-1}} \subseteq \Lambda_W\) and the set of regular Serre weight with central character \(\mu - \eta\). By Proposition 2.4.2 this map induces a bijection between \(s(\Sigma) \subseteq \Lambda_{W}^{\mu-\eta}\) and the set \(W(\mathfrak{p})\), and by Proposition 2.4.3 this map induces a bijection between \(sw^{-1}(\Sigma - \mathfrak{p}) \subseteq \Lambda_{W}^{\mu-\eta}\) and the set \(\text{JH} (\sigma(\tau_\omega) \otimes_F N_{k/\mathbb{F}_p} \circ \text{det})\). (Note that Propositions 2.4.2, 2.4.3 apply as soon as \(\mu - \eta\) is \(2\)-deep in alcove \(C_0\), and we have \(N \geq 2\).)

We conclude that the statement of the proposition is equivalent to: there is a unique bijection \(\theta^\Sigma : \Sigma \rightarrow \{t_{(2,1)}, t_{(1,2)}\}^f\) such that for \(\omega \in \Sigma\) and \(\tilde{w} \in \text{Adm}(\mathfrak{t}_{(2,1)})\) we have

\[
\omega \in w^{-1}(\Sigma - \mathfrak{p}) \iff (\tilde{w}_j \neq \theta^\Sigma(\omega)j \forall j).
\]

Thus \(\theta^\Sigma(\omega)_j\) only depends on \(\omega_j\), so we may assume that \(f = 1\). In that case,

\[
\tilde{w} \in \text{Adm}(\mathfrak{t}_{(2,1)}) = \{t_{(2,1)}, wt_{(2,1)}, t_{(1,2)}\}
\]

and note that correspondingly

\[
(w, \mathfrak{p}) \in \{(1, \eta), (w, \eta), (1, -\eta)\}.
\]

As \(w = -1\) on \(\Lambda_W\), we see from Figure 1 and \ref{fig:extension_graph} that \(\theta^\Sigma(0) = t_{(1,2)}\) and \(\theta^\Sigma(\eta) = t_{(2,1)}\) is the desired unique bijection.

\[\square\]
4.2. Deformation rings I: single type. We now compute some Galois deformation rings of \( \overline{\rho} \) for a single type \( \tau \) and Hodge–Tate weights \( \leq (3,0) \), meaning Hodge–Tate weights \( (3,0) \) or \( (2,1) \).

We suppose that \( \overline{\rho} \) is as in \( \S 4.1 \). Fix now \( \tilde{w} \in \text{Adm}^{\vee}(t(2,1)) \) and \( \overline{M} \in Y^{\leq (3,0), \tau_\emptyset}(\mathbb{F}) \) semisimple of shape \( \tilde{w} \) such that \( T_d^\ast(\overline{M}) \cong \mathbb{F} \). By the proof of Lemma 4.1.1 \( \overline{M} \) is such that the associated matrix \( A^{(j)} \) is \( D_j \tilde{w}_j \) for some \( D_j \in T(\mathbb{F}) \) and some choice of an eigenbasis for \( \overline{M} \).

We use the notation

\[
D_{f-1-j} = \begin{cases} 
\begin{pmatrix} e^{(j)}_{11} & 0 \\
0 & d^{(j)}_{22} \end{pmatrix} & \text{if } \tilde{w}_{f-1-j} = t_{(2,1)}, \\
\begin{pmatrix} d^{(j)}_{12} & 0 \\
0 & d^{(j)}_{21} \end{pmatrix} & \text{if } \tilde{w}_{f-1-j} = w t_{(2,1)}, \\
\begin{pmatrix} d^{(j)}_{11} & 0 \\
0 & e^{(j)}_{22} \end{pmatrix} & \text{if } \tilde{w}_{f-1-j} = t_{(1,2)}. 
\end{cases}
\]

(See Tables 1, 3 where the superscript \( (j) \) is omitted for readability.)

Let \( R^{\leq (3,0), \tau_\emptyset}_p \) denote the maximal reduced, \( \mathcal{O} \)-flat quotient of \( R^{\leq (3,0)}_p \) that parametrizes lifts of \( \overline{\rho} \) of Hodge–Tate weights \( \leq (3,0) \) in each embedding and tame inertial type \( \tau_\emptyset \). For each dominant character \( \lambda \in X^\ast(T) \) let \( R^{\lambda, \tau_\emptyset}_p \) denote the maximal reduced, \( \mathcal{O} \)-flat quotient of \( R^{\leq (3,0)}_p \) that parametrizes lifts of \( \overline{\rho} \) of Hodge–Tate weights \( \lambda_j \) in the \( j \)-th embedding \( \sigma_j \) for all \( j \) and tame inertial type \( \tau_\emptyset \).

**Proposition 4.2.1.** We have an isomorphism

\[
R^{\leq (3,0), \tau_\emptyset}_p \llbracket X_1, \ldots, X_{2f} \rrbracket \cong \left( \bigotimes_{0 \leq j \leq f-1} \frac{R^{(j)}_p}{I^{(j)}_p} \right) \llbracket Y_1, \ldots, Y_4 \rrbracket,
\]

where the rings \( R^{(j)}_p \) and their ideals \( I^{(j)}_p \) are found in Tables 2, 3. The irreducible components of \( \text{Spec } R^{\leq (3,0), \tau_\emptyset}_p \) are given by the \( \text{Spec } R^{\lambda, \tau_\emptyset}_p \), where \( \lambda = \lambda_j \in \{ (3,0), (2,1) \}^f \).

More precisely, via the isomorphism, for any choice of \( \lambda = (\lambda_j) \in \{ (3,0), (2,1) \}^f \) the kernel of the natural surjection \( R^{\leq (3,0), \tau_\emptyset}_p \llbracket X_1, \ldots, X_{2f} \rrbracket \twoheadrightarrow R^{\lambda, \tau_\emptyset}_p \llbracket X_1, \ldots, X_{2f} \rrbracket \) is generated by the prime ideals \( p^{(j), \lambda_j-1-j}, 0 \leq j \leq f-1 \), that are found in Tables 2, 3.

**Remark 4.2.2.** To obtain Proposition 4.2.1 we cannot use directly the results of [LLHLM], namely Theorem 7.3.2(2) there. In fact, on the one hand we need the precise equations for the ideals \( I^{(j)}_p \) to perform the computations in Proposition 4.3.2 (where we check that \( p \) is contained in suitably chosen ideals in multi Hodge-type deformation rings). On the other hand, we are not appealing to Elkik’s approximation theorem, which is used in the proof of [LLHLM] Theorem 7.3.2(2)]. This lets us have less stringent conditions on the tame inertial types appearing in Proposition 4.2.1 above, in that the genericity of \( \tau_\emptyset \) is the explicit requirement that \( \mu(\tau) \) is \( 8 \)-deep in \( C_0 \), rather than a condition on an inexplicit polynomial \( P_{\tau_\emptyset} \in \mathbb{Z}[X_1, X_2] \) such that \( P_{\tau_\emptyset}(\mu(\tau)_{(j)}) \neq 0 \) (mod \( p \)) for all \( j \in J \) (cf. the genericity condition of [LLHLM] §1.2.1]).

**Proof.** We let \( \tau \overset{\text{def}}{=} \tau_\emptyset \) for short.
As $\overline{A}^{(j)} = D_j \hat{\omega}_j$, the standard basis $\overline{\beta}$ is a gauge basis of $\overline{M}$ in the sense of [LLHL19 Def. 3.2.23]. (There, $\overline{M} \in Y^{0,\tau}(\mathbb{F})$ but $\eta$ plays no role.) For $R$ a complete noetherian local $\mathcal{O}$-algebra with residue field $\mathbb{F}$ define $D_{\overline{M},\overline{\beta}}(\bigotimes_{\mathbb{F}})$ to be the groupoid of triples $(\overline{M}, \overline{\beta}, j)$, where $\overline{M} \in Y^{\leq \tau}(\mathbb{F})$, $\beta$ is a gauge basis of $\overline{M}$ (Definition 3.1.6) and $j : \overline{M} \otimes_R \mathbb{F} \sim \overline{M}$ sending $\beta$ to $\overline{\beta}$. From the definition of a gauge basis, for any lift $(\overline{M}, \overline{\beta}, j) \in D_{\overline{M},\overline{\beta}}(\bigotimes_{\mathbb{F}})$ the corresponding matrices $A^{(j)}$ are given in row 1 of Tables 1–3, where the entries $c^{(j)}_{11}, c^{(j)}_{12}, \ldots$ are in $R$, subject to $A^{(j-f)}$ reducing to our fixed $A^{(j-1)}$ modulo $m_R$.

By the analog of [LLHLM18 Prop. 4.18] the finite height conditions are given by

$$\det A^{(f-1)} \in R^\times (v + p)^3 \forall j,$$

giving rise to the generators of the ideal $I^{(j)} \leq \tau$ in row 4 of Tables 1–3. As in [LLHLM18 Thm. 4.17], $D_{\overline{M},\overline{\beta}}(\bigotimes_{\mathbb{F}})$ is represented by the maximal reduced $p$-flat quotient of $\bigotimes_{\mathbb{F}}$ and $A^{(j)}$ is in $R$, we also denote by $R_{\overline{M},\overline{\beta}}(\bigotimes_{\mathbb{F}})$.

By Proposition 3.1.9 (applied with $h = 3$) the monodromy conditions are given by

$$\left(\frac{d}{dv}\right)^t \left|_{v=0} \right. \left[P_N(A^{(f-1)}) + O(p^{N-3}) \right] = 0 \quad \forall 0 \leq t \leq 1, \ 0 \leq j \leq f - 1,$$

where the $O(p^{N-3})$ denote specific (but not explicit) elements of $p^{N-3} M_2(R)$. Note that

$$P_N(A^{(f-1)}) = \left[ -e' v \frac{d}{dv} A^{(f-1)} + A^{(f-1)} \binom{b^{(j)}}{c^{(j)}} (v + p)^3 (A^{(f-1)})^{-1} \right]$$

modulo $(v + p)^3 M_2(R[v])$, where $(b^{(j)}, c^{(j)}) \overset{\text{def}}{=} (s'_0, f_{-j})^{-1}(A^{(f-1)}(v + p)^3 (A^{(f-1)})^{-1} \in \mathbb{Z}_p$ (Note that the “other” term $(v + p)^3 M_2(R[v])$ from the Lie bracket in equation 15 is in $(v + p)^3 M_2(R[v])$)

Combining this, the monodromy condition is

$$\left(\frac{d}{dv}\right)^t \left|_{v=0} \right. \left[ -v \frac{d}{dv} A^{(f-1)} - A^{(f-1)} \binom{a^{(j)}}{c^{(j)}} (v + p)^3 (A^{(f-1)})^{-1} \right] + O(p^{N-3}) = 0$$

for all $0 \leq t \leq 1, 0 \leq j \leq f - 1$. The entries of the left-hand side give rise to the eight generators in row 5 of Tables 1–3, where we denote $a^{(j)}$ by $a^{(j)}_1, a^{(j)}_2, a^{(j)}_3$ respectively, and the $O(p^{N-3})$ denote elements of $p^{N-3} R$.

By [LLHLM19 §3.2] we have

$$(b^{(j)}, c^{(j)}) \equiv (s'_{0, f_{-j}})^{-1}(s(\tau), \mu(\tau), f_{-j}) \equiv s(\tau)^{-1}(\mu(\tau) + \eta)_j \equiv (ws^{-1}(\mu) - \nu)_j \pmod{p},$$

recalling that $(s(\tau), \mu(\tau)) = (sw^{-1}(\mu) - \nu(\tau) - \eta)$. Hence $a^{(j)} \equiv -(ws^{-1}(\mu) - \nu)_j \pmod{p}$.

As $\mu_j = (r_j + 2, 1)$, this gives us the explicit formulas for $a^{(j)} \pmod{p}$ listed below Tables 1–3.
Let \( R_{\leq (3,0), \tau, \nu} \) be the maximal reduced and \( \mathcal{O} \)-flat quotient of \( R_{\leq (3,0), \tau} / \sum_j (I^{(j), \leq (3,0)} + I^{(j), \nu}) \).

As in [LLHLM18, §5], using that \( \text{ad}(\overline{p}) \) is cyclotomic free we get

\[
R_{\overline{p}}^{\leq (3,0), \tau}[X_1, \ldots, X_{2f}] \cong R_{\leq (3,0), \tau, \nu}[Y_1, \ldots, Y_4].
\]

(See in particular Thm. 5.12, Cor. 5.13, and Diagram (5.9) in [LLHLM18], noting that for us \( n = 2 \), so the addition of the gauge basis requires \( 2f \) instead of \( 3f \) variables and the framing of the Galois deformation requires \( 2^2 = 4 \) instead of \( 3^2 = 9 \) variables. Note also that \( T_3 \) should be \( T_0 \) in [LLHLM18] Cor. 5.13, cf. the errata in [LLHLM20] §6]. Finally note that we allow deformations with any Hodge–Tate weights \( \leq (3,0) \), so we do not have a restriction on the shape as in [LLHLM18] Cor. 5.13.)

We now compute the \( p \)-saturation \( I^{(j)} \overset{\text{def}}{=} \left( I^{(j), \leq (3,0)} + I^{(j), \nu} \right)^{p, \text{sat}} \), justifying row 6 of Tables 1 and 3. For short, we will let \( D \overset{\text{def}}{=} d_{11} \) in case of Table 1, \( D \overset{\text{def}}{=} d_{11}d_{22} + pd_1^*d_2^* \) in case of 2 and \( D \overset{\text{def}}{=} d_{22} \) in case of Table 3.

In the following, we will focus on Table 2 (the other cases being similar). Let us label the elements on the right side of row 4 by \((H_i)\) (\(1 \leq i \leq 3\)), of row 5 by \((M_i)\) (\(1 \leq i \leq 8\)), and of row 6 by \((G_i)\) (\(1 \leq i \leq 5\)). Then, omitting superscripts \((j)\) for simplicity,

\[
\frac{1}{p}\begin{pmatrix}
-(M_7) + \frac{1}{p}(M_8)
\end{pmatrix} = d_{12}^*c_{21} + (a_2 - 2)(c_{12}d_{21}^* + d_{12}^*c_{21}) + D + O(p^{N-5})
\]

\[
= -c_{12}d_{21}^* + (a_2 - 1)(c_{12}d_{21}^* + d_{12}^*c_{21}) + D + O(p^{N-5}),
\]

so replacing \( c_{12}d_{21}^* + d_{12}^*c_{21} \) by \( D \) using \((H_1)\) we see that \((G_1), (G_2) \in I^{(j)}\). From \((M_3)\) and \((G_2)\) we get \((G_3) \in I^{(j)}\), as \( a_2 \not\equiv -1 \pmod{p} \).

From \( \frac{1}{p}[-(M_5) + \frac{1}{p}(M_8)] \) and \((G_1)\) we get \((G_4) \in I^{(j)}\), as \( a_2 \not\equiv 2 \pmod{p} \). Replacing \( c_{12}, c_{21}, c_{11} \) in \( \frac{1}{p}(M_8) \) by the elements \((G_1), (G_2), (G_3)\) and as \( a_2 \not\equiv 0, -1 \pmod{p} \) we get

\[
(d_{11}d_{22} + pd_1^*d_2^*) \left( d_{11}d_{22} + p\frac{(a_2 - 2)(a_2 + 1)}{a_2(a_2 - 1)}d_{12}^*d_{21}^* \right) + O(p^{N-5}) \in I^{(j)}.
\]

As \( D = d_{11}d_{22} + pd_1^*d_2^* \), we can rewrite this as

\[
D \left( D - \frac{2p}{a_2(a_2 - 1)}d_{12}^*d_{21}^* \right) + p^{N-5}f \in I^{(j)},
\]

for some \( f \in R^{(j)} \). Since we know that \((G_i) \in I^{(j)}\) for \(1 \leq i \leq 4\), we may assume that \( f \in \mathcal{O}[d_{11}, d_{22}, x_{12}^*, x_{21}^*] \). (Recall that \( d_{12}^* = [d_{12}^*] + x_{12}^* \text{ and } d_{21}^* = [d_{21}^*] + x_{21}^* \))

Consider the surjective homomorphism obtained as composition

\[
\psi : \bigotimes_{j \in \mathbb{J}} R^{(j)} \twoheadrightarrow R_{\leq (3,0), \tau, \nu} \rightarrow R_{\leq (2,1), \tau, \nu}.
\]

The second and third rings are reduced and \( \mathcal{O} \)-flat, of relative dimension \( 3f \) over \( \mathcal{O} \) (by [21], [Kis08] Thm. (3.3.8]) and the analogous results for Hodge–Tate weights \((2,1)\)). By the finite height conditions, all entries of \( A^{(j)} \) now have to be divisible by \( v + p \), so we deduce that \( p^{(j),(2,1)} \overset{\text{def}}{=} (c_{11}, c_{12}, c_{21}, c_{22}, D) \subseteq \ker(\psi) \) and similarly in the other embeddings.
By Lemma 3.3.1 \( R^{(j)}/p^{(j),(2,1)} \) is geometrically integral, reduced and \( \mathcal{O} \)-flat of relative dimension 3 over \( \mathcal{O} \). Hence \( \psi \) induces an isomorphism
\[
\bigotimes_{\mathcal{O}, j} R^{(j)}/p^{(j),(2,1)} \cong R^{(3,0),\tau,\sigma}_{\mathfrak{P}}
\]
by [Cai18, Lemma 2.6] and [BLGHT11, Lemma 3.3]. From (23) we show that its kernel is generated by the ideals \( I^{(j)} \). Therefore the set of prime ideals \( \mathcal{O}[d_{11}, d_{22}, x_{12}^\ast, x_{21}^\ast] \). Hence \( \mathcal{O}[d_{11}, d_{22}, x_{12}^\ast, x_{21}^\ast] \). Therefore (G5) \( \in I^{(j)} \) by (22).

Let \( J^{(j)} \subseteq I^{(j)} \) be the ideal of \( R^{(j)} \) generated by \( (G_1)-(G_5) \). Again by Lemma 3.3.1 \( R^{(j)}/J^{(j)} \) is reduced, \( \mathcal{O} \)-flat, with two geometrically integral irreducible components of relative dimension 3 over \( \mathcal{O} \). By [Cai18, Lemma 2.6] and [BLGHT11, Lemma 3.3] the surjection
\[
\bigotimes_{\mathcal{O}, j} R^{(j)}/J^{(j)} \to R^{(3,0),\tau,\sigma}_{\mathfrak{P}}
\]
is an isomorphism (and hence \( J^{(j)} = I^{(j)} \) for all \( j \)), provided that \( R^{(3,0),\tau,\sigma}_{\mathfrak{P}} \), or equivalently \( R^{(3,0),\tau}_{\mathfrak{P}} \) by (21), has at least \( 2^f \) irreducible components. To see this, it suffices to show that for any choice of \( \lambda \in \{(3,0),(2,1)\}^f \), \( \mathfrak{p} \) admits a potentially crystalline lift \( \rho \) of type \( \tau \) with \( H_{\lambda}(\rho) = \lambda_j \) for all \( j \). This in turn follows from [GHLST17, Thm. D], provided
\[
JH(\sigma(\tau) \otimes E \bigotimes_{E,j} V_E(\lambda_j - (1,0))^{(j)}) \cap W(\mathfrak{p}) \neq 0.
\]

The left-hand side contains \( JH(\sigma(\tau) \otimes E \bigotimes_{E,j} V_E((1,1))^{(j)}) \cap W(\mathfrak{p}) \) as \( L(a,b) \otimes \mathfrak{p} L(2,0) \cong L(a+2,b) \oplus L(a+1,b+1) \oplus L(a,b+2) \) if \( 2 \leq a-b \leq p-3 \). (Note that the highest weights of the elements of \( JH(\sigma(\tau)) \) are \( 7 \)-deep, as follows from Proposition 2.4.3 and Remark 2.4.3(iv)). Hence (24) follows from Lemma 4.1.2.

By the above argument that (23) is an isomorphism, we know that the irreducible components of \( \text{Spec} R^{(3,0),\tau,\sigma}_\mathfrak{P} \) are in bijection with the set \( \{(3,0),(2,1)\}^f \), explicitly given by sending a component \( C \) to the labeled Hodge-Tate weights of the framed deformation corresponding to any closed point of the generic fiber of \( C \). So the components are indeed given by the \( \text{Spec} R^{(3,0),\tau,\sigma}_\mathfrak{P} \), where \( \lambda = (\lambda_j) \in \{(3,0),(2,1)\}^f \).

To establish the final claim identifying irreducible components, for any \( \lambda \in \{(3,0),(2,1)\}^f \) consider the surjective homomorphism \( \bigotimes_{\mathcal{O}, j} R^{(j)} \to R^{(3,0),\tau,\sigma}_{\mathfrak{P}} \to R^{(3,0),\lambda,\tau,\sigma}_{\mathfrak{P}} \) as above. Exactly as for (23) we show that its kernel is generated by the ideals \( p^{(j),(2,1)} \) if \( \lambda_{f-1-j} = (2,1) \) and \( I^{(j)} \) otherwise. Therefore the set of prime ideals \( (p^{(j),(2,1)} \lambda_{f-1-j} : 0 \leq j \leq f - 1) \) for \( \lambda' \leq \lambda \) matches the set of components \( \text{Spec} R^{(3,0),\lambda,\tau,\sigma}_\mathfrak{P} \) for \( \lambda' \leq \lambda \). We conclude the matching by induction on the number of \( j \) such that \( \lambda_j = (3,0) \).

4.3. Deformation rings II: multiple types. Inspired by the techniques of [Le19, §3.2] we now compute some multi-type deformation rings.
We suppose that $p$ is as in \textsection4.1. For $σ ∈ W(\mathcal{P})$ let $R_σ^{(3,0),\sigma}$ denote the maximal reduced, $\mathcal{O}$-flat quotient of $R_σ^{(3,0)}$ that parametrizes lifts of $p$ of Hodge–Tate weights $≤ (3, 0)$ in each embedding and tame inertial type $τ$ for some $τ$ such that $σ ∈ JH \left(Ω(τ) ⊗_F N_{k/p} ∘ \text{det} \right)$. Letting $\tilde{w}_σ \overset{\text{def}}{=} θ(σ)$ via the bijection $θ$ of Lemma 4.1.2 and
\[
X(σ) \overset{\text{def}}{=} \{\tilde{w} ∈ \text{Adm}^V(t_{(2,1)}): \tilde{w}_j \neq (\tilde{w}_σ)_j \ ∀ j\},
\]
we see that $\text{Spec } R_σ^{(3,0),\sigma}$ is the flat closure of $∪_{\tilde{w} \in X(σ)} \text{Spec } R_σ^{(3,0),\sigma}[1/p]$ inside $\text{Spec } R_σ^{\square}$. Also, define a bijection $i: \text{Adm}^V(t_{(2,1)}) → \{1, 2, 3\}$ by $i(t_{(2,1)}) = 1$, $i(wt_{(2,1)}) = 2$, $i(t_{(1,2)}) = 3$.

**Proposition 4.3.1.** We have an isomorphism
\[
R_σ^{(3,0),\sigma}[X_1, \ldots, X_2] ∼= \left(\bigotimes_{0 ≤ j ≤ f−1} S^{(j)} / J^{(j)}\right)[Y_1, \ldots, Y_4],
\]
where the ring $S^{(j)}$ is as Table 4 and $J^{(j)} \overset{\text{def}}{=} I_1^{(j)} \cap I_2^{(j)}$ if $w_σ)_j ≠ (\tilde{w}_σ)_j$, whereas $S^{(j)}$ is as Table 5 and $J^{(j)} \overset{\text{def}}{=} I_2^{(j)} \cap I_3^{(j)}$ if $w_σ)_j = (\tilde{w}_σ)_j$. The irreducible components of $\text{Spec } R_σ^{(3,0),\sigma}$ are given by the Spec $R_σ^{λ,τ_σ}$, where $λ = (λ_j) ∈ \{(3, 0), (2, 1)\}$ and $\tilde{w} ∈ X(σ)$.

More precisely, via the isomorphism, for any choice of $λ = (λ_j) ∈ \{(3, 0), (2, 1)\}$ and $\tilde{w} ∈ X(σ)$ the kernel of the natural surjection $R_σ^{(3,0),\sigma}[X_1, \ldots, X_2] → R_σ^{λ,τ_σ}[X_1, \ldots, X_2]$ is generated by the prime ideals $p^{(j,λ−1−j)}_{\tilde{w}_j − 1−j}$, $0 ≤ j ≤ f−1$, that are found in Tables 4–5.

**Proof.** Recall that $\overline{p}_{|K} \overset{\text{def}}{=} F(s, μ)$. The proof of Lemma 4.1.1 shows that the étale $ϕ$-module associated to $\overline{p}_{|G_{K∞}}$ is given by $\text{Mat}(ϕ(\lambda)) = (D s^* t_μ)_j$ in some basis, for some $D = (D_j) ∈ T(F)$. Define $δ^{(j)}_{12}, δ^{(j)}_{21} ∈ \mathcal{O}^\times$ to be the Teichmüller lifts of the diagonal entries of $D_{j−1−j}$. Also let $μ_j \overset{\text{def}}{=} μ_j − (1, 1) = (r_j + 1, 0)$.

Let $S \overset{\text{def}}{=} \bigotimes_{(j)} S^{(j)} / J^{(j)}$. Consider the étale $ϕ$-module $M$ over $O_{E,S}$ given by
\[
\text{Mat}(ϕ_M^{(j−1−j)}) = \left(\begin{array}{cc}
(v + p)(δ_{12}^{(j)} + x_{12}^{(j)}) + c_{12}^{(j)} + \frac{b_{12}^{(j)}}{v} & \frac{1}{v}(v + p)(1 + c_{12}^{(j)}) \\
(v + p)(δ_{22}^{(j)} + x_{22}^{(j)}) + c_{22}^{(j)} & \frac{1}{v}(v + p)(1 + c_{22}^{(j)}) + \frac{b_{22}^{(j)}}{v}
\end{array}\right) s_j^{-1} μ_j'
\]
in a suitable basis, where $b_{21}^{(j)} \overset{\text{def}}{=} 0$ if $w_σ)_j ≠ (\tilde{w}_σ)_j$ and $b_{12}^{(j)} \overset{\text{def}}{=} 0$ if $w_σ)_j = (\tilde{w}_σ)_j$. Write $S[Y] = S[Y_1, \ldots, Y_4]$ for short and define the $ϕ$-module $M_{S[Y]} \overset{\text{def}}{=} M ⊗_S S[Y]$ over $O_{E,S,Y}$. (Recall that $O_ε$ denotes the $p$-adic completion of $W(k)[v][1/v]$ and $O_{E,S,Y} \overset{\text{def}}{=} O_E \widehat{⊗}_{\mathbb{Z}_p} S[Y]$.)

Let $M_{F} \overset{\text{def}}{=} M ⊗_S F$. As every variable in $S^{(j)}$ gets sent to zero in $F$ and $μ_j = (r_j + 2, 1)$, we see that $V_{K}^{*}(M_{F}) \overset{\text{def}}{=} p|G_{K∞}$. Fix an $F$-basis $γ_F$ of $V_{K}^{*}(M_{F})$ together with (1 + $Y_1$ $Y_2$ $Y_3$ $Y_4$) $γ ⊗ 1$ gives rise to a homomorphism $ψ_0: R_σ^{\square} → S[Y]$. Fix an $S$-basis $γ$ of $V_{K}^{*}(M)$ that lifts $γ_F$. Then the $G_{K∞}$-representation $V_{K}^{*}(M_{S[Y]})$ together with basis (1 + $Y_1$ $Y_2$ $Y_3$ $Y_4$) $γ ⊗ 1$ gives rise to a homomorphism $ψ_0: R_σ^{\square} → S[Y]$.
For notational convenience, rename the variables \((X_1, \ldots, X_f)\) as \(X' \overset{\text{def}}{=} (X'_0, \ldots, X'_{f-1})\) and \((X_{f+1}, \ldots, X_{2f})\) as \(X'' \overset{\text{def}}{=} (X''_0, \ldots, X''_{f-1})\). Extend \(\psi_0\) to a homomorphism \(\psi : R_{\rho|G_{K_\infty}}^\square [X', X''] \to S[Y]\) as follows:

\[
\psi(X'_j) = \begin{cases} x_{12}^{(j)} & \text{if } 0 \leq j < f - 1 \text{ or } \rho \text{ is irreducible;} \\ Y_1 & \text{if } j = f - 1 \text{ and } \rho \text{ is reducible;}
\end{cases}
\]

\[
\psi(X''_j) = \begin{cases} x_{21}^{(j)} & \text{if } 0 \leq j < f - 1; \\ Y_4 & \text{if } j = f - 1.
\end{cases}
\]

On the other hand we have surjections

\[
R_{\rho|G_{K_\infty}}^\square \to R_{\overline{\rho}}^\square \to R_{\overline{\rho}}^{(3, 0), \sigma}.
\]

(For the first, see [LLHLM18, Prop. 3.12] and use that \(\text{ad}(\overline{\rho})\) is cyclotomic free.)

**Claim 1.** The map \(\psi : R_{\rho|G_{K_\infty}}^\square [X', X''] \to S[Y]\) is surjective.

We will check it is injective on reduced tangent vectors, i.e. on \(\mathbb{F}[\varepsilon]/(\varepsilon^2)\)-points. Pick any continuous homomorphism \(t : S[Y] \to \mathbb{F}[\varepsilon]/(\varepsilon^2)\), let \(t_0 : S[Y] \to \mathbb{F} \to \mathbb{F}[\varepsilon]/(\varepsilon^2)\) be the zero vector, and suppose that \(t \circ \psi = t_0 \circ \psi\). Abusing notation, we will write \(t(b_{ik}^{(j)}) = \varepsilon b_{ik}^{(j)}\) for some \(b_{ik}^{(j)} \in \mathbb{F}\) on the right, and similarly \(t(c_{ik}^{(j)}) = \varepsilon c_{ik}^{(j)}\), \(t(d_{ik}^{(j)}) = \varepsilon d_{ik}^{(j)}\), \(t(x_{12}^{(j)}) = \varepsilon x_{12}^{(j)}, t(Y_1) = \varepsilon y_i\).

From the definition of \(\psi\) and \(t \circ \psi = t_0 \circ \psi\) we deduce \(x_{12}^{(j)} = x_{21}^{(j)} = 0\) for \(0 \leq j < f - 1, y_4 = 0,\) and

\[
\begin{align*}
\left\{ \begin{array}{ll}
x_{12}^{(j-1)} = 0 & \text{if } \rho \text{ is irreducible,} \\
y_1 = 0 & \text{if } \rho \text{ is reducible.}
\end{array} \right.
\]

Also, there is an isomorphism

\[
\lambda : \mathcal{M}_{S[Y]} \otimes_{S[Y]} \mathbb{F}[\varepsilon]/(\varepsilon^2) \overset{\sim}{\longrightarrow} \mathcal{M}_{S[Y]} \otimes_{S[Y]} t_0 \mathbb{F}[\varepsilon]/(\varepsilon^2)
\]

such that \(\mathbb{V}_K^*(\lambda \mod \varepsilon)\) sends the basis \((1 + \varepsilon \left( y_1 \begin{array}{c} y_2 \\ y_3 \\ y_4 \end{array} \right)) (\gamma \otimes 1)\) to \(\gamma \otimes 1\). In particular \(\mathbb{V}_K^*(\lambda \mod \varepsilon)\) is the identity of \(\mathcal{M}_\mathbb{F}\).

Hence the isomorphism \(\lambda\) is realized by change of basis matrices of the form

\[
1 + \varepsilon M_{f-1-j} \in \text{GL}_2(\mathcal{O}_{\mathbb{F}, E}[\varepsilon]/(\varepsilon^2)),
\]

for some \(M_{f-1-j} \in \text{M}_2(\mathcal{O}_{\mathbb{F}, E}) = \text{M}_2(\mathbb{F}(v))\). In other words,

\[
\left(1 + \varepsilon M_{j-1}\right) \left( \begin{array}{cc}
\delta_{12}^{(j)} & \delta_{21}^{(j)} \\
\delta_{12}^{(j)} & \delta_{21}^{(j)}
\end{array} \right) s_j^{-1} v'^j(1 - \varepsilon \varphi(M_j)) = 
\]

\[
\left( \begin{array}{cc}
\delta_{12}^{(j)} + \varepsilon (x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2}) & \varepsilon (d_{21}^{(j)} v^{-1} + c_{11}^{(j)} v^{-2}) \\
\varepsilon (d_{21}^{(j)} v^{-1} + c_{11}^{(j)} v^{-2}) & \delta_{21}^{(j)} + \varepsilon (x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2})
\end{array} \right) s_j^{-1} v'^j,
\]

where we have divided by \(v\), and \(j\) is considered in \(\mathbb{Z}/f\mathbb{Z}\), as usual.
Let $k_j \in \mathbb{Z}$ be minimal such that $v^{k_j} \mathcal{M}_j \in \mathcal{M}_2(\mathbb{F}[v])$. Consider
\[
1 - \varepsilon \varphi(M_j) = v^{-\nu_j^j} s_j \left( \frac{\delta_{12}^{(j)}}{\delta_{21}^{(j)}} \right)^{-1} (1 - \varepsilon M_{j-1}) - \left( \frac{\delta_{12}^{(j)} + \varepsilon(x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2})}{\delta_{21}^{(j)} + \varepsilon(x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2})} \right) s_j^{-1} v^{\nu_j^j}.
\]

Then multiplying the right-hand side by $v^{j+1} \cdot v^{k_j-1} \cdot v^2$ makes it $v$-integral, hence $pk_j \leq k_j - 1 + r_j + 3 < k_j - 1 + p - 1$ by genericity. This implies $p \max k_j < \max k_j + p - 1$, so $\max k_j < 1$, meaning $\mathcal{M}_j \in \mathcal{M}_2(\mathbb{F}[v])$ for all $j$.

From (27) we get by multiplying on the right by $v^{-\nu_j^j} s_j$:
\[
M_{j-1} \left( \frac{\delta_{12}^{(j)}}{\delta_{21}^{(j)}} \right) - \left( \frac{\delta_{12}^{(j)} + \varepsilon(x_{12}^{(j)} + c_{12}^{(j)} v^{-1} + b_{12}^{(j)} v^{-2})}{\delta_{21}^{(j)} + \varepsilon(x_{21}^{(j)} + c_{21}^{(j)} v^{-1} + b_{21}^{(j)} v^{-2})} \right) s_j^{-1} v^{\nu_j^j} \varphi(M_j) v^{-\nu_j^j} s_j = 0.
\]

Recall that we assumed $s_j = 1$ for all $0 < j \leq f - 1$, and hence $s_0 = 1$ if and only if $\mathcal{P}$ is reducible (due to our genericity assumption).

As the $(1, 1)$ and $(2, 2)$-entries of the left-hand side of (28) are $v$-integral, we deduce that $c_{12}^{(j)} = b_{12}^{(j)} = c_{21}^{(j)} = b_{21}^{(j)} = 0$. From the $(1, 2)$-entry of (28) when $s_j = 1$ (resp. the $(2, 1)$-entry of (28) when $s_j \neq 1$) and from $2 < r_j + 1 < p$ we deduce that $v \mid (M_j)_{21}$ for all $j$. This implies that the left-hand side of (28) is $v$-integral and its $(2, 1)$-entry is divisible by $v$. In particular, $d_{11}^{(j)} = c_{11}^{(j)} = d_{22}^{(j)} = c_{22}^{(j)} = 0$ for all $j$.

If $s_j = 1$ (e.g. if $j \neq 0$) we have by (28) and the previous paragraph
\[
\begin{cases}
  x_{12}^{(j)} = \delta_{12}^{(j)}((M_{j-1})_{11} - (M_j)_{11})_{v=0}, \\
  x_{21}^{(j)} = \delta_{21}^{(j)}((M_{j-1})_{22} - (M_j)_{22})_{v=0}.
\end{cases}
\]

In particular, as $x_{12}^{(j)} = x_{21}^{(j)} = 0$ for $0 \leq j < f - 1$, we conclude that
\[
(M_j)_{11} |_{v=0}, \quad (M_j)_{22} |_{v=0}
\]

are independent of $j$.

If $s_j \neq 1$ then we have by (28) and the previous paragraph
\[
\begin{cases}
  x_{12}^{(j)} = \delta_{12}^{(j)}((M_{j-1})_{11} - (M_j)_{22})_{v=0}, \\
  x_{21}^{(j)} = \delta_{21}^{(j)}((M_{j-1})_{22} - (M_j)_{11})_{v=0}.
\end{cases}
\]

If $\mathcal{P}$ is reducible (i.e. $s_0 = 1$) we deduce by (29) and (30) that $x_{12}^{(j)} = x_{21}^{(j)} = 0$ for all $j$.

Otherwise (i.e. $s_0 \neq 1$), we deduce from (25), (30), and (31) that $x_{12}^{(j)} = x_{21}^{(j)} = 0$ for all $j$. As a result, the right-hand side of (28) vanishes and we conclude that $(M_{f-1-j})_j \in \text{End}_{\mathcal{P} \text{-mod}}(\mathcal{M}_F)$. Denote this endomorphism by $\xi$. From (26) we have $(1 + \varepsilon \mathcal{V}_K^\ast(\xi))((1 + \varepsilon \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix})(\gamma \otimes 1)) = \gamma \otimes 1$, so
\[
\mathcal{V}_K^\ast(\xi) = -\begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}
\]

with respect to the basis $\gamma_\mathcal{F}$. On the other hand, $\text{End}_{\mathcal{P} \text{-mod}}(\mathcal{M}_F) \cong \text{End}_{G_K}(\mathcal{P})$ by Lemma 3.2.8.
at the type deformation of types $G$ via the first change of variables in Figure 2, where we omit the superscripts corresponding to $\text{Spec}_{\text{subschemes}}$ integral and of relative dimension in Figure 2. It is straightforward in case of Table 3 and ideal $I_3$. We have shown that $t = t_0$, completing the proof of Claim 1.

Claim 2. The map $\psi_0 : \mathcal{R}_{\mathcal{O}|G_{K_{\infty}}} \to \mathcal{S}[\mathcal{Y}]$ factors through the surjection $\mathcal{R}_{\mathcal{O}|G_{K_{\infty}}} \to \mathcal{R}_{\mathcal{O}|G_{K_{\infty}}}^{(3,0),\sigma}$.

By $\mathcal{O}$-flatness it is enough to check that any closed point $x$ of $\text{Spec} \mathcal{S}[\mathcal{Y}]/1/p$ is sent to the closed subscheme $\text{Spec} \mathcal{R}_{\mathcal{O}|G_{K_{\infty}}}^{(3,0),\sigma}/1/p$ of $\text{Spec} \mathcal{R}_{\mathcal{O}|G_{K_{\infty}}}/1/p$. Let $p_x$ be the maximal ideal of $\mathcal{S}[\mathcal{Y}]/1/p$ corresponding to $x$. Its residue field $\kappa(x)$ is a finite extension of $E$.

By Lemma 3.3.2 and the definition of $J^{(j)}$ we deduce that

$$\bigcap_{\tilde{w} \in \mathcal{X}(\sigma)} (I_{\tilde{w}(f \mathfrak{f})^{-1}}, \ldots, I_{\tilde{w}(w)}^{(f \mathfrak{f})^{-1}}) = 0$$

in $S$, hence there exists some $\tilde{w} \in \mathcal{X}(\sigma)$ such that $(I_{\tilde{w}(f \mathfrak{f})^{-1}}, \ldots, I_{\tilde{w}(w)}^{(f \mathfrak{f})^{-1}}) \subseteq p_x$.

Thus the $\varphi$-module $\mathcal{M}_{\mathcal{S}[\mathcal{Y}]} \otimes_{\mathcal{S}[\mathcal{Y}]} \kappa(x)$ is one of the $\varphi$-modules described in Tables 1 and 3 for the type $\tau_{\tilde{w}}$, at least after replacing $\mathcal{O}$ by $\mathcal{O}_{\kappa(x)}$. (To see this, note that we can identify the $\varphi$-module at the $(f - 1 - j)$-th embedding in Table 1 over $R^{(j)}/I^{(j)}$ with the one in Table 4 over $S^{(j)}/I^{(j)}_3$, via the first change of variables in Figure 2 where we omit the superscripts $(j)$ for readability.

The comparison for Table 3 and ideal $I_3^{(j)}$ in Table 5 is similar, via the second change of variables in Figure 2. It is straightforward in case of Table 3 and ideal $I_3^{(j)}$ in Tables 3 and 5.

In particular, by the proof of Proposition 4.2.1 we know that $\mathcal{V}_K^* (\mathcal{M}_{\mathcal{S}[\mathcal{Y}]} \otimes_{\mathcal{S}[\mathcal{Y}]} \kappa(x))$ is the restriction to $G_{K_{\infty}}$ of a potentially crystalline representation $\rho_x$ of $G_K$ over $\kappa(x)$, of inertial types $\tau_{\tilde{w}}$ and Hodge–Tate weights $\leq (3,0)$. Together with the basis $\gamma \otimes_{\mathfrak{x}} 1$, $\rho_x|G_{K_{\infty}}$ is a framed deformation of $\mathcal{P}|G_{K_{\infty}}$. By Corollary 3.2.9, $\rho_x$ is a framed deformation of $\mathcal{P}$, completing the proof of Claim 2.

Claim 3. The $\mathcal{O}$-flat ring $S^{(j)}/J^{(j)}$ has four irreducible components, each being geometrically integral and of relative dimension 3 over $\mathcal{O}$.

Suppose that $(\tilde{w})_{f-1-j} = t_{1,2}$, so that $J^{(j)} = I^{(j)}_1 \cap I^{(j)}_2$. It suffices to show that (a power of) $p$ is contained in $I^{(j)}_1 + I^{(j)}_2$, for then by $\mathcal{O}$-flatness the set of irreducible components of $S^{(j)}/J^{(j)}$ is

<table>
<thead>
<tr>
<th>Table 1</th>
<th>$e_{11}^r$</th>
<th>$d_{11}$</th>
<th>$c_{11}$</th>
<th>$d_{21}$</th>
<th>$c_{12}$</th>
<th>$d_{22}$</th>
<th>$c_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Table 2</td>
<td>$d_{11}$</td>
<td>$c_{11}$</td>
<td>$d_{12}$</td>
<td>$c_{12}$</td>
<td>$d_{21}$</td>
<td>$c_{22}$</td>
<td></td>
</tr>
<tr>
<td>Table 3</td>
<td>$d_{11}$</td>
<td>$c_{11}$</td>
<td>$d_{12}$</td>
<td>$c_{12}$</td>
<td>$d_{21}$</td>
<td>$c_{22}$</td>
<td></td>
</tr>
<tr>
<td>Table 4</td>
<td>$d_{11}$</td>
<td>$c_{11}$</td>
<td>$d_{12}$</td>
<td>$c_{12}$</td>
<td>$d_{21}$</td>
<td>$c_{22}$</td>
<td></td>
</tr>
</tbody>
</table>

If $\mathcal{P}$ is (absolutely) irreducible, then $\text{End}_{\mathcal{O} \text{-mod}}(\mathcal{M}_{\mathcal{P}}) = \mathbb{F}$. As $y_4 = 0$ we conclude from the formula for $V_k^*(\xi)$ that $y_i = 0$ for all $i$.

If $\mathcal{P}$ is reducible, then $\text{End}_{\mathcal{O} \text{-mod}}(\mathcal{M}_{\mathcal{P}}) \cong \mathbb{F} \times \mathbb{F}$. By our condition that $\gamma_{\mathcal{P},1}$, $\gamma_{\mathcal{P},2}$ each span $G_{K_{\infty}}$-stable lines, we conclude that $y_3 = y_4 = 0$. Using (25) we also have $y_1 = y_1 = 0$. The comparison for Table 3 and ideal $I_3$ in Table 5 is similar, via the second change of variables in Figure 2. It is straightforward in case of Table 3 and ideal $I_3^{(j)}$ in Tables 3 and 5.

In particular, by the proof of Proposition 4.2.1 we know that $V_{K}^* (\mathcal{M}_{\mathcal{S}[\mathcal{Y}]} \otimes_{\mathcal{S}[\mathcal{Y}]} \kappa(x))$ is the restriction to $G_{K_{\infty}}$ of a potentially crystalline representation $\rho_x$ of $G_K$ over $\kappa(x)$, of inertial types $\tau_{\tilde{w}}$ and Hodge–Tate weights $\leq (3,0)$. Together with the basis $\gamma \otimes_{\mathcal{O}^*} 1$, $\rho_x|G_{K_{\infty}}$ is a framed deformation of $\mathcal{P}|G_{K_{\infty}}$. By Corollary 3.2.9, $\rho_x$ is a framed deformation of $\mathcal{P}$, completing the proof of Claim 2.

Claim 3. The $\mathcal{O}$-flat ring $S^{(j)}/J^{(j)}$ has four irreducible components, each being geometrically integral and of relative dimension 3 over $\mathcal{O}$.
By 

\[ \text{(see the explicit formulas below Tables } 1, 2) \]

and these rings are identified with \( R^{(j)}(I_{1}^{(j)}) \) as in Table 1 (resp. Table 2), and \( N \geq 7 \) we deduce that \( p \in I_{1}^{(j)} + I_{2}^{(2)} \).

The case where \( (\tilde{\omega})_{f-1-j} = t_{(2,1)} \) is analogous, checking that \( p \in I_{1}^{(j)} + I_{3}^{(2)} \) by using the two elements of the form \( c_{21} + \ldots \) from Table 5. This establishes Claim 3.

**Conclusion of the proof.** By Claims 1 and 2 we have a surjective morphism \( R_{\tilde{\sigma}}^{\leq (3,0),\sigma} [X', X'''] \to S[Y] \). By Gelfand–Kirillov Thm. (3.3.8) the ring \( R_{\tilde{\sigma}}^{\leq (3,0),\sigma} [X', X'''] \) is reduced, \( \mathcal{O} \)-flat, and each irreducible component is of relative dimension \( f + 4 \) over \( \mathcal{O} \). By Proposition 4.2.1 it has precisely \( 4f \) irreducible components. On the other hand, each ring \( S^{(j)}/J^{(j)} \) is reduced and \( \mathcal{O} \)-flat, so \( S \) is reduced and \( \mathcal{O} \)-flat by [Cal18, Lemma 2.6]. By Claim 3 and [BLGHT11, Lemma 3.3] we know that \( S \) has \( 4f \) irreducible components, each of relative dimension \( 3f \) over \( \mathcal{O} \). We deduce that \( R_{\tilde{\sigma}}^{\leq (3,0),\sigma} [X', X'''] \cong S[Y] \).

The identification of irreducible components follows from Proposition 4.2.1 as for any \( \tilde{\omega} \in X(\sigma) \) the isomorphism \( R_{\tilde{\sigma}}^{\leq (3,0),\sigma} [X', X'''] \cong S[Y] \) factors through the isomorphism \( R_{\tilde{\sigma}}^{\leq (3,0),\sigma} [X', X'''] \cong \otimes_{\mathcal{O}} S^{(j)}/I_{1}^{(j)} \otimes_{\mathcal{O}} [Y] \) of Proposition 4.2.1 (keeping in mind the change of variables discussed in the proof of Claim 2).

Recall that \( \bar{\sigma} : G_{K} \to \text{GL}_{2}(\mathbb{F}) \) is such that \( \bar{\sigma}|_{I_{K}} \cong \tau(s, \mu) \), where \( \mu - \eta \) is \( N \)-deep with \( N \geq 9 \) (see item (iii) in §4.1).

**Proposition 4.3.2.** Keep the hypotheses of Proposition 4.3.1. Then for \( 0 \leq j \leq f - 1 \) we have \( p \in p_{1}^{(j),(2,1)} \cap p_{2}^{(j),(2,1)} + p_{1}^{(j),(3,0)} \cap p_{2}^{(j),(3,0)} + p_{2}^{(j),(3,0)} \cap p_{3}^{(j),(3,0)} \) if \( (\tilde{\omega}_{\sigma})_{f-1-j} = t_{(1,2)} \) and \( p \in p_{2}^{(j),(2,1)} \cap p_{2}^{(j),(2,1)} + p_{2}^{(j),(3,0)} \cap p_{3}^{(j),(3,0)} \) if \( (\tilde{\omega}_{\sigma})_{f-1-j} = t_{(2,1)} \).

**Proof.** Suppose that \( (\tilde{\omega}_{\sigma})_{f-1-j} = t_{(1,2)} \). We will systemically omit superscripts \( (j) \) for readability.

From Table 4 note that the following elements are in \( p_{1}^{(3,0)} \):

\[
\begin{align*}
b_{12} - p^{2}d_{12} - (a_{1} - 1)d_{11}d_{22}^{*} \left( \frac{(a_{1} - 1)(a_{1} - 2)}{a_{1}} \right) d_{11}d_{22}^{*} - p & + O(p^{N-5}), \\
c_{21} - (a_{1} - 1)d_{11}d_{22}^{*} & + O(p^{N-5}), \\
d_{11}d_{22} - \frac{2p}{(a_{1} - 1)(a_{1} - 2)} d_{11}d_{22}^{*} & + O(p^{N-5}).
\end{align*}
\]
By eliminating $d_{11}d_{22}$ using the last element we get
\[
b_{12} - p^2 \left( 1 - \frac{2}{a_1} \right) d_{12}^* + O(p^{N-5}) \in p_1^{(3,0)},
\]
\[
\text{c}_{21} - \frac{2p}{a_1 - 2} d_{21}^* + O(p^{N-5}) \in p_1^{(3,0)}.
\]

Similarly we obtain
\[
b_{12}, \quad c_{21} + \frac{2p}{a_2} d_{21}^* + O(p^{N-5}) \in p_2^{(3,0)}.
\]

Hence,
\[
b_{12} \left( c_{21} - \frac{2p}{a_1 - 2} d_{21}^* \right) - \left( b_{12} - p^2 \left( 1 - \frac{2}{a_1} \right) d_{12}^* \right) \left( c_{21} + \frac{2p}{a_2} d_{21}^* \right) + O(p^{N-5}) \in p_1^{(3,0)} \cap p_2^{(3,0)}.
\]

Equivalently,
\[
\frac{2p^3}{a_2} \left( 1 - \frac{2}{a_1} \right) d_{12}^* d_{21}^* - 2p \left( \frac{1}{a_1 - 2} + \frac{1}{a_2} \right) b_{12} d_{21}^* + p^2 \left( 1 - \frac{2}{a_1} \right) d_{12}^* c_{21} + O(p^{N-5}) \in p_1^{(3,0)} \cap p_2^{(3,0)}.
\]

Letting $y = \left( \frac{1}{a_1 - 2} + \frac{1}{a_2} \right) (1 - p^{-1}) a_1^{-1} a_2 = a_1 (a_1 + a_2 - 2) p^{-1} a_2$ which belongs to $\mathbb{Z}_p$, as $a_1 + a_2 \equiv 2 \pmod{p}$, we get by dividing by $p^2 (1 - \frac{2}{a_1}) d_{12}^* d_{21}^*$:
\[
\frac{2p}{a_2} - \frac{2y b_{12}}{d_{12}^*} + \frac{c_{21}}{d_{21}^*} + O(p^{N-7}) \in p_1^{(3,0)} \cap p_2^{(3,0)}.
\]

Noting that $b_{12} - pc_{12}$ and $c_{21}$ are in $p_1^{(3,0)} \cap p_2^{(3,0)}$ we deduce that
\[
p_1^{(3,0)} \cap p_2^{(3,0)} + p_1^{(2,1)} \cap p_2^{(2,1)} \geq \frac{2p}{a_2} - \frac{2y b_{12}}{d_{12}^*} + O(p^{N-7})
\]
\[
= p \left( \frac{2}{a_2} - \frac{2y b_{12}}{d_{12}^*} + O(p^{N-8}) \right).
\]

As $N \geq 9$, the factor in parentheses is a unit in $S^{(j)}$, so we obtain $p \in p_1^{(3,0)} \cap p_2^{(3,0)} + p_1^{(2,1)} \cap p_2^{(2,1)}$.

The case $(\tilde{w}_\sigma)_{f-1-j} = t_{(2,1)}$ is completely analogous, using from Table 5 that
\[
b_{21}, \quad c_{12} - \frac{2p}{a_2 - 1} d_{12}^* + O(p^{N-5}) \in p_2^{(3,0)},
\]
\[
b_{21} - p^2 \frac{a_3 + 1}{a_3 - 1} d_{21}^* + O(p^{N-5}), \quad c_{12} + \frac{2p}{a_3 + 1} d_{12}^* + O(p^{N-5}) \in p_3^{(3,0)},
\]
\[
b_{21} - pc_{21}, \quad c_{12} \in p_2^{(2,1)} \cap p_3^{(2,1)},
\]

and that $a_2 + a_3 \equiv 0 \pmod{p}$. (Alternatively, we mention that the element $\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ normalizing the Iwahori interchanges shapes $t_{(2,1)}$ and $t_{(1,2)}$ and preserves $\mathfrak{m} \ell_{(2,1)}$. It can then be seen that Tables 1 and 3, and likewise Tables 4 and 5, are interchanged under the transformation sending $c_{ik}, d_{ik}, \ldots$ to $c_{3-k,3-k}, d_{3-k,3-k}, \ldots$ and $a_i$ to $1 - a_{4-i}$. In this way we can reduce the second case of this proposition to the first.)
Table 1. \( \tilde{\phi}_{f-1-j} = t_{(2,1)}, \) i.e. \( \tilde{\phi}_{(f-1-j)} = \begin{pmatrix} c_{11}v^2 & 0 \\ 0 & d_{22}v \end{pmatrix}. \)

<table>
<thead>
<tr>
<th>( A^{(f-1-j)} )</th>
<th>( \begin{pmatrix} (v + p)^2 e_{11}^* + (v + p)d_{11} + c_{11} &amp; c_{12} \ v((v + p)d_{21} + c_{21}) &amp; (v + p)d_{22} + c_{22} \end{pmatrix} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \varphi )-module at the ( (f - 1 - j) )-th embedding</td>
<td>( \begin{pmatrix} \frac{1}{v}(v + p)^2 e_{11}^* + (v + p)d_{11} + c_{11} \ (v + p)d_{21} + c_{21} \end{pmatrix} \begin{pmatrix} c_{12} \ (v + p)d_{22} + c_{22} \end{pmatrix} s_j^{-1} \begin{pmatrix} v^r + 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
</tr>
<tr>
<td>( R^{(j)} )</td>
<td>( \mathcal{O}[c_{11}, d_{11}, x_{11}^<em>, c_{12}, c_{21}, d_{21}, c_{22}, x_{22}^</em>] )</td>
</tr>
<tr>
<td>( f^{(j), \leq(3,0)} )</td>
<td>( c_{11}c_{22} + pc_{12}c_{21}, \ d_{11}c_{22} - c_{12}c_{21} + c_{11}d_{22}^* + pc_{12}d_{21}, \ e_{11}^<em>c_{22} + d_{11}d_{22}^</em> - c_{12}d_{21} )</td>
</tr>
<tr>
<td>( f^{(j), \neq} )</td>
<td>( (a_1 - 1)d_{11}c_{22} + a_1c_{11}d_{22}^* + p(d_{11}d_{22}^* + 2e_{11}^<em>c_{22}) + O(p^{N-3}), \ c_{22}(a_1c_{11} + pd_{11}) + O(p^{N-3}), \ c_{12}(a_1 - 1)d_{11} + 2pe_{11}^</em> + O(p^{N-3}), \ c_{12}(a_1c_{11} + pd_{11}) + O(p^{N-3}), \ (a_1 - 1)c_{21}c_{22} - p((a_1 - 3)d_{21}c_{22} + (a_1 - 1)c_{21}d_{22}) + O(p^{N-3}), \ p((a_1 - 1)c_{21}c_{22} + p(d_{21}c_{22} - c_{21}d_{22})) + O(p^{N-3}), \ (a_1 - 1)c_{12}c_{21} + c_{11}d_{22}^* - p((a_1 - 3)c_{12}d_{21} + d_{11}d_{22}^* + O(p^{N-3}), \ p((a_1 - 1)c_{12}c_{21} + c_{11}d_{22}^* + pc_{12}d_{21}) + O(p^{N-3}) )</td>
</tr>
<tr>
<td>( p^{(j),(2,1)} )</td>
<td>( f^{(j)} + (c_{12}) = (c_{11}, c_{12}, c_{21}, c_{22}, d_{11}) )</td>
</tr>
<tr>
<td>( p^{(j),(3,0)} )</td>
<td>( f^{(j)} + (a_1 - 1)(a_1 - 2)c_{12}d_{21} - 2pe_{11}^*d_{22} + O(p^{N-5}) )</td>
</tr>
</tbody>
</table>

Here, \( a_1 \in \mathbb{Z}_{(p)} \) and \( a_1 \equiv -s_j^{-1}(\mu_j) - (2,1), a_j^\ast \equiv -\text{sgn}(s_j)(r_j + 1) + 1 \pmod{p} \). For readability we write \( c_{11}, c_{21}, \) etc. instead of \( \tilde{a}_1^{(j)}, c_{ik}^{(j)} \), etc. Also, note that \( x_{11}^* \equiv -e_{11}^* - [e_{11}] \) and \( x_{22}^* \equiv d_{22} - [d_{22}] \).
| \( A^{(f-1-j)} \) | \[
\begin{pmatrix}
(v + p)d_{11} + c_{11} & (v + p)d_{12}^{*} + c_{12} \\
(v + p)(d_{21}^{*}) & (v + p)d_{22} + c_{22}
\end{pmatrix}
\] |
|---|---|
\( \varphi \)-module at the \((f - 1 - j)\)-th embedding | \[
\begin{pmatrix}
(v + p)d_{12}^{*} + c_{12} \\
(v + p)d_{22} + c_{22}
\end{pmatrix}
\]  
\[
\frac{1}{v} \left( \frac{d_{21}^{*} + c_{11}}{v + p} \right) s_{j}^{-1} \begin{pmatrix} v_{r+1} & 0 \\ 0 & 1 \end{pmatrix}
\] |
| \( R^{(j)} \) | \( \mathcal{O}[c_{11}, d_{11}, c_{12}, x_{12}^{*}, c_{21}, x_{21}, c_{22}, d_{22}] \) |
| \( f(j)_{\leq (3,0)} \) | \[
\begin{align*}
d_{11}d_{22} - (c_{12}d_{21}^{*} + d_{12}^{*}c_{21}) + pd_{12}^{*}d_{21}, \\
c_{12}c_{21} - d_{11}c_{22} - c_{11}d_{22} - p(c_{12}d_{21}^{*} + d_{12}^{*}c_{21}), \\
c_{11}c_{22} + pc_{12}c_{21}
\end{align*}
\] |
| \( f(j), \nabla \) | \[
\begin{align*}
(a_{2} - 1)d_{11}c_{22} + a_{2}c_{12}d_{22} + p(d_{11}d_{22} - 2d_{12}^{*}c_{21} + pd_{12}^{*}d_{21}) + O(p^{N-3}), \\
a_{2}c_{11}c_{22} + p(d_{11}c_{22} + pd_{12}^{*}c_{21}) + O(p^{N-3}), \\
(a_{2} + 1)c_{11}d_{12}^{*} + (a_{2} - 1)d_{11}c_{12} + O(p^{N-3}), \\
a_{2}c_{11}c_{12} + p(d_{11}c_{12} - c_{11}d_{12}^{*}) + O(p^{N-3}), \\
(a_{2} - 1)c_{21}c_{22} - p((a_{2} - 3)d_{21}^{*}c_{22} + (a_{2} + 1)d_{21}c_{22}) + O(p^{N-3}), \\
p((a_{2} - 1)c_{21}c_{22} + p(d_{21}^{*}c_{22} - c_{21}d_{22})) + O(p^{N-3}), \\
(a_{2} - 1)c_{12}c_{21} + c_{11}d_{22} - p((a_{2} - 3)c_{12}d_{21} + (a_{2} - 1)d_{12}^{*}c_{21} + d_{11}d_{22} + pd_{12}^{*}d_{21}) + O(p^{N-3}), \\
p((a_{2} - 1)c_{12}c_{21} + c_{11}d_{22} + pc_{12}d_{21}) + O(p^{N-3})
\end{align*}
\] |
| \( f(j) \stackrel{\text{def}}{=} (f(j), \nabla)_{\leq (3,0)} \) | \[
\begin{align*}
c_{21} + (a_{2} - 1)d_{21} \left( \frac{d_{11}d_{22}}{d_{12}^{*}d_{21}} + p \right) + O(p^{N-5}), \\
c_{12} - a_{2}d_{12}^{*} \left( \frac{d_{11}d_{22}}{d_{12}d_{21}} + p \right) + O(p^{N-5}), \\
c_{11} + \frac{a_{2}(a_{2} - 1)}{a_{2} + 1}d_{11} \left( \frac{d_{11}d_{22}}{d_{12}d_{21}} + p \right) + O(p^{N-5}), \\
c_{22} - \frac{a_{2}(a_{2} - 1)}{a_{2} - 2}d_{22} \left( \frac{d_{11}d_{22}}{d_{12}d_{21}} + p \right) + O(p^{N-5}), \\
\left( \frac{d_{11}d_{22}}{d_{12}d_{21}} + p \right) \left( \frac{a_{2}(a_{2} - 1)}{(a_{2} - 2)(a_{2} + 1)}d_{12}^{*}d_{21} + p + O(p^{N-5}) \right)
\end{align*}
\] |
| \( p(j),(2,1) \) | \[
I^{(j)} + \frac{d_{11}d_{22}}{d_{12}^{*}d_{21}} + p = \left( c_{11}, c_{12}, c_{21}, c_{22}, \frac{d_{11}d_{22}}{d_{12}^{*}d_{21}} + p \right)
\] |
| \( p(j),(3,0) \) | \[
I^{(j)} + \left( \frac{a_{2}(a_{2} - 1)}{(a_{2} - 2)(a_{2} + 1)} \frac{d_{11}d_{22}}{d_{12}^{*}d_{21}} + p + O(p^{N-5}) \right)
\] |

Here, \( a_{2} \in \mathbb{Z}_{(p)} \) and \( a_{2} \equiv -(w_{j}^{-1}(a_{j}) - (2,1), a_{j}^{(i)} \equiv \text{sgn}(s_{j})(r_{j} + 1) + 1 \pmod{p}) \). For readability we write \( a_{2}, c_{ik}, \) etc. instead of \( a_{2}^{(j)}, c_{ik}^{(i)} \), etc. Also, note that \( x_{12}^{*} \equiv d_{12}^{*} - \left[ \frac{d_{12}^{*}}{d_{21}} \right] \) and \( x_{21}^{*} \equiv d_{21}^{*} - \left[ \frac{d_{21}^{*}}{d_{21}} \right] \).
Table 3. Shape $\hat{w}_{f-1-j} = t_{(1,2)}$, i.e. $\mathcal{A}^{(f-1-j)} = \begin{pmatrix} d_{11}^* v & 0 \\ 0 & e_{22}^* v^2 \end{pmatrix}$.

<table>
<thead>
<tr>
<th>$A^{(f-1-j)}$</th>
<th>$(v + p)d_{11}^* + c_{11} \ (v + p)^2 e_{22}^* + (v + p)d_{12} + c_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi$-module at the $(f - 1 - j)$-th embedding</td>
<td>$(v + p)d_{11}^* + c_{11} \ \frac{1}{v}(v + p)d_{12} + c_{12} \ (v + p)^2 e_{22}^* + (v + p)d_{22} + c_{22}$</td>
</tr>
<tr>
<td>$R^{(j)}$</td>
<td>$\mathcal{O}[c_{11}, x_{11}^<em>, c_{12}, d_{12}, c_{21}, c_{22}, d_{22}, e_{22}^</em>]$</td>
</tr>
<tr>
<td>$f^{(j), \leq (3,0)}$</td>
<td>$c_{11}c_{22} + pc_{12}c_{21}, \ c_{11}d_{22} - c_{12}c_{21} + d_{11}^<em>c_{22} + pd_{12}c_{21}, \ c_{11}e_{22}^</em> + d_{11}d_{22} - d_{12}c_{21}$</td>
</tr>
<tr>
<td>$I^{(j), \nabla}$</td>
<td>$a_3c_{11}d_{22} + (a_3 - 1)d_{11}^*c_{22} - p(d_{11}^*d_{22} + 2c_{11}^*e_{22}) + O(p^{N-3}), \ c_{11}((a_3 - 1)c_{22} + pd_{22}) + O(p^{N-3}), \ c_{21}((a_3 - 1)c_{22} + pd_{22}) + O(p^{N-3}), \ a_3c_{11}c_{12} - p((a_3 + 2)c_{11}d_{12} + (a_3 - 2)d_{11}^*c_{12}) + O(p^{N-3}), \ p(a_3c_{11}c_{12} - p(c_{11}d_{12} - d_{11}^*c_{12})) + O(p^{N-3}), \ a_3c_{12}c_{21} - d_{11}^*c_{22} - p((a_3 + 2)d_{12}c_{21} - d_{11}^*d_{22}) + O(p^{N-3}), \ p(a_3c_{12}c_{21} - d_{11}^*c_{22} - pd_{12}c_{21}) + O(p^{N-3})$</td>
</tr>
<tr>
<td>$f^{(j)} \overset{\text{def}}{=} (I^{(j), \nabla} + f^{(j), \leq (3,0)})_{p\text{-sat}}$</td>
<td>$d_{22} - (a_3 + 1)d_{12}c_{21} + O(p^{N-5}), \ c_{11} + a_3d_{12}c_{21} + O(p^{N-5}), \ c_{12} - \frac{a_3(a_3 + 1)(d_{12}^<em>c_{21})}{a_3 - 1} + O(p^{N-5}), \ c_{22} - \frac{d_{12}c_{21}}{d_{11}^</em>} + O(p^{N-5}), \ c_{21}(a_3(a_3 + 1)d_{12}c_{21} - 2pd_{11}^*c_{22} + O(p^{N-5}))$</td>
</tr>
<tr>
<td>$p^{(j), (2,1)}$</td>
<td>$I^{(j)} + (c_{21}) = (c_{11}, c_{12}, c_{21}, c_{22}, d_{22})$</td>
</tr>
<tr>
<td>$p^{(j), (3,0)}$</td>
<td>$I^{(j)} + (a_3(a_3 + 1)d_{12}c_{21} - 2pd_{11}^*c_{22} + O(p^{N-5}))$</td>
</tr>
</tbody>
</table>

Here, $a_3 \in \mathbb{Z}_p$ and $a_3 \equiv -s_j^{-1}(\mu_j) - (1, 2), \alpha_j' \equiv -\text{sgn}(s_j)(r_j + 1) - 1 \pmod{p}$. For readability we write $a_3, c_{ik},$ etc. instead of $a_3^{(j)}, c_{ik}^{(j)},$ etc. Also, note that $x_{11}^* \overset{\text{def}}{=} d_{11} - |d_{11}^*|$ and $x_{22}^* \overset{\text{def}}{=} e_{22}^* - |e_{22}^*|$. 

Table 4. Multi-type deformations: shapes $\tilde{w}_{f-1-j} = t_{(2,1)}$ and $\tilde{w}_{f-1-j} = w_{t_{(2,1)}}$.

<table>
<thead>
<tr>
<th>Multi-type $\varphi$-module at the $(f-1-j)$-embedding</th>
<th>$S^{(j)}$</th>
<th>$O\square c_{11}, d_{11}, b_{12}, c_{12}, x_{12}^<em>, c_{21}, x_{21}^</em>, c_{22}, d_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{1}^{(j)}$</td>
<td>$c_{11} + pd_{11}$, $c_{12} - pd_{12}^* + (a_{1} - 2) \frac{d_{11}^* d_{22}}{d_{21}^<em>} + O(p^{N-5})$, $c_{21} - (a_{1} - 1) \frac{d_{11} d_{22}}{d_{21}^</em>} + O(p^{N-5})$, $c_{22} + \frac{(a_{1} - 1)(a_{1} - 2)}{a_{1}} \frac{d_{11} d_{22}^2}{d_{21}^* d_{22}} + O(p^{N-5})$, $b_{12} - pc_{12} - \frac{d_{11} d_{22}^<em>}{d_{21}^</em>} \left(\frac{(a_{1} - 1)^2(a_{1} - 2)}{a_{1}} \frac{d_{11} d_{22}}{d_{21}^<em>} - p\right) + O(p^{N-5})$, $d_{11} \left((a_{1} - 1)(a_{1} - 2)d_{11} d_{22}^</em> - 2pd_{12} d_{22}^* + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$I_{2}^{(j)}$</td>
<td>$b_{12}$, $c_{21} + (a_{2} - 1)d_{12}^* \left(\frac{d_{11} d_{22}}{d_{12}^* d_{21}^<em>} + p\right) + O(p^{N-5})$, $c_{12} - a_{2} d_{12}^</em> \left(\frac{d_{11} d_{22}}{d_{12}^* d_{21}^<em>} + p\right) + O(p^{N-5})$, $c_{11} + \frac{a_{2}(a_{2} - 1)}{a_{2} + 1} d_{11} \left(\frac{d_{11} d_{22}^</em>}{d_{12}^* d_{21}^<em>} + p\right) + O(p^{N-5})$, $c_{22} - \frac{a_{2}(a_{2} - 1)}{a_{2} - 2} d_{22} \left(\frac{d_{11} d_{22}^</em>}{d_{12}^* d_{21}^<em>} + p\right) + O(p^{N-5})$, $\left(\frac{d_{11} d_{22}}{d_{12}^</em> d_{21}^<em>} + p\right) \left(\frac{a_{2}(a_{2} - 1)}{a_{2} - 2}(a_{2} + 1) \frac{d_{11} d_{22}}{d_{12}^</em> d_{21}^*} + p + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{1}^{(j),(2,1)}$</td>
<td>$I_{1}^{(j)} + (d_{11}) = \left(b_{12} - pc_{12}, c_{11}, c_{12} - pd_{12}^*, c_{21}, c_{22}, d_{11}\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{1}^{(j),(3,0)}$</td>
<td>$I_{1}^{(j)} + \left((a_{1} - 1)(a_{1} - 2)d_{11} d_{22} - 2pd_{12} d_{22}^* + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{2}^{(j),(2,1)}$</td>
<td>$I_{2}^{(j)} + \left(\frac{d_{11} d_{22}}{d_{12}^* d_{21}^<em>} + p\right) = \left(b_{12}, c_{11}, c_{12}, c_{21}, c_{22}, d_{11} d_{22}^</em> + p\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{2}^{(j),(3,0)}$</td>
<td>$I_{2}^{(j)} + \left(\frac{a_{2}(a_{2} - 1)}{a_{2} - 2}(a_{2} + 1) \frac{d_{11} d_{22}}{d_{12}^* d_{21}^*} + p + O(p^{N-5})\right)$</td>
<td></td>
</tr>
</tbody>
</table>

For readability we write $a_{i}$, $c_{ik}$, etc. instead of $a_{i}^{(j)}$, $c_{ik}^{(j)}$, etc. Also, note that $x_{12}^{(j)} \equiv d_{12}^* - [d_{12}^*]$ and $x_{21}^{(j)} \equiv d_{21}^* - [d_{21}^*]$, where $d_{12}^*$, $d_{21}^*$ $\in \mathbb{F}^\times$. The unspecified constants $O(p^{N-5})$ come from Tables 3 and 4 by the change of variables in Figure 2.
Table 5. Multi-type deformations: shapes $\tilde{w}_{f-1-j} = \mathfrak{w}_{(2,1)}$ and $\tilde{w}_{f-1-j} = t_{(1,2)}$.

<table>
<thead>
<tr>
<th>Multi-type $\varphi$-module at the $(f - 1 - j)$-embedding</th>
<th>$S^{(j)}$</th>
<th>$O[[c_{11}, d_{11}, c_{12}, x_{12}^{<em>}, b_{21}, c_{21}, x_{21}^{</em>}, c_{22}, d_{22}]]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_{2}^{(j)}$</td>
<td>$b_{21}$, $c_{21} + (a_{2} - 1)d_{21}^{<em>}\left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right) + O(p^{N-5})$, $c_{12} - a_{2}d_{12}^{</em>}\left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right) + O(p^{N-5})$, $c_{11} + a_{2}(a_{2} - 1)\frac{d_{12}d_{21}^{<em>}}{a_{2} + 1}d_{11}\left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right) + O(p^{N-5})$, $c_{22} - a_{2}(a_{2} - 1)\frac{d_{12}d_{21}^{</em>}}{a_{2} - 2}d_{22}\left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right) + O(p^{N-5})$, $\left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right)\left(\frac{a_{2}(a_{2} - 1)}{(a_{2} - 2)(a_{2} + 1)}\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$I_{3}^{(j)}$</td>
<td>$c_{22} + pd_{22}$, $c_{21} - pd_{21}^{<em>} - (a_{3} + 1)\frac{d_{11}d_{22}}{d_{12}^{2}} + O(p^{N-5})$, $c_{12} + a_{3}\frac{d_{11}d_{22}}{d_{21}^{2}} + O(p^{N-5})$, $c_{11} - a_{3}(a_{3} + 1)\frac{d_{11}d_{22}}{a_{3} - 1}d_{11}\frac{d_{12}d_{21}^{</em>}}{d_{12}d_{21}^{2}} + O(p^{N-5})$, $b_{21} - pc_{21} - \frac{d_{11}d_{22}}{d_{12}^{2}}\left(\frac{(a_{3} + 1)}{a_{3} - 1}\frac{d_{11}d_{22}}{d_{12}d_{21}^{*}} - p\right) + O(p^{N-5})$, $d_{22}\left(a_{3}(a_{3} + 1)d_{11}d_{22} - 2pd_{12}d_{21} + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{2}^{(j),(2,1)}$</td>
<td>$I_{2}^{(j)} + \left(\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p\right) = (b_{21}, c_{11}, c_{12}, c_{21}, c_{22}, \frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p)$</td>
<td></td>
</tr>
<tr>
<td>$p_{2}^{(j),(3,0)}$</td>
<td>$I_{2}^{(j)} + \left(\frac{a_{2}(a_{2} - 1)}{(a_{2} - 2)(a_{2} + 1)}\frac{d_{11}d_{22}}{d_{12}d_{21}^{2}} + p + O(p^{N-5})\right)$</td>
<td></td>
</tr>
<tr>
<td>$p_{3}^{(j),(2,1)}$</td>
<td>$I_{3}^{(j)} + (d_{22}) = (b_{21} - pc_{21}, c_{11}, c_{12}, c_{21} - pd_{21}^{*}, c_{22}, d_{22})$</td>
<td></td>
</tr>
<tr>
<td>$p_{3}^{(j),(3,0)}$</td>
<td>$I_{3}^{(j)} + \left(a_{3}(a_{3} + 1)d_{11}d_{22} - 2pd_{12}d_{21} + O(p^{N-5})\right)$</td>
<td></td>
</tr>
</tbody>
</table>

For readability we write $a_{i}$, $c_{ik}$, etc. instead of $a_{i}^{(j)}$, $c_{ik}^{(j)}$, etc. Also, note that $x_{12} \equiv d_{12} - [d_{12}]$, and $x_{21} \equiv d_{21} - [d_{21}]$, where $d_{12}, d_{21} \in \mathbb{F}^{x}$. The unspecified constants $O(p^{N-5})$ come from Tables 2, 3 by the change of variables in Figure 2.
5. Gelfand–Kirillov dimension and representations of the Iwahori

We introduce an analog of the Gelfand–Kirillov dimension for smooth modulo $p$ representations of $p$-adic analytic groups and prove Corollary 5.3.5 which gives an upper bound for this dimension in the case of representations of the Iwahori subgroup of $GL_2(L)$, $L$ unramified, satisfying a “multiplicity one” assumption in the first three layers of their socle filtration.

Let $F$ be a finite field of characteristic $p$. If $H$ is a compact $p$-adic analytic group, we define
\[
\mathbb{Z}_p[H] \overset{\text{def}}{=} \varprojlim_{H' \subseteq H} \mathbb{Z}_p[H/H'], \quad F[H] \overset{\text{def}}{=} F \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[H],
\]
for $H'$ varying among open normal subgroups of $H$. If $H$ is moreover a pro-$p$-group, $F[H]$ is a complete noetherian local ring whose maximal ideal is denoted by $m_H$. We let $\text{gr}_m F[H]$ be the graded ring of $F[H]$ for the $m_H$-adic filtration
\[
\text{gr}_m F[H] \overset{\text{def}}{=} \bigoplus_{n \geq 0} m_H^n/m_H^{n+1}.
\]

5.1. Review of Gelfand–Kirillov dimension. We recall the notion of Gelfand–Kirillov dimension of an admissible smooth $F$-representation of a $p$-adic analytic group. General references for this part are [Ven02] and [AB06]. We recall here some useful definitions and results for the reader.

Let $H$ be a compact $p$-adic analytic group and let $M$ be a finitely generated $F[H]$-module. Its grade $j_H(M)$ is the smallest integer $d$ such that $\text{Ext}^d_{F[H]}(M, F[H]) \neq 0$ (with the convention that the smallest element of the empty set is $+\infty$). Moreover, if $M \neq 0$, we have
\[
0 \leq j_H(M) \leq \dim(H),
\]
where $\dim(H)$ is the dimension of $H$ as a $q_p$-analytic variety. This is a consequence of the following two facts:

(i) if $H' \subseteq H$ is an open subgroup of $H$, the $F[H']$-module $M$ is finitely generated and we have $j_H(M) = j_{H'}(M)$, as follows from [Ven02 Prop. 2.7];
(ii) if $H$ is $p$-torsion free, $F[H]$ is of finite injective dimension equal to $\text{cd}_p(H)$ [Ven02 Thm. 3.30(ii)] and $\text{cd}_p(H) = \dim(H)$ [Ser65 Cor. 1].

We also define a dimension function by $\text{dim}_H(M) \overset{\text{def}}{=} \dim(H) - j_H(M)$.

When $H$ is a uniform pro-$p$-group, the graded $F$-algebra $\text{gr}_m F[H]$ is commutative isomorphic to the polynomial algebra in $\dim(H)$ variables over $F$ (see the paragraph after Remark 3.31 in [Ven02]). If $M$ is a finitely generated $F[H]$-module, its graded module $\text{gr}_m M$ for the $m_H$-adic filtration is a finitely generated $\text{gr}_m F[H]$-module and $\text{dim}_H(M)$ is equal to the dimension of the support of $\text{gr}_m M$ in $\text{Spec}(\text{gr}_m F[H])$ (see [Ven02 Thm. 3.21.(ii)]).

Let $G$ be a $p$-adic analytic group and $\pi$ an admissible smooth $F$-representation of $G$. For each compact open subgroup $H$ of $G$, the dual $\pi^\vee \overset{\text{def}}{=} \text{Hom}_F(\pi, F)$ of $\pi$ is a finitely generated $F[H]$-module. Its grade does not depend on the choice of $H$ and is denoted $j_G(\pi^\vee)$. The dimension, or Gelfand–Kirillov dimension, of $\pi$ is then $\text{dim}_G(\pi) \overset{\text{def}}{=} \dim(G) - j_G(\pi^\vee) = \dim_H(\pi^\vee)$.

Remark 5.1.1. Let $H$ be some open uniform subgroup of $G$. Then $\text{dim}_G(\pi)$ is the Gelfand–Kirillov dimension of the graded module of $\pi^\vee$ for the $m_H$-adic topology (see [AB06 §5.4]) but
it does not coincide in general with the Gelfand–Kirillov dimension of $\pi^V$ as an $\mathbb{F}[H]$-module [loc. cit., §5.6]. However we have the following description of $\dim_G(\pi)$ (see [EP20, Prop. 2.18]). For $n \geq 1$, let $H^{p^n}$ be the subgroup of $p^n$-th powers of elements of $H$. There exist real numbers $a \geq b \geq \frac{1}{\dim_G(\pi)}$ such that

\begin{equation}
bp_n\dim_G(\pi) + O(p^n(\dim_G(\pi) - 1)) \leq \dim \left( \pi^{H^{p^n}} \right) \leq ap^n\dim_G(\pi) + O(p^n(\dim_G(\pi) - 1)).
\end{equation}

For this reason, the integer $0 \leq \dim_G(\pi) \leq \dim(G)$ (or $-\infty$ if $\pi = 0$) is also called the Gelfand–Kirillov dimension of $\pi$.

**Lemma 5.1.2.** Let $G$ be a $p$-adic analytic group and $N$ a closed normal subgroup of $G$. Let $\pi$ be an admissible smooth $\mathbb{F}$-representation of $G$ such that $N$ acts trivially on $\pi$. Then we have $\dim_G(\pi) = \dim_{G/N}(\pi)$.

**Proof.** By replacing $G$ by an open subgroup and $N$ by the intersection we may assume that $G$ is uniform [DdSMS99, Cor. 8.34]. Then by Exercise 14 in [DdSMS99, §4] there exists an open uniform pro-$p$-group $H \subseteq G$ such that $H \cap N$ is uniform. The result is then a direct consequence of the characterization given by (32). \hfill $\square$

**Lemma 5.1.3.** Let $G$ be an analytic pro-$p$-group without $p$-torsion. Assume that the graded ring $\operatorname{gr}_m \mathbb{F}[G]$ is Auslander-regular (see for example [LvO96, §III.2.1, Def. 7] for the precise definition). Let $I$ be a two-sided ideal of $\operatorname{gr}_m \mathbb{F}[G]$ generated by a sequence of $r$ central elements which is $\operatorname{gr}_m \mathbb{F}[G]$-regular (where $\operatorname{gr}_m \mathbb{F}[G]$ is considered as a module over its center) and such that $\operatorname{gr}_m \mathbb{F}[G]/I$ is isomorphic to a polynomial ring in $\dim(G) - r$ variables. Let $M$ be a finitely generated $\mathbb{F}[G]$-module such that $\operatorname{gr}_m M$ is annihilated by $I$. Then $\dim_G(M)$ is equal to the dimension of the support of $\operatorname{gr}_m M$ in $\operatorname{Spec}(\operatorname{gr}_m \mathbb{F}[G]/I)$.

**Proof.** For a ring $A$ and a left $A$-module $N$, we recall the notation

\[ j_A(N) \overset{\text{def}}{=} \min \{ n \in \mathbb{N} : \operatorname{Ext}^n_A(N, A) \neq 0 \} \]

(with the usual convention that the minimum of the empty set is $+\infty$). Let $A \overset{\text{def}}{=} \operatorname{gr}_m \mathbb{F}[G]$. It follows from [LvO96, §III.2.5, Thm. 2] that $j_G(M) = j_A(\operatorname{gr}_m M)$ if $M$ is a finitely generated $\mathbb{F}[G]$-module. (Note that $\mathbb{F}[G]$ is a left and right Zariski ring by [LvO96, II.2.2, Prop. 1].)

As $A/I$ is a polynomial ring in $\dim(G) - r$ variables, it follows from [LvO96, §III.4.1, Thm. 7] that $j_{A/I}(\operatorname{gr}_m M)$ is equal to $\dim(G) - r - \dim_K(\operatorname{Supp}_{\operatorname{Spec}(A/I)}(\operatorname{gr}_m M))$, where $\dim_K$ denotes the Krull dimension.

Since $\operatorname{gr}_m M$ is annihilated by $I$, there is a spectral sequence

\[ E_2^{p,q} = \operatorname{Ext}^p_{A/I}(\operatorname{gr}_m M, \operatorname{Ext}^q_{A/I}(A/I, A)) \Rightarrow \operatorname{Ext}^{p+q}_A(\operatorname{gr}_m M, A). \]

Let $(h_1, \ldots, h_r)$ be an $A$-regular generating sequence of central elements in $I$. For all $i \in \mathbb{Z}$, we have $\operatorname{Ext}^i_A(A, A) \cong A$ if $i = 0$ and 0 if $i \neq 0$. By induction on $r$, we can use the long exact sequence of cohomology to prove that $\operatorname{Ext}^i_A(A/I, A) \cong A/I$ if $i = r$ and 0 if $i \neq r$. This implies that the spectral sequence degenerates and that $\operatorname{Ext}^p_{A/I}(\operatorname{gr}_m M, A/I) \cong \operatorname{Ext}^{p+r}_A(\operatorname{gr}_m M, A)$ for all $p \in \mathbb{Z}$. We deduce that $j_{A/I}(\operatorname{gr}_m M) = j_A(\operatorname{gr}_m M) - r$. Consequently we have

\[ j_A(\operatorname{gr}_m M) = \dim(G) - \dim_K \left( \operatorname{Supp}_{\operatorname{Spec}(A/I)}(\operatorname{gr}_m M) \right). \]
and we deduce
\[ \dim_G(M) = \dim(G) - j_G(M) = \dim(G) - j_A(\gr M) = \dim_K \left( \Supp_{\Spec(A/I)} (\gr M) \right). \]

5.2. Recollection of results of Lazard. Let \( G \) be a group with unit element \( e_G \). A \( p \)-valuation \([\text{Laz65}, \text{III.2.1.2}]\) on \( G \) is a map
\[ \omega : G \longrightarrow \mathbb{R}_{>0} \cup \{+\infty\} \]
such that, for all \( x, y \in G \),
\begin{itemize}
  \item \( \omega(xy^{-1}) \geq \min(\omega(x), \omega(y)) \);
  \item \( \omega(x^{-1}y^{-1}xy) \geq \omega(x) + \omega(y) \);
  \item \( \omega(x) = +\infty \iff x = e_G \);
  \item \( \omega(x) > \frac{p}{p-1} \);
  \item \( \omega(x^p) = \omega(x) + 1 \).
\end{itemize}

A \( p \)-valuation \( \omega \) on \( G \) is saturated \([\text{Laz65}, \text{III.2.1.5}]\) if, for all \( x \in G \),
\[ \omega(x) > \frac{p}{p-1} \iff \exists y \in G, \ y^p = x. \]

Now we assume that there exists, and we fix it, a saturated \( p \)-valuation \( \omega \) on \( G \). For \( \nu \in \mathbb{R}_{>0} \), we define
\[ G_{\nu} \overset{\text{def}}{=} \{ x \in G : \omega(x) \geq \nu \}, \quad G_{\nu}^+ \overset{\text{def}}{=} \{ x \in G : \omega(x) > \nu \}, \quad \gr_{\nu} G \overset{\text{def}}{=} G_{\nu}/G_{\nu}^+. \]
The sets \( G_{\nu} \) and \( G_{\nu}^+ \) are normal subgroups of \( G \). They form a fundamental system of neighborhoods of \( e_G \) for a structure of topological group on \( G \). The direct sum \( \gr G \overset{\text{def}}{=} \bigoplus_{\nu} \gr_{\nu} G \) is a graded Lie algebra \([\text{Laz65}, \text{II.1.1.7}]\). If \( x \in G \setminus \{ e_G \} \), we define \( \gr(x) \) as being the image of \( x \) in \( \gr_{\omega(x)} G \subseteq \gr G \). We assume that the topological group \( G \) is compact so that \( \omega(G) \) is discrete in \( \mathbb{R}_{>0} \cup \{+\infty\} \) \([\text{Laz65}, \text{Prop. III.2.2.6}]\).

Let \( \mathbb{Z}_p[G] \overset{\text{def}}{=} \varprojlim_{\nu} \mathbb{Z}_p[G/G_{\nu}] \) be the completed group algebra of \( G \). Note that when \( G \) is a compact \( p \)-adic analytic group, the topology induced by a \( p \)-valuation is the profinite topology of \( G \) \([\text{Laz65}, \text{III.3.1.4}]\).

The map \( \gr(x) \mapsto \gr(x^p) \) from \( \gr_{\nu} \) to \( \gr_{\nu+1} \) induces an endomorphism of degree 1 of the graded Lie algebra \( \gr G \). Let \( \mathbb{F}_p[\varepsilon] \) be the graded polynomial algebra in \( \varepsilon \) with \( \varepsilon \) in degree 1. Then there is a unique structure of graded \( \mathbb{F}_p[\varepsilon] \)-Lie algebra on \( \gr G \) such that \( \varepsilon \) acts via \( \gr(x) \mapsto \gr(x^p) \). The graded \( \mathbb{F}_p[\varepsilon] \)-module \( \gr G \) is then a graded-free \( \mathbb{F}_p[\varepsilon] \)-module \([\text{Laz65}, \text{III.2.1.3}]\). If \( G \) is a compact \( p \)-adic analytic group, this \( \mathbb{F}_p[\varepsilon] \)-module has finite rank \( d = \dim(G) \) \([\text{Laz65}, \text{Prop. III.3.1.3}]\).

From now on we assume that \( G \) is a compact \( p \)-adic analytic group (and still that it has a saturated \( p \)-valuation). We fix a family \( (x_i)_{1 \leq i \leq d} \) of elements of \( G \) such that \( (\gr(x_i))_{1 \leq i \leq d} \) is a basis of the \( \mathbb{F}_p[\varepsilon] \)-module \( \gr G \) (so that \( x_i \neq 1 \) for all \( i \)). We call the family \( (x_i)_{1 \leq i \leq d} \) an ordered basis of \( G \).

Let \( \alpha = (\alpha_i)_{1 \leq i \leq d} \in \mathbb{N}^d \). We define \( z^\alpha \overset{\text{def}}{=} \prod_{i=1}^d (x_i - 1)^{\alpha_i} \in \mathbb{Z}_p[G] \) and \( \tau(\alpha) \overset{\text{def}}{=} \sum_{i=1}^d \alpha_i \omega(x_i) \). Following Lazard, we define a valuation \( w : \mathbb{Z}_p[G] \rightarrow \mathbb{R}_{>0} \cup \{+\infty\} \) as the (pointwise) infimum of the set of all \( \mathbb{Z}_p \)-valued valuations \( w \), such that, for all \( x \in G \), \( w(x - 1) \geq \omega(x) \). Actually Lazard takes the (pointwise) infimum of all filtrations \([\text{Laz65}, \text{III.2.3.1.2}]\) but in our case this last infimum is a valuation, so that our definition is equivalent \([\text{Laz65}, \text{Thm. III.2.3.3, Cor. III.2.3.4}]\).
Moreover by *loc. cit.*, the $\mathbb{Z}_p$-algebra $\mathbb{Z}_p[G]$ is isomorphic to the completion of $\mathbb{Z}_p[G]$ for $w$. We have the following description of $\mathbb{Z}_p[G]$ and $w$ [Laz65 III.2.3.8, III.2.3.9]:

$$\mathbb{Z}_p[G] = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha z^\alpha : \lambda_\alpha \in \mathbb{Z}_p \right\};$$

$$w \left( \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha z^\alpha \right) = \inf \{ v_\nu (\lambda_\alpha) + \tau (\alpha) \}.$$ 

The valuation $w$ extends immediately to $\mathbb{Q}_p[G]$ and we define $D_G$ as the completion of $\mathbb{Q}_p[G]$ for the valuation $w$ (or equivalently for the multiplicative norm $|| \cdot || = p^{-w(\cdot)}$) which extends canonically to $D_G$. This is the $\mathbb{Q}_p$-algebra named $\text{Sat} \mathbb{Z}_p[G]$ in [Laz65 IV.1.2.7]. We deduce from the previous description that:

$$D_G = \left\{ \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha z^\alpha : \lambda_\alpha \in \mathbb{Q}_p, \ v_\nu (\lambda_\alpha) + \tau (\alpha) \to +\infty \text{ as } \tau (\alpha) \to +\infty \right\}$$

and that the closure of $\mathbb{Z}_p[G]$ in $D_G$ is isomorphic to the completed group algebra $\mathbb{Z}_p[G]$.

Let $U_{\mathbb{F}_p[\varepsilon]}(\text{gr} G)$ be the enveloping algebra of the $\mathbb{F}_p[\varepsilon]$-Lie algebra $\text{gr} G$. As $\text{gr} G$ is graded, the $\mathbb{F}_p[\varepsilon]$-algebra $U_{\mathbb{F}_p[\varepsilon]}(\text{gr} G)$ is canonically a graded $\mathbb{F}_p[\varepsilon]$-algebra. Namely the tensor algebra $T_{\mathbb{F}_p[\varepsilon]}(\text{gr} G)$ of the $\mathbb{F}_p[\varepsilon]$-module $\text{gr} G$ inherits a grading from $\text{gr} G$ (see [Laz65 I.3.3.2]) and, for $x, y \in \text{gr} G$ two homogeneous elements, the element $x \otimes y - y \otimes x - [x, y]$ is homogeneous in $T_{\mathbb{F}_p[\varepsilon]}(\text{gr} G)$. Consequently $U_{\mathbb{F}_p[\varepsilon]}(\text{gr} G)$ is a quotient of a graded algebra by a homogeneous ideal and is a graded algebra (see [Laz65 IV.2.1.4]).

Let $\text{gr} \mathbb{Z}_p[G]$ be the graded algebra of $\mathbb{Z}_p[G]$ with respect to the valuation $w$ which is naturally a graded $\mathbb{F}_p[\varepsilon]$-algebra [Laz65 I.2.3.2, I.2.3.11]. By definition of $w$, there is a morphism of graded $\mathbb{F}_p[\varepsilon]$-Lie algebras $\text{gr} G \to \text{gr} \mathbb{Z}_p[G]$ given by $\text{gr}(g) \mapsto \text{gr}([g] - 1)$ for $g \in G$ [Laz65 III.2.3.2]. In particular, we have $\text{gr}(g^t) \mapsto \varepsilon \text{gr}([g] - 1)$ for $g \in G$. By the universal property of the enveloping algebra, it extends to a morphism of graded algebras $U_{\mathbb{F}_p[\varepsilon]}(\text{gr} G) \to \text{gr} \mathbb{Z}_p[G]$. It follows from [Laz65 Thm. III.2.3.3] that this morphism is an isomorphism. As $\mathbb{Z}_p[G]$ is the completion of $\text{gr} \mathbb{Z}_p[G]$ for the valuation $w$, we can identify $\text{gr} \mathbb{Z}_p[G]$ and $\text{gr} \mathbb{Z}_p[G]$.

We have $\mathbb{F}_p[G] = \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

Let $\overline{w}$ be the quotient filtration (in the sense of [Laz65 I.2.1.7]) on $\mathbb{F}_p[G]$. It is defined by $\overline{w}(x) \overset{\text{def}}{=} \sup \{ w(\hat{x}) \in \mathbb{R} \cup \{ +\infty \} : \hat{x} \in \mathbb{Z}_p[G], \ \hat{x} \equiv x \mod p \}$. We have

$$\overline{w} \left( \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha z^\alpha \right) = \inf \{ \tau (\alpha) : \lambda_\alpha \neq 0 \}.$$ 

If $x \in \mathbb{Z}_p[G]$, we have $w(p x) = w(x) + 1$ so that $\text{gr}(p x) = \varepsilon \text{gr}(x)$ and finally $\text{gr}(p \mathbb{Z}_p[G]) = \mathbb{Z}_p[G]$ in $\text{gr}(\mathbb{Z}_p[G])$. This implies that the short exact sequence of filtered modules is strict [Laz65 I.2.3.8.2]

$$0 \longrightarrow (p \mathbb{Z}_p[G], w|_{p \mathbb{Z}_p[G]}) \longrightarrow (\mathbb{Z}_p[G], w) \longrightarrow (\mathbb{F}_p[G], \overline{w}) \longrightarrow 0.$$
Combined with the isomorphism $U_{F_p}[\mathfrak{g}](\text{gr } G) \cong \text{gr } \mathcal{Z}_p[G]$, this implies the existence of an isomorphism of graded algebras

$$U_{F_p}[\varepsilon](\text{gr } G) \otimes_{F_p[\varepsilon]} F_p \cong \text{gr } F_p[G].$$

Let $\overline{\text{gr } G}$ be the graded Lie algebra $\text{gr } G \otimes_{F_p[\varepsilon]} F_p$. We deduce an isomorphism of graded algebras

$$(33) \quad U_{\varepsilon}(\overline{\text{gr } G}) \cong \text{gr } F_p[G].$$

We now give a convenient way to compute $\overline{\text{gr } G}$. Actually we rather compute $\text{gr } G$ and deduce $\overline{\text{gr } G}$ after quotienting by $\varepsilon$.

Let $\mathcal{L}$ be a $\mathbb{Z}_p$-Lie algebra. A $p$-valuation on $\mathcal{L}$ is a map $w: \mathcal{L} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ such that for all $\lambda \in \mathbb{Z}_p$ and $x, y \in \mathcal{L}$:

- $w(\lambda x) = v_p(\lambda) + w(x)$;
- $w(x + y) \geq \inf(w(x), w(y))$;
- $w([x, y]) \geq w(x) + w(y)$.

If $(\mathcal{L}, w)$ is a $p$-valued Lie algebra, the set $\text{gr } \mathcal{L}$ has a canonical structure of graded Lie algebra. Moreover the map $\text{gr}(x) \to \text{gr}(px)$ extends to a degree 1 morphism $\text{gr } \mathcal{L} \to \text{gr } \mathcal{L}$ and to a structure of graded $F_p[\varepsilon]$-Lie algebra on $\text{gr } \mathcal{L}$.

If $x \in G$, the series

$$\log_{D_G}(x) \overset{\text{def}}{=} \sum_{n \geq 0} \frac{(-1)^{n-1}}{n}(x - 1)^n$$

converges in $D_G$. The associative algebra $D_G$ with its valuation $w$ is a $p$-valued Lie algebra. The subset $\mathcal{L}_G \overset{\text{def}}{=} \{ \log_{D_G}(x) : x \in G \}$ of $D_G$ is then a $p$-valued sub-$\mathbb{Z}_p$-Lie algebra of $D_G$. Moreover there is canonical isomorphism of graded $F_p[\varepsilon]$-Lie algebras $\text{gr } \mathcal{L}_G \cong \text{gr } G$ (this is a consequence of [Laz65, Thm. IV.3.2.5 and IV.1.3.5]).

5.3. The case of the pro-$p$-Iwahori of $\text{GL}_2$. We compute the graded ring of the completed group algebra of the pro-$p$-Iwahori subgroup $I_1$ of $\text{GL}_2(L)$ for unramified $L$ and introduce an interesting ideal which allows us to control the Gelfand–Kirillov dimension of representations of $I_1$.

Let $L$ be an unramified extension of $\mathbb{Q}_p$ of degree $f$ with ring of integers $\mathcal{O}_L$ and residue field $k$. We are interested in the particular case of the group $I_1/\mathbb{Z}_1$ which is the quotient of the (upper) pro-$p$-Iwahori subgroup of $\text{GL}_2(\mathcal{O}_L)$ by its center. This group is isomorphic to the subgroup $G \overset{\text{def}}{=} I_1 \cap \text{SL}_2(\mathcal{O}_L)$ of $I_1$ since $p > 2$. The following results can also be deduced from [Clo17]. However we prefer to follow [Laz65] in order to emphasize that the graded module naturally has the structure of an enveloping algebra (see [16]).

We follow [Laz65, III.3.2.7] to define a saturated $p$-valuation on $G$. We assume that $p > 3$. Let $L' = L(\sqrt{p})$ and $v: M_2(L') \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ be the valuation defined by

$$v((m_{i,j})) \overset{\text{def}}{=} \min\{v_p(m_{i,j})\}.$$ 

Let $D$ be the diagonal matrix $\left( \begin{array}{cc} 1 & 0 \\ 0 & \sqrt{p} \end{array} \right)$ in $M_2(\mathcal{O}_{L'})$. We define, for $x \in G$:

$$\omega(x) \overset{\text{def}}{=} v(D^{-1}xD - I_2).$$
It follows from [Laz65] III.3.2.7 that \( \omega \) is a saturated \( p \)-valuation on \( G \) (here we are using that \( p > 3 \)). Explicitly, for \( a, b, c, d \in \mathcal{O}_L \) such that \( (1 + pa)(1 + pd) - pbc = 1 \):

\[
\omega\left(\frac{1 + pa}{pc} \frac{b}{1 + pd}\right) = \min\{1 + v_p(a), \frac{1}{2} + v_p(b), \frac{1}{2} + v_p(c), 1 + v_p(d)\}.
\]

Let \( \mathfrak{g}_{Z_p} \) be the sub-\( Z_p \)-Lie algebra of \( \mathfrak{sl}_2 \), defined by

\[
\mathfrak{g}_{Z_p} = \{ \left( \frac{pa}{pc} \frac{b}{-pa}\right) : (a, b, c) \in Z_p^3 \}.
\]

**Lemma 5.3.1.** We have an isomorphism of \( p \)-valued Lie algebras \( \mathcal{L}_G \cong \mathcal{O}_L \otimes_{Z_p} \mathfrak{g}_{Z_p} \) with valuation, for \( a, b, c \in \mathcal{O}_L \),

\[(34) \quad w\left(\frac{pa}{pc} \frac{b}{-pa}\right) = \min\{1 + v_p(a), \frac{1}{2} + v_p(b), \frac{1}{2} + v_p(c)\}.
\]

**Proof.** Let \( G' \) be the subgroup of \( GL_2(L') \) defined by

\[
G' = \left\{ x \in M_2(L') : v(x - I_2) \geq \frac{1}{2} \right\}.
\]

As \( p - 1 > 2 \), it follows from [Bou72] II.8.4, Prop. 4] that \( \log_{M_2(L')}(G') \) is the sub-Lie algebra of \( M_2(L') \) defined by

\[
\log_{M_2(L')}(G') = \left\{ x \in M_2(L') : v(x) \geq \frac{1}{2} \right\}.
\]

For \( x \in G' \), we have \( \log_{M_2(L')}(\text{Ad}(D)x) = \text{Ad}(D)\log_{M_2(L')}(x) \). As \( G = \text{Ad}(D)(G') \cap M_2(L) \), we have

\[
\log_{M_2(L')}(G) = \left\{ x \in M_2(L) : v(\text{Ad}(D)^{-1}x) \geq \frac{1}{2} \right\} = \mathcal{O}_L \otimes_{Z_p} \mathfrak{g}_{Z_p}.
\]

We use the notation to denote the valuation on \( D_G \) associated to \( \omega \) as in section 5.2. Let \( \log_{D_G} \) be the logarithm map on \( D_G \):

\[
\left\{ x \in D_G : w(x - 1) > \frac{1}{p - 1} \right\} \rightarrow \left\{ x \in D_G : w(x) > \frac{1}{p - 1} \right\}.
\]

The inclusion \( G \subseteq M_2(\mathcal{O}_L) \) is continuous and extends to a continuous morphism of \( Z_p \)-algebras \( h : Z_p[G] \rightarrow M_2(\mathcal{O}_L) \) and a morphism of \( \mathbb{Q}_p \)-algebras \( Q_p[G] \rightarrow M_2(L') \). By definition of \( w \), we have the inequality \( w(x) \leq v(\text{Ad}(D^{-1})h(x)) \) for \( x \in Z_p[G] \), since \( v \circ \text{Ad}(D^{-1}) \circ h \) is a valuation \( w' \) on \( Z_p[G] \) such that \( w'(x - 1) = \omega(x) \) for \( x \in G \) and \( w \) is defined as the pointwise infimum of valuations \( w'' \) with \( w''(x - 1) \geq \omega(x) \) for \( x \in G \). As \( w \) and \( v \) are valuations of \( \mathbb{Q}_p \)-algebras, we deduce that this inequality is true for all \( x \in Q_p[G] \). As \( M_2(L') \) is complete, we can extend \( h \) to a morphism of valued \( \mathbb{Q}_p \)-algebras \( (D_G, w) \rightarrow (M_2(L'), v \circ \text{Ad}(D)^{-1}) \). Now, by continuity of \( h \), the composite

\[
G \xrightarrow{\log_{D_G}} D_G \xrightarrow{h} M_2(L')
\]

is the logarithm computed in \( M_2(L') \). This implies that the restriction of \( h \) to \( \log_{D_G}(G) \) is an isomorphism of Lie algebras

\[
\mathcal{L}_G = \log_{D_G}(G) \cong \log_{M_2(L')}(G).
\]
Finally both valuations $w$ and $v \circ \text{Ad}(D)^{-1}$ take value $\omega(x)$ at $x - 1$ for $x \in G$. By [Laz65, III.1.1.5] the condition $\omega(x) > \frac{1}{p-1}$ for $x \in G$ implies then

$$w(\log_{\mathfrak{g}_G}(x)) = \omega(x) = v(\text{Ad}(D)^{-1}) \log_{M_2(L')} (x),$$

proving that (36) is an isomorphism of valued Lie algebras. The conclusion follows from (35) and from the fact that the valuation $v \circ \text{Ad}(D)^{-1}$ restricted to $\log_{M_2(L')} (G) = \mathcal{O}_L \otimes_{\mathbb{Z}_p} \mathfrak{g}_{\mathbb{Z}_p}$ is given by (34).

We endow the Lie algebra $\mathfrak{g}_{\mathbb{Z}_p}$ with the restriction of the valuation $w$ and we let $\mathfrak{g} \overset{\text{def}}{=} \text{gr} \mathfrak{g}_{\mathbb{Z}_p}$. The Lie algebra $\mathcal{L}_G$ is an $\mathcal{O}_L$-Lie algebra and, for $a \in \mathcal{O}_L$ and $x \in \mathcal{L}_G$, we have $w(ax) = v_p(a) + w(x)$. Hence the graded $\mathbb{F}_p[\varepsilon]$-Lie algebra $\text{gr} G \cong \text{gr} \mathcal{L}_G$ has the structure of a $k[\varepsilon]$-graded Lie algebra and is isomorphic to $k \otimes_{\mathbb{F}_p} \mathfrak{g}$. Consequently the graded $\mathbb{F}_p$-Lie algebra $\overline{\text{gr} G} = \text{gr} G \otimes_{\mathbb{F}_p[\varepsilon]} \mathbb{F}_p$ is isomorphic to $k \otimes_{\mathbb{F}_p} \overline{\mathfrak{g}}$, where $\overline{\mathfrak{g}} \overset{\text{def}}{=} \mathbb{F}_p \otimes_{\mathbb{F}_p[\varepsilon]} \mathfrak{g}$, and has a natural structure of graded $k$-Lie algebra.

We want to show that $\text{gr} \mathbb{F}_p[G]$, defined by the valuation $\overline{\omega}$ associated to $\omega$, and $\text{gr}_m \mathbb{F}_p[G]$ (the graded ring for the $m$-adic filtration of $\mathbb{F}_p[G]$) are isomorphic up to rescaling indices. We will need the following lemma:

**Lemma 5.3.2.** Let $G$ be a pro-$p$-group. Then for $g$ and $h$ in $G$, we have

$$gh - 1 \equiv (g - 1) + (h - 1) \mod m^2_G, \quad (g^{-1} - 1) \equiv -(g - 1) \mod m^2_G$$

in $\mathbb{F}_p[G]$. Moreover if $g \in G$, $(g^p - 1) \in m^p_G$.

**Proof.** The first two assertions are consequences of the equality $(g - 1)(h - 1) = (gh - 1) - (g - 1) - (h - 1)$ and from the fact that $g - 1 \in m_G$. The last one comes from $(g^p - 1) = (g - 1)^p$. □

**Proposition 5.3.3.** We have, for $j \in \frac{1}{2}\mathbb{N}$,

$$m^2_j = \{ x \in \mathbb{F}_p[G] : \overline{\omega}(x) \geq j \}.$$

**Proof.** Let $a \in \mathcal{O}_L$ such that $\mathbb{F}_p[a] = k$, hence $\mathcal{O}_L = \mathbb{Z}_p[a]$. Using Lemma 5.3.1 (and its proof) we see that we can choose an ordered basis $(x_1, \ldots, x_{3f})$ of $G$ whose elements are

$$E_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_i = \begin{pmatrix} 1 & 0 \\ a^i & 1 \end{pmatrix}, \quad H_i = \begin{pmatrix} (1-a^i)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

for $0 \leq i \leq f - 1$.

For $j \in \frac{1}{2}\mathbb{N}$, $\{ x \in \mathbb{F}_p[G] : \overline{\omega}(x) \geq j \}$ is the ideal generated by monomials $z^{\alpha} = \prod_{i=1}^{3f} x_i^{\alpha_i}$ with $\tau(\alpha) = \sum_{i=1}^{3f} \omega(x_i) \alpha_i \geq j$. For $0 \leq i \leq f - 1$, we have $E_i - 1 \in m_G$, $F_i - 1 \in m_G$. Let’s prove that $H_i - 1 \in m^2_G$. We have

$$E_i F_0 E_i^{-1} F_0^{-1} = H_i \begin{pmatrix} 1 & -(1-pa^i)a^{2i} \\ 0 & 1 \end{pmatrix} ^p \begin{pmatrix} 1 & 0 \\ pa^i(1-pa^i)^{-1} & 1 \end{pmatrix} ^p.$$

Using Lemma 5.3.2 this implies that

$$E_i F_0 E_i^{-1} F_0^{-1} - 1 \equiv H_i - 1 \mod m^2_G$$

and finally that

$$H_i - 1 \equiv E_i - 1 + F_0 - 1 - (E_i - 1) - (F_0 - 1) \mod m^2_G,$$

$$\equiv 0 \mod m^2_G.$$
Since $\omega(E_i) = \omega(F_i) = 1/2$ and $\omega(H_i) = 1$, this proves that $z^\alpha \in m^2_{G}$ when $\tau(\alpha) \geq j$, i.e. 
\{ x \in F_p[G] : \omega(x) \geq j \} \subseteq m^2_{G}.

Noticing that $m_G = \{ x \in F_p[G] : \omega(x) \geq 1/2 \}$, we have, conversely, 
\[ m^j_G \subseteq \{ x \in F_p[G] : \omega(x) \geq 1/2 \} \cap \{ x \in F_p[G] : \omega(x) \geq j/2 \}, \]
the last inclusion being deduced from the properties of a valuation.

Proposition 5.3.3 suggests that we should rescale the gradings of $g$ and $\overline{g}$ by replacing the valuation $w$ on $g_{Z_p}$ with $2w$, and this is what we do from now on. Therefore, the multiplication by $\varepsilon$ on $g$ now has degree 2. We deduce from Proposition 5.3.3 and isomorphism (33) that we have an isomorphism of $F_p$-Lie algebras
\[ \text{gr}_\omega F_p[G] \cong U_{F_p}(k \otimes_{F_p} \overline{g}). \]

We now determine $\overline{g}$ explicitly. The $Z_p$-Lie algebra $g_{Z_p}$ has a $Z_p$-basis given by 
\[ e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}, \quad h = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \]
with relations 
\[ [e, f] = h, \quad [h, e] = 2pe, \quad [h, f] = -2pf \]
and valuations $2w(e) = 2w(f) = 1$, $2w(h) = 2$. Hence the graded $F_p[\varepsilon]$-Lie algebra $g = \text{gr} g_{Z_p}$ is 
\[ g = F_p[\varepsilon]e \oplus F_p[\varepsilon]f \oplus F_p[\varepsilon]h \]
with $e$ and $f$ in degree 1 and relations 
\[ [e, f] = h, \quad [h, e] = 2\varepsilon e, \quad [h, f] = -2\varepsilon f, \]
and the graded $F_p$-Lie algebra $\overline{g}$ is 
\[ \overline{g} = F_pe \oplus F_pf \oplus F_ph \]
with $e$ and $f$ in degree 1, $h$ in degree 2 and relations 
\[ [e, f] = h, \quad [h, e] = [h, f] = 0. \]

Let $H$ be the (prime-to-$p$) torsion subgroup of the diagonal torus of $GL_2(O_L)$. Then $H$ is a finite subgroup of the “upper” Iwahori subgroup $I$ of $GL_2(O_L)$. It normalizes $I_1$ and $G$. Therefore the group $H$ acts on every object considered so far: $F_p[G]$, $\mathcal{L}_G$, $g$, $\overline{g}$, ... and the isomorphism (37) is equivariant for this action of $H$. Note that the action of $H$ on $\mathcal{L}_G$, $g$ and $\overline{g}$ is $k$-linear. More precisely, we have, for $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in H$, and $\alpha \in k$: 
\[ g(\alpha \otimes e) = (ad^{-1}\alpha) \otimes e, \quad g(\alpha \otimes f) = ((ad^{-1})^{-1}\alpha) \otimes f, \quad g(\alpha \otimes h) = \alpha \otimes h. \]

Let $F$ be a field of characteristic $p$. Recall from the introduction that if $F$ is an extension of $F_p$ such that $k$ embeds into $F$, we label the embeddings $\sigma_j = \sigma_0 \circ \varphi^j$, so the set $J$ of embeddings $k \rightarrow F$ is identified with $\{0, \ldots, f-1\}$. In this case, for $0 \leq j \leq f-1$, we define $g_j \equiv F \otimes_{\sigma_j, k} \text{gr} G$ and $\overline{g}_j \equiv \overline{F} \otimes_{\sigma_j, k} \overline{\text{gr} G}$. Then we have a decomposition
\[ F \otimes_{F_p} \text{gr} G \cong \bigoplus_{j=0}^{f-1} \overline{g}_j \]
and canonical isomorphisms \( g_j \cong F \otimes_{\mathbb{F}_p} g \) as well as \( \overline{g}_j \cong F \otimes_{\mathbb{F}_p} \overline{g} \). Using also \( (37) \) we deduce an isomorphism of graded \( F \)-algebras

\[
\gr_m F[G] \cong F \otimes_{\mathbb{F}_p} \gr_m F_p[G] \cong \bigotimes_{j=0}^{f-1} U_{\mathbb{F}_p}(\overline{g}_j) \cong U_{\mathbb{F}_p}(\overline{g})^f.
\]

For \( 0 \leq j \leq f - 1 \) let \( e_j, f_j, h_j \in \overline{g}_j \) denote the images of \( 1 \otimes e, 1 \otimes f, 1 \otimes h \) under the isomorphism \( F \otimes_{\mathbb{F}_p} \overline{g} \cong \overline{g}_j \). Then we have, for \( g = \left( \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \right) \in H \), and for \( 0 \leq j \leq f - 1 \),

\[
ge_j = \sigma_j(ad^{-1})e_j, \quad gf_j = \sigma_j(ad^{-1})^{-1}f_j, \quad gh_j = h_j.
\]

Let \( I_G \) be the left ideal of \( \gr_m F[G] \) generated by the elements \((1 \otimes e)(1 \otimes f)\) and \((1 \otimes h)\) (of degree 2). We easily see that \( I_G \) is in fact a 2-sided ideal of \( \gr_m F[G] \). If \( k \) embeds in \( F \), then \( I_G \) is the left ideal generated by \((e_j f_j, h_j; 0 \leq j \leq f - 1)\) via the isomorphism \( (40) \).

**Theorem 5.3.4.** Let \( F \) be a field of characteristic \( p \). The graded ring \( \gr_m F[G] \) is Auslander-regular and \((\gr_m F[G])/I_G \) is a commutative Cohen–Macaulay \( F \)-algebra of dimension \( f \). More precisely, if we assume moreover that \( k \) embeds in \( F \), then

(i) the sequence \((h_0, \ldots, h_{f-1})\) is a regular sequence of central elements of \( \gr_m F[G] \) and \((\gr_m F[G]/(h_0, \ldots, h_{f-1})) \) is isomorphic to \( F[e_j, f_j; 0 \leq j \leq f - 1] \), a polynomial ring in \( 2f \) variables;

(ii) we have an isomorphism

\[
(\gr_m F[G])/I_G \cong F[e_j, f_j; 0 \leq j \leq f - 1]/(e_j f_j; 0 \leq j \leq f - 1).
\]

**Proof.** By [LVO96, §III.2.4.4], the graded ring \( \gr_m F[G] \) is Auslander-regular since it is isomorphic to an enveloping algebra. Assume now that \( k \) embeds in \( F \).

(i) It follows from \( (38) \) that \((h_0, \ldots, h_{f-1})\) are central elements of \( \gr_m F[G] \). For \( 0 \leq i \leq f - 1 \), the ring \((\gr_m F[G])/I_G \) is isomorphic to the enveloping algebra of the quotient of the Lie algebra \( F \otimes_{\mathbb{F}_p} \gr G \) by the ideal generated by \( h_0, \ldots, h_i \) and is therefore a ring without zero divisors by the Poincaré–Birkhoff–Witt Theorem. This proves that \( h_{i+1} \) is a regular element of \((\gr_m F[G])/I_G \) and that \((h_0, \ldots, h_{f-1})\) is a regular sequence of central elements of \( \gr_m F[G] \). The last assertion is clear by \( (38) \).

(ii) Using the isomorphism of \( F \)-algebras

\[
(\gr_m F[G])/I_G \cong \bigotimes_{0 \leq j \leq f-1} (U_{\mathbb{F}_p}(\overline{g}_j))/(e_j f_j, h_j),
\]

the assertion is a consequence of (i). The sequence \((e_j f_j; 0 \leq j \leq f - 1)\) is a regular sequence in \( F[e_j, f_j; 0 \leq j \leq f - 1] \), so the ring \((\gr_m F[G])/I_G \) is Cohen–Macaulay of dimension \( f \).

In general (if \( k \) does not embed in \( F \)), we can find a finite extension \( F' \) such that \( k \) embeds in \( F' \). By what precedes, the ring \( F' \otimes_{F} (\gr_m F[G]/I_G) \cong \gr_m (F'[G]/(F' \otimes_{F} I_G)) \) is Cohen–Macaulay of dimension \( f \), hence so is \((\gr_m F[G])/I_G \) \[\text{[Grob5]}\ Cor. (6.7.8)]\.

**Corollary 5.3.5.** Let \( \pi \) be an admissible smooth representation of \( I/Z_1 \) over \( F \). Assume that for each character such that \( \text{Hom}_I(\chi, \pi) \neq 0 \), the natural injection

\[
\text{Hom}_I(\chi, \pi) \hookrightarrow \text{Hom}_I(W_{\chi,3}, \pi)
\]

is an isomorphism, where \( W_{\chi,3} \) is defined in \( (42) \). Then \( \dim_I(\pi) = \dim_{I/Z_1}(\pi) \leq f \).
Proof. By increasing $F$ we may assume that $k$ embeds in $F$. As $\pi$ is an admissible representation of $I/Z_1$, it is an admissible representation of $G \cong I_1/Z_1$ and $\pi^\vee$ is a finitely generated $F[G]$-module. Moreover the socle filtration on $\pi$ coincides with the socle filtration on $\pi|_G$ and with the dual of the $m_G$-adic filtration on $\pi^\vee$ so that $(soc_i \pi / soc_{i-1} \pi)^{\vee} \cong gr^i_m \pi^\vee$. Moreover the graded $gr_m F[G]$-module $gr_m \pi^\vee$ is generated by its homogeneous elements of degree 0.

Let $I_G$ be the graded ideal of $gr_m F[G]$ defined above and let $I_G^{(2)}$ be its homogeneous component of degree 2. Note that $H$ acts trivially on $I_G^{(2)}$. If $\text{Hom}(\chi, gr^0_m \pi^\vee) \neq 0$, then by assumption $\text{Hom}(\chi, gr^0_m \pi^\vee) = 0$, so we have $I_G^{(2)}(gr^0_m \pi^\vee) = 0$. As $gr_m \pi^\vee$ is generated by $gr^0_m \pi^\vee$ and $I_G$ by $I_G^{(2)}$, we deduce that $I_G(gr_m \pi^\vee) = 0$ and that $gr_m \pi^\vee$ is actually a $gr_m F[G]/I_G$-module. Theorem 5.3.4 implies that the dimension of its support is $\leq f$. We can therefore apply Lemma 5.1.3 (with $I = (h_0, \ldots, h_{f-1})$) to conclude that $\dim_{I/Z_1}(\pi) = \dim_G(\pi) \leq f$. The equality $\dim_I(\pi) = \dim_{I/Z_1}(\pi)$ follows from Lemma 5.1.2. □

Using (40) and the Poincaré–Birkhoff–Witt Theorem, we can write down explicitly the structure of the first three graded pieces of $gr_m F[I_1/Z_1]$ as $I$-representations, assuming that $k$ embeds in $F$:

$$
gr^0_m F[I_1/Z_1] = F, \quad gr^1_m F[I_1/Z_1] \cong \bigoplus_{i=0}^{f-1} (F\alpha_i \oplus F\alpha_i^{-1}),$$

$$
gr^2_m F[I_1/Z_1] \cong F^{2f} \oplus \bigoplus_{0 \leq i \leq f-1} F\alpha_i \alpha_j + \bigoplus_{0 \leq i \leq f-1} F\alpha_i^{-1} \alpha_j^{-1} + \bigoplus_{0 \leq i \neq j \leq f-1} F\alpha_i \alpha_j^{-1},$$

where $\alpha_j$ is the character $(\alpha_0, \alpha_d) \mapsto \sigma_j(ad^{-1})$. As a consequence, each nontrivial character appears with multiplicity at most one as a Jordan–Hölder factor of $F[I_1/Z_1]/m^3_{I_1/Z_1}$.
6. On smooth representations of $\GL_2$

The aim of this section is to prove Theorem 6.4.7 below which provides a useful criterion for bounding the dimension of an admissible smooth representation of $\GL_2(L)$.

We keep the notation of §5.3: $L$ is a finite unramified extension of $\mathbb{Q}_p$ of degree $f$ with ring of integers $\mathcal{O}_L$ and residue field $k$, $I$ (resp. $I_1$) is the upper (resp. upper pro-$p$) Iwahori subgroup of $K \overset{\text{def}}{=} \GL_2(\mathcal{O}_L)$ and $Z_1$ is the center of $I_1$. We set $K_1 \overset{\text{def}}{=} 1 + p \mathcal{M}_2(\mathcal{O}_L) \subseteq I_1$.

If $H$ is a compact $p$-adic analytic group and if $V$ is an admissible smooth $\mathbb{F}$-rational representation of $H$ we denote $\Ind_H^G V$ an injective envelope of $V$ in the category of admissible smooth representations of $H$; it is unique up to nonunique isomorphism. As an $\mathbb{F}[H]$-module, the dual $V^\vee$ is finitely generated and we denote by $\Proj_H V^\vee$ a projective envelope of $V^\vee$ in the category of pseudocompact $\mathbb{F}[H]$-modules. The radical $\mathbb{m}_H M$ of a pseudocompact $\mathbb{F}[H]$-module is the submodule $\mathbb{m}_H M$.

If $G$ is a $p$-adic analytic group, $H$ a closed subgroup of $G$ and $V$ a smooth $H$-representation over $\mathbb{F}$, we denote by $\Ind_H^G V$ the $\mathbb{F}$-vector space of smooth functions $f : G \to V$ such that $f(hg) = hf(g)$ for all $g \in G$ and $h \in H$. The group $G$ acts on $\Ind_H^G V$ by translation on the right. If $H$ is cocompact in $G$, the representation $\Ind_H^G V$ is smooth and if moreover $V$ is admissible, it is admissible.

If $\lambda \in X^*(T)$ we use the notation $\chi_\lambda$ to denote the character $T(k) \to \mathbb{T}(\mathbb{F}) \overset{\lambda}{\to} \mathbb{F}^\times$, where the first map is the inclusion. We use the same notation $\chi_\lambda$ to denote the character of $I$ obtained by composition with $I \to T(k)$. Equivalently $\chi_\lambda$ is the character of $I$ acting on $F(\lambda)^I$.

In this section, we always assume that $p > 2$.


Let $\alpha_i : T(k) \to \mathbb{F}^\times$ denote also the character $\chi_{\alpha_i}$, i.e. the character sending $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T(k)$ to $\sigma_i(ad^{-1})$. In particular, $\alpha_i = \alpha_0^{p^i}$ as characters of $T(k)$ for $0 \leq i \leq f - 1$.

We let $\chi : I \to \mathbb{F}^\times$ be a smooth character. For any $n \geq 1$, we set

$$W_{\chi,n} \overset{\text{def}}{=} (\Proj_{I/Z_1} \chi)/m_{I_1}^n.$$  

(Note that via the natural map $\mathbb{F}[I] \to \mathbb{F}[I/Z_1]$ the actions of $m_{I_1}$ and $m_{I_1/Z_1}$ coincide on $\Proj_{I/Z_1} \chi$; similar comment will apply later on for pseudocompact $\mathbb{F}[K/Z_1]$-modules.)

Let $\chi_0$ be the trivial character of $I$. As any smooth character $\chi : I \to \mathbb{F}^\times$ is trivial on $I_1$, there is an isomorphism of $\mathbb{F}[I/Z_1]$-modules

$$\Proj_{I/Z_1} \chi \cong \chi \underset{\mathbb{F}}{\otimes} \Proj_{I/Z_1} \chi_0$$

and an isomorphism of $\mathbb{F}[I/Z_1]$-modules $\Proj_{I/Z_1} \chi_0 \cong \mathbb{F}[I_1/Z_1]$. (Note that the decomposition $I = I_1 \times H$ with $H$ as in §5.3 gives a natural left action of $I$ on $\mathbb{F}[I_1/Z_1]$, where $I_1$ acts by left translation and $H$ by conjugation.) Consequently for any $n \geq 1$, we have an isomorphism of $I$-representations $W_{\chi,n} \cong \chi \underset{\mathbb{F}}{\otimes} (\mathbb{F}[I_1/Z_1]/m_{I_1}^n)$. From the description of $\text{gr}_m \mathbb{F}[I_1/Z_1]$ in (11), we can deduce the following result.

**Lemma 6.1.1.** We keep the above hypotheses.
(i) For any $\chi' \neq \chi$, $[W_{\chi,3} : \chi'] \leq 1$.
(ii) Suppose that $\chi, \chi' : I \to \mathbb{F}^\times$ are smooth characters such that $\text{Ext}_{I/Z_1}^1(\chi, \chi') \neq 0$. Then $\chi' \in \{\alpha_i^{\pm 1} : 0 \leq i \leq f - 1\}$ and we have $\dim \mathbb{F} \text{Ext}_{I/Z_1}^1(\chi, \chi') = 1$. Letting $E_{\chi', \chi}$ denote the unique nonsplit $I$-extension
\begin{equation}
0 \to \chi' \to E_{\chi', \chi} \to \chi \to 0,
\end{equation}
the group $K_1$ acts trivially on $E_{\chi', \chi}$ if and only if $\chi' = \chi \alpha_i$ for some $0 \leq i \leq f - 1$.

**Proof.** Part (i) follows from equation (41) by twisting and part (ii) follows from [Hu10, Lemma 2.4] (i) and (ii). □

Now, let $\chi'$ be a character such that $\text{Ext}_{I/Z_1}^1(\chi, \chi') \neq 0$. Since $[W_{\chi,3} : \chi'] = 1$ and $\chi'$ occurs as a subquotient in $\text{rad} I_1(W_{\chi,3})$ which is killed by $m_{I_1}^3$, there is a unique (up to scalar) nonzero $I$-equivariant morphism $W_{\chi', 2} \to W_{\chi, 3}$.

**Lemma 6.1.2.** If $\text{Ext}_{I/Z_1}^1(\chi, \chi') \neq 0$, then any nonzero morphism $W_{\chi', 2} \to W_{\chi, 3}$ is injective.

**Proof.** By twisting, it is sufficient to consider the case where $\chi$ is the trivial character $\chi_0$. In this case, there is an $I$-equivariant isomorphism $\mathbb{F}[I_1/Z_1] \cong \text{Proj}_{I_1/Z_1} \chi_0$. Let $e \in \text{gr}_m ^I \mathbb{F}[I_1/Z_1]$ be an eigenvector of weight $\chi'$. There is a unique degree 1 morphism of graded $\text{gr}_m \mathbb{F}[I_1/Z_1]$-modules $f : \text{gr}_m \mathbb{F}[I_1/Z_1] \to \text{gr}_m \mathbb{F}[I_1/Z_1]$ sending 1 to $e$. As $\text{gr}_m \mathbb{F}[I_1/Z_1]$ is isomorphic to an enveloping algebra over a field by [10], the Poincaré–Birkhoff–Witt Theorem implies that it has no zero divisor so that the map $f$ is injective. Let $\tilde{e} \in m_{I_1/Z_1}$ such that $\text{gr}_m(\tilde{e}) = e$. We define a degree 1 morphism of filtered $\mathbb{F}[I_1/Z_1]$-modules $\tilde{f} : \mathbb{F}[I_1/Z_1] \to \mathbb{F}[I_1/Z_1]$ sending $x$ to $x\tilde{e}$. Obviously, we have $f = \text{gr}_m(\tilde{f})$. Moreover, if we choose for $\tilde{e}$ a $\chi'$-eigenvector for the action of the group $H$, then $\tilde{f}$ induces an $H$-equivariant map $\tilde{f}' : \chi' \otimes_{\mathbb{F}} \mathbb{F}[I_1/Z_1] \to \mathbb{F}[I_1/Z_1]$. As $I = I_1 \times H$, the map $\tilde{f}'$ is $I$-equivariant. Since $\tilde{f}'$ is injective on graded modules for the $m_{I_1}$-adic filtration, it induces an $I$-equivariant injective map $W_{\chi', 2} = \text{Proj}_{I_1/Z_1} \chi'/m_{I_1}^3 \hookrightarrow \text{Proj}_{I_1/Z_1} \chi_0/m_{I_1}^3 = W_{\chi_0, 3}$. □

For an integer $0 \leq \ell \leq q - 1$ we let $\ell_i$ denote the $i$-th base $p$ digit of $\ell$, so $\ell = \sum_{i=0}^{\ell-1} \ell_i p^i$.

**Lemma 6.1.3.** Let $I_{\chi} \overset{\text{def}}{=} \text{Ind}_{B(k)}^I \chi$. Then $I_{\chi}$ has socle and cosocle isomorphic to $\chi$, and its remaining Jordan–Hölder factors $\chi \alpha_0^{-j}$, $0 < j < q - 1$, occur with multiplicity 1. Its submodule structure is determined by the following property: the unique proper submodule of $I_{\chi}$ with cosocle $\chi \alpha_0^{-j}$ $(0 \leq j < q - 1)$ has Jordan–Hölder factors $\chi \alpha_0^{-\ell}$, where $0 \leq \ell < q - 1$ and $\ell_i \leq j_i$ for all $i$.

**Proof.** The claim about socle and cosocle are true for injective envelopes of any finite group.

We first observe that $I_{\chi} \cong \text{Ind}_{T(k)}^B \chi$. The latter representation is injective by Frobenius reciprocity (as any $T(k)$-representation is injective). It has the correct socle and cosocle by Frobenius reciprocity, hence indeed $I_{\chi} \cong \text{Ind}_{T(k)}^B \chi$.

As the kernel of $B(k) \to T(k)$ is a normal $p$-subgroup, every irreducible $B(k)$-representation is trivial on it. To determine Jordan–Hölder factors we may thus restrict to $T(k)$. By Mackey’s formula, $(\text{Ind}_{T(k)}^B \chi)|_{Z(k)} \cong \chi \oplus (\text{Ind}_{Z(k)}^T \chi)|_{Z(k)}$, where $Z$ is the center of $\text{GL}_2$. Thus the irreducible
constituents of $\mathcal{I}_\chi$ are all the characters $\chi'$ of $T(k)$ such that $\chi'|_{Z(k)} = \chi|_{Z(k)}$, or equivalently $\chi' = \chi\alpha_0^{-j}$ for some $0 \leq j < q - 1$, as well as one more copy of $\chi$.

As in [BP12, §2] we define $f_j \defeq \sum_{\lambda \in k} \lambda'(0,1)x$, where $e \in \text{Ind}_{T(k)}^{B(k)} \chi$ is some function whose support equals $T(k)$. It follows that $f_j$ is a $T(k)$-eigenvector with eigenvalue $\chi\alpha_0^{-j}$.

Assume now that $j < q - 1$. An explicit calculation shows that $(1, x)_{\lambda, e} f_j = \sum_{\ell=0}^{j} \binom{j}{\ell}(-x)^{j-\ell} f_\ell$. Hence the $B(k)$-representation $W$ generated by $f_j$ has basis $f_\ell$ for all $i$. In particular, $W \not\cong \mathcal{I}_\chi$ since $j < q - 1$. On the other hand, $W$ is a quotient of $\text{Ind}_{T(k)}^{B(k)} \chi\alpha_0^{-j}$, so $W$ is the unique proper subrepresentation of $\mathcal{I}_\chi$ with cosocle $\chi\alpha_0^{-j}$. □

The element $(0,1) \in \text{GL}_2(L)$ normalizes $I$ and its square is central. Let $\chi^s$ denote the conjugate of $\chi$ by $(0,1) \in \text{GL}_2(L)$. By conjugating $\mathcal{I}_\chi$ by $(0,1) \in \text{GL}_2(L)$ we obtain the following corollary.

**Corollary 6.1.4.** Given $\chi : T(k) \to \mathbb{F}^\times$ there is a (finite-dimensional) smooth representation $\mathcal{J}_\chi$ of $I$ with the following properties. The socle and cosocle of $\mathcal{J}_\chi$ are isomorphic to $\chi^s$, and the remaining Jordan–Hölder factors of $\mathcal{J}_\chi$ are $\chi^s\alpha_0^0$ for $0 < j < q - 1$, each occurring with multiplicity 1. The unique proper submodule of $\mathcal{J}_\chi$ with cosocle $\chi^s\alpha_0^0$ (0 ≤ $j < q - 1$) has Jordan–Hölder factors $\chi^s\alpha_0^\ell$, where 0 ≤ $\ell < q - 1$ and $\ell_i < j_i$ for all $i$. Moreover, $\mathcal{J}_\chi$ admits a central character.

**Remark 6.1.5.** On $\mathcal{J}_\chi$ the action of $I$ does not factor through its quotient $B(k)$, contrary to the case $\mathcal{I}_\chi$ (cf. Lemma 6.1.1).

6.2. On some indecomposable representations of $K$.

We will use again the notation of section 2.4. In particular, recall that we have identified $\mathcal{J} = \text{Hom}(k, \mathbb{F})$ with $\{0, 1, \ldots, f - 1\}$ and that $\eta_j \defeq \sum_{i \in J} \eta_i$ for $J \subseteq \mathcal{J}$. Also, for $\lambda \in X^+(T)$ recall the injective map

$$t_\lambda : \Lambda_\mathfrak{t}^\lambda \to X_{\text{reg}}(T)/(p - \pi)X^0(T).$$

Let $\sigma'$ be a Serre weight appearing in $\text{Inj}_{\text{GL}_2(k)} F(\lambda)$. It follows from [BP12, Cor. 3.12] that there exists a unique subrepresentation of $\text{Inj}_{\text{GL}_2(k)} F(\lambda)$, denoted by $I(\lambda, \sigma')$, with cosocle $\sigma'$ and such that $[I(\lambda, \sigma') : F(\lambda)] = 1$. Moreover, $I(\lambda, \sigma')$ is multiplicity-free. As a consequence, if $W$ is a subrepresentation of $\text{Inj}_{\text{GL}_2(k)} F(\lambda)$ such that $[W : \sigma'] \neq 0$, then $W$ contains $I(\lambda, \sigma')$ as a subrepresentation. Dually, we have similar statements for quotients of $\text{Proj}_{\text{GL}_2(k)} F(\lambda)$.

**Lemma 6.2.1.** We keep the above hypotheses.

(i) Suppose that $0 < \langle \lambda, \alpha_i^\vee \rangle < p - 1$ for all $i$. Then $\text{Ind}_{T}^{K} \chi^s$ is multiplicity-free with Jordan–Hölder factors $\{F(t_{\lambda}(\sum_{j \in J} a_j \eta_j)) : J \subseteq \mathcal{J}\}$.

(ii) Suppose that $0 < \langle \lambda, \alpha_i^\vee \rangle < p - 2$ for all $i$. The Jordan–Hölder factors of $\text{Inj}_{\text{GL}_2(k)} F(\lambda)$ are the $\{F(t_{\lambda}(\sum_{j \in J} a_j \eta_j)) : (a_i)_{i \in \mathcal{J}} \subseteq \{0, \pm 1\}^J, a_i 

(iii) Suppose that $0 < \langle \lambda, \alpha_i^\vee \rangle < p - 2$ for all $i$. Let $\sigma' = F(t_{\lambda}(\sum_{j \in J} a_j \eta_j))$ for some $(a_i) \in \{0, \pm 1\}^J$. The Jordan–Hölder factors of $I(\lambda, \sigma')$ are $\{F(t_{\lambda}(\sum_{j \in J} a_j \eta_j)) : J \subseteq \mathcal{J}\}$. As a consequence, the length of $I(\lambda, \sigma')$ is equal to $2|\{i \in J : a_i \neq 0\}|$. 
By Remark 2.4.3(iii) the condition on \( \lambda \) in (i) is precisely that all weights \( t_\lambda(-\eta_J) \) lie in \( C_0 \). Also note in part (iii) that the Jordan–Hölder factors correspond via \( t_\lambda \) precisely to the weights lying on geodesics between 0 and \( \sum_{i \in J} a_i \eta_i \).

**Proof.** Part (i) is almost a special case of Proposition 2.4.3 (with \( sw^{-1} = 1, \nu = \eta, \) and \( \mu - \eta = \lambda \)), but the hypothesis is weaker here.

If \( \nu \in X^0(\mathcal{T}) \), then from the definition, \( F(t_{\lambda + \nu} (\omega)) \equiv F(t_\lambda(\omega)) \otimes_F F(\nu) \). (Note that \( F(\nu) \) is one-dimensional.) We may therefore assume that \( \lambda_i \) is of the form \((a_i, 0)\) for some integers \( 0 < a_i < p - 1 \).

Recall from Remark 2.4.3(i) the notation \( w_{0,i} = \prod_{i+1 \in J} w_i \in W \), where \( w_i \) denotes the Weyl group element which is nontrivial exactly in the \( i \)-th embedding. We first calculate \( t_\lambda(-\eta_J) \equiv \mu_J \mod (p - \pi)X^0(\mathcal{T}) \), where \( \mu_J = (t_{\pi^{-1}(\eta_J)} w_{0,i}) (\lambda - \eta_J) \in X^*(\mathcal{T}) \). We have

\[
\begin{align*}
\mu_J & = \left\{ \begin{array}{ll}
\lambda_i - \delta_J(i)(1, 0) & \text{if } i + 1 \notin J, \\
w_0 \cdot (\lambda_i + (0, p) - \delta_J(i)(1, 0)) & \text{if } i + 1 \in J,
\end{array} \right.
\end{align*}
\]

where \( \delta_J \) is the characteristic function of \( J \) (cf. equation (12)). Replacing \( J \) by the set \( K \equiv \{ i \in J : i + 1 \notin J \} \), we obtain precisely the formula for the composition factors listed in [Dia07, Prop. 1.1].

Part (ii) follows similarly from [BPT12, Lemma 3.2], and part (iii) follows from [BPT12, Cor. 4.11].

**Proposition 6.2.2.** Fix \( \lambda \in X^*(\mathcal{T}) \). Suppose that integers \( B_i \in \mathbb{Z}_{\geq 0} \) and signs \( \varepsilon_i \in \{ \pm 1 \} \) (\( 0 \leq i < f - 1 \)) satisfy the following conditions:

(i) \( B_i \equiv 1 - \varepsilon_i - 1 \pmod{2} \);

(ii) if \( \varepsilon_i = -1 \), then \( B_i \leq \langle \lambda, \alpha_i^\vee \rangle \leq p - 2 - \frac{1 + \varepsilon_i - 1}{2} \);

(iii) if \( \varepsilon_i = 1 \), then \( B_i \leq p - 2 - \langle \lambda, \alpha_i^\vee \rangle \leq p - 2 - \frac{1 + \varepsilon_i - 1}{2} \).

Then there exists a multiplicity-free representation \( V \) of \( K/Z_1 \) with Jordan–Hölder constituents \( \sigma_\alpha \equiv F(t_\lambda(\sum \varepsilon_i a_i \eta_i)) \), where \( 0 \leq a_i \leq B_i \) and whose submodule structure is determined as follows: the unique subrepresentation with cosocle \( \sigma_\gamma \) has constituents \( \sigma_\alpha \) for all \( \alpha \) such that \( 0 \leq b_i \leq a_i \) for all \( i \). In particular, the socle of \( V \) is isomorphic to \( F(\lambda) \).

**Proof.** As a first step we consider the case where \( \varepsilon_i = -1 \) for all \( i \). Let \( b_i \equiv B_i - 1 \in \mathbb{Z}_{\geq 0} \) for \( 0 \leq i \leq f - 1 \). Note that \( t_\lambda(-\sum_i a_i \eta_i) \in C_0 \) for all \( 0 \leq a_i \leq B_i \) is equivalent to condition (ii) (cf. Remark 2.4.5(iii)). Let \( \chi \equiv t_\lambda \). Corollary 6.1.4 gives us a representation \( W \subseteq \mathcal{F}_\chi \) of \( I \) with constituents \( \chi^i a_i^\vee \), where \( 0 \leq j_i \leq b_i \) for all \( i \), and such that the unique subrepresentation of \( W \) with cosocle \( \chi^i a_i^\vee \) has constituents \( \chi^i \alpha_i^\vee \), where \( 0 \leq \ell_i \leq j_i \) for all \( i \). Let \( V \equiv \text{Ind}_I^K W \). By Lemma 6.2.1 and Remark 2.4.5(ii) this representation is multiplicity-free with constituents \( F(t_\lambda(-\sum_i c_i \eta_i)) \), where \( 0 \leq c_i \leq 2b_i + 1 = B_i \) for all \( i \).
To determine the submodule structure, by Lemma 2.4.6 it is enough to show that for any 
\((c_i)_j\) as above and any \(j\) such that \(c_j < 2b_j + 1\) there exists a length 2 subquotient with socle
\(F(t_{\lambda}(-\sum c_i\eta_i))\) and cosocle \(F(t_{\lambda}(-\eta_j - \sum c_i\eta_i))\). To see this, write \(c_i = 2d_i + r_i\) with \(0 \leq r_i \leq 1\). Observe that
\[
F(t_{\lambda}(-\sum c_i\eta_i)) = F(t_{\lambda}(-\sum r_i\eta_i - \sum d_i\alpha_i)) = F(t_{\lambda}\sum d_i\alpha_i(-\sum r_i\eta_i))
\]
by applying Remark 2.4.7(ii). By Lemma 6.2.1 this is a constituent of \(\text{Ind}_I^K \chi^{s}\), where \(\chi^{s} = \chi^{s}_{\lambda}\sum d_i\alpha_i = \chi^{s}_{\lambda}\sum d_if^{p}\).

If \(r_j = 0\), then \(F(t_{\lambda}(-\eta_j - \sum c_i\eta_i))\) is a constituent of \(\text{Ind}_I^K \chi^{s}\) as well, and we are done by Lemma 6.2.1 as \(V\) admits \(\text{Ind}_I^K \chi^{s}\) as subquotient.

If \(r_j = 1\), then \(F(t_{\lambda}(-\eta_j - \sum c_i\eta_i))\) is a constituent of \(\text{Ind}_I^K \chi^{s}\alpha_0^{p'}\). Letting the other \(r_i\) vary in \(\{0,1\}\), we need to check the existence of the \(2^{f-1}\) nonsplit extensions inside \(V\) between constituents of \(\text{Ind}_I^K \chi^{s}\alpha_0^{p'}\) and \(\text{Ind}_I^K \chi^{s}\) given by Lemma 2.4.6. When \(f = 1\) this is obvious, as we can compute the cosocle of \(\text{Ind}_I^K(E^{s'}\chi^{s}\alpha_0^{p'})\) by Frobenius reciprocity (cf. Lemma 6.3.1).

When \(f \geq 2\) then [HN10] Lemma 2.12(i) confirms there are \(2^{f-1}\) nonsplit extensions, as required (in the notation of that reference the condition is \(J(\lambda) = J(\theta) \sqcup \{j - 1\}\)).

Finally we treat the general case. Let \(J \overset{\text{def}}{=} \{0 \leq i \leq f - 1 : \varepsilon_i = 1\}\). Set \(\mu = t_{\lambda}(w_{0,J}(\eta_j))\). Using Lemma 2.4.4 and Remark 2.4.7(ii) we compute \(t_{\lambda}(\sum \varepsilon_i a_i\eta_i) = t_{\mu}(-\sum (a_i + \delta_j(i))\eta_i)\) for integers \(a_i\). Note that \(\delta_j(i) = \frac{1 + \varepsilon_i - 1}{2}\).

We apply the first step of the proof with the weight \(\mu\), the bounds \(B_i + \delta_j(i)\) and all signs \(-1\). We obtain a representation \(V'\) with socle \(F(\mu)\) satisfying the desired hypotheses with signs \(-1\) for all \(i\) and \(B_i + \delta_j(i)\) in place of \(B_i\). We note that its unique quotient with socle \(F(\lambda)\) has the desired properties with signs \(\varepsilon_i\) and bounds \(B_i\). We just have to check that we can apply the first step in this case. Namely it suffices to check that \(t_{\mu}(-\sum a_i'\eta_i) \in C_0\) for \(0 \leq a_i' \leq B_i + \delta_j(i)\), noting that \(B_i + \delta_j(i) = B_i + \frac{1 + \varepsilon_i - 1}{2}\) is odd for all \(i\). Equivalently, we need that \(t_{\lambda}(\sum \varepsilon_i a_i\eta_i) \in C_0\) for \(-\delta_j(i) \leq a_i \leq B_i\), i.e. \(0 \leq \langle \lambda, a_i^{s} \rangle + \varepsilon_i a_i \leq p - 2\) for \(-\delta_j(i) \leq a_i \leq B_i\) and all \(i\). This is equivalent to conditions (ii) and (iii) that we assumed.

Assume that \(\lambda\) is 1-deep in alcove \(C_0\), i.e. \(1 \leq \langle \lambda, a_i^{s} \rangle \leq p - 3\) for all \(i\). Let \(V\) be the representation of Proposition 6.2.2 with \(B_i \in \{0,1\}\) for all \(i\). Let \(q\) be such that \(0 \leq a_i \leq B_i\) for all \(i\). Then the subrepresentation of \(V\) with cosocle \(\sigma\) of Proposition 6.2.2 is isomorphic to the representation \(I(F(\lambda), \sigma)\) of [BP12, Cor. 3.12].

**Lemma 6.2.3.** Suppose that \(V\) is a finite-dimensional smooth representation of \(K\) that has irreducible \(K\)-socle \(\sigma = F(\lambda)\) with \(3 \leq \langle \lambda, a_i^{s} \rangle \leq p - 4\) for all \(i\). If \(|V : \sigma| = 1\) and all constituents of \(V\) occur in \(\text{Ind}_{\text{GL}_2(k)} \sigma\), then \(V\) is \(K_1\)-invariant.

**Proof.** By writing \(V\) as a quotient of \(\text{Proj}_K(\text{cosoc}_K V)\) and decomposing \(\text{cosoc}_K V\) as a direct sum of irreducible representations, we see that \(V\) is the sum of all subrepresenations with irreducible cosocle. We may thus assume that \(V\) itself has irreducible cosocle \(\tau\), and we argue by induction on the length \(\ell(V)\) of \(V\). If \(\ell(V) = 1\) there is nothing to show. By induction, \(\text{rad} V = K_1\)-invariant, so \(V[\text{m}_{K_1}^2] = V\). By [HW] Thm. 2.22 we know that \(V\) is \(K_1\)-invariant. (Note that the bounds on \(\lambda\) ensure that the argument there goes through.)
Proposition 6.2.4. Fix \( \lambda \in X^*(T) \). Suppose that integers \( B_i \in \mathbb{Z}_{\geq 0} \) and signs \( \varepsilon_i \in \{ \pm 1 \} \) \((0 \leq i \leq f - 1)\) satisfy the following conditions:

(i) \( B_i \equiv \frac{1 - \varepsilon_i - 1}{2} \) (mod 2);
(ii) if \( \varepsilon_i = -1 \), then \( 3 + 2 |B_i/2| \leq \langle \lambda, \alpha_i^\vee \rangle \leq p - 4; \)
(iii) if \( \varepsilon_i = 1 \), then \( 3 \leq \langle \lambda, \alpha_i^\vee \rangle \leq p - 4 - 2|B_i/2| \).

Let \( V \) be the \( K \)-representation defined by this choice of \( \lambda, B_i, \varepsilon_i \) in Proposition 6.2.2.

Then for \( 0 \leq n - 1 \leq \sum |B_i/2| \) we have that \( V|^{m_K^2}_{K_1} \) is the unique subrepresentation of \( V \) with cosocle \( \oplus \sigma_a \), where the sum runs over all \( a \) such that \( 0 \leq a_i \leq B_i \) and

(i) \( a_i \) is odd or \( a_i = B_i \),
(ii) \( \sum |a_i/2| = n - 1 \).

Proof. We proceed by induction on \( n \geq 1 \) and denote by \( V_n \) the unique subrepresentation in the statement. For convenience let \( V_0 = 0 \). We need to show that \( V_n/V_{n-1} = (V/V_{n-1})^{K_1} \). The constituents of \( V_n/V_{n-1} \) (resp. \( V/V_{n-1} \)) are all Serre weights \( \sigma_a \) with \( 0 \leq a_i \leq B_i \) and \( \sum |a_i/2| = n - 1 \) (resp. \( \sum |a_i/2| \geq n - 1 \)). Using the submodule structure of \( V \) given by Proposition 6.2.2, we see that \( V_n/V_{n-1} \) is a direct sum of indecomposable representations \( W_a \), where the index set is the same as in the statement of the proposition and the constituents of \( W_a \) are all \( \sigma_a \) with \( 0 \leq b_i \leq B_i \) and \( |b_i/2| = |a_i/2| \) for all \( i \) (and the submodule structure is described by the usual partial order). Note that \( \text{soc}_K W_a \cong \sigma_a \), where \( b_i = 2 |a_i/2| \).

By Lemma 6.2.3, \( V_n/V_{n-1} \) is \( K_1 \)-invariant (the given bounds guarantee that the lemma applies by Remark 2.4.3iii, see also Lemma 6.2.1ii). On the other hand, \( (V/V_{n-1})^{K_1} \) has to inject into the injective envelope \( \text{Ind}_{GL_2(k)}(\text{soc}_K(V/V_{n-1})) \). By Lemma 6.2.1ii we deduce that \( (V/V_{n-1})^{K_1} \subseteq V_n/V_{n-1} \). (Note that our genericity bounds are stronger.) \( \square \)

6.3. A result on maximal representations of \( K \) with prescribed socle. In this section, we prove a structure result for certain representations of \( K \) killed by \( m_K^2 \).

We begin with some preliminary lemmas concerning Jordan–Hölder factors of subrepresentations of some parabolically induced representations. Recall from [43] the representation \( E_{\lambda', \lambda} \) for two characters \( \lambda, \lambda' \) of \( I \) such that \( \text{Ext}_I^1(\lambda, \lambda') \neq 0 \).

Lemma 6.3.1. Assume \( \lambda' = \lambda \alpha_i^{-1} \) for some \( 0 \leq i \leq f - 1 \). The cosocle of \( \text{Ind}_I^K E_{\lambda', \lambda} \) is equal to the cosocle of \( \text{Ind}_I^K \lambda \).

Proof. Let \( \sigma \) be a Serre weight and assume there exists a surjection \( f : \text{Ind}_I^K E_{\lambda', \lambda} \twoheadrightarrow \sigma \). Then Frobenius reciprocity induces a nonzero \( I \)-equivariant morphism \( f' \in \text{Hom}_I(E_{\lambda', \lambda'}, \sigma|_I) \). Since \( K_1 \) acts trivially on \( \sigma \) but not on \( E_{\lambda', \lambda} \) (see Lemma 6.1.1ii), \( f' \) cannot be injective. In other words, \( f' \) factors through \( E_{\lambda', \lambda} \twoheadrightarrow \sigma \hookrightarrow \sigma|_I \), i.e. \( f \) factors through \( \text{Ind}_I^K E_{\lambda', \lambda} \hookrightarrow \text{Ind}_I^K \lambda \). \( \square \)

Remark 6.3.2. For the explicit structure of \( \text{Ind}_I^K E_{\lambda', \lambda} \) when \( \lambda' = \lambda \alpha_i^{-1} \), see [BP12, §18].

Given \( \lambda \) satisfying \( \lambda \neq \lambda^g \), we denote by \( \sigma_\lambda \) the unique Serre weight such that \( I \) acts on \( \sigma_\lambda^I \) via \( \chi \). Recall that in this case \( \text{Ind}_I^K \lambda \) has irreducible cosocle \( \sigma_\lambda \) and irreducible socle \( \sigma_\lambda^* \) (see e.g. [BP12, Thm. 2.4]). Given a Serre weight \( \sigma \), we denote by \( \chi_\sigma \) the character of \( I \) acting on \( \sigma^I \).
Lemma 6.3.3. Suppose that \( \chi = \chi_\lambda \) with \( 2 < \langle \lambda, \alpha_i^\vee \rangle < p - 3 \) for all \( 0 \leq i \leq f - 1 \). Then the \( K \)-representation \( \Ind^K I \chi_{2} \) is multiplicity-free, where \( W_{\chi,2} \) is defined in (42).

Proof. This is a direct check using Remark [2.4.7][ii] and Lemma [6.2.1][i]. The assumption on \( \lambda \) ensures that the hypothesis of Lemma [6.2.1][i] applies to all \( \Ind^K I \chi \) with \( \chi' \in \JH(W_{\chi,2}) \). □

From now on we fix \( \chi = \chi_\lambda \) with \( \lambda \in X_1(P) \) such that \( 2 < \langle \lambda, \alpha_i^\vee \rangle < p - 3 \) for all \( 0 \leq i \leq f - 1 \).

Let now \( \chi' \defeq \chi_{\alpha_i} \) for some \( i \in J \), so \( \Ext^{1}_{I/Z}(\chi, \chi') \neq 0 \). As \( E_{\chi', \chi} \) is a quotient of \( W_{\chi,2} \), Lemma [6.3.3] implies that \( \Ind^K I E_{\chi', \chi} \) is multiplicity-free. On the other hand, \( K_1 \) acts trivially on \( \Ind^K I E_{\chi', \chi} \) by Lemma [6.1.3][ii]. Hence there is a unique (up to scalar) nonzero map \( K : \Proj_{GL_2(k)} \sigma_\chi \rightarrow \Ind^K I E_{\chi', \chi} \). Observe that the composite map

\[
\Proj_{GL_2(k)} \sigma_\chi \xrightarrow{f} \Ind^K I E_{\chi', \chi} \rightarrow \Ind^K I \chi
\]

is surjective, since it is surjective on \( K \)-cosocles.

Lemma 6.3.4. Suppose that \( \chi = \chi_\lambda \) with \( 2 < \langle \lambda, \alpha_i^\vee \rangle < p - 3 \) for all \( 0 \leq i \leq f - 1 \). Assume \( \chi' = \chi_{\alpha_i} \) for some \( i \in J \). We have

\[
\JH(\Im(f)) = \JH(\Ind^K I E_{\chi', \chi}) \cap \JH(\Proj_{GL_2(k)} \sigma_\chi).
\]

Proof. Observe that the \( K \)-socle of \( \Ind^K I E_{\chi', \chi} \) is isomorphic to \( \sigma_{\chi'} \oplus \sigma_{\chi^s} \), i.e. the direct sum of the socles of \( \Ind^K I \chi' \) and \( \Ind^K I \chi \). Indeed, it is clear that

\[
\sigma_{\chi'} \subseteq \soc_K(\Ind^K I E_{\chi', \chi}) \subseteq \sigma_{\chi'} \oplus \sigma_{\chi^s},
\]

so it suffices to prove that \( \Hom_K(\sigma_{\chi^s} \oplus \sigma_{\chi^s}, \Ind^K I E_{\chi', \chi}) \neq 0 \), or equivalently \( \Hom_I(\sigma_{\chi^s} \oplus \sigma_{\chi^s}, \Ind^K I \chi') \neq 0 \), by Frobenius reciprocity. This can be checked directly, by writing down the standard basis of \( \sigma_{\chi^s} \).

Let \( V \defeq \Im(f) \). We claim that \( V \cap \Ind^K I \chi' \neq 0 \). Otherwise, the composite morphism \( V \hookrightarrow \Ind^K I E_{\chi', \chi} \hookrightarrow \Ind^K I \chi \) would be injective, and also surjective as remarked before the lemma. Thus, we would have a \( K \)-equivariant decomposition \( \Ind^K I E_{\chi', \chi} \cong \Ind^K I \chi \oplus \Ind^K I \chi' \), which is not possible (see for example [Alp86], §8, Lemma 6(5)). As a consequence of the claim, \( \sigma_{\chi^s} \) appears in \( V \) (as a subobject), and therefore \( V \) admits a quotient isomorphic to \( I(\sigma_{\chi^s}, \sigma_\chi) \) (we recall that this representation was defined in §6.2).

Now we prove (44). The inclusion \( \subseteq \) is obvious. Let \( \sigma \) be a Serre weight lying in the right-hand side of (44). If \( \sigma \) is in \( \JH(\Ind^K I \chi) \), then clearly \( \sigma \in \JH(V) \) because \( \Ind^K I \chi \) is a quotient of \( V \). So we may assume \( \sigma \in \JH(\Ind^K I \chi') \). Then, by Lemma [6.2.1][i] and Remark [2.4.3][ii] \( \sigma \) is of the form \( F(t_{\lambda + \alpha_i}(-\eta_j)) = F(t_{\lambda}(2\eta_i - \eta_j)) \) for some \( J \subseteq J \). It follows from Lemma [6.2.1][ii][iii] and Remark [2.4.3][ii] that such a Serre weight is a Jordan–Hölder factor of \( \Proj_{GL_2(k)} \sigma_\chi \) if and only if it is a Jordan–Hölder factor of \( I(\sigma_{\chi^s}, \sigma_\chi) \). (Note that \( \sigma_{\chi^s} \cong F(\lambda) \) and \( \sigma_{\chi^s} \cong F(t_\lambda(2\eta_i - \eta_j)) \).)

Since \( I(\sigma_{\chi^s}, \sigma_{\chi^s}) \) is a quotient of \( V \), this finishes the proof. □

Lemma 6.3.5. Suppose that \( \chi = \chi_\lambda \) with \( 2 < \langle \lambda, \alpha_i^\vee \rangle < p - 3 \) for all \( 0 \leq i \leq f - 1 \). Assume \( \chi' = \chi_{\alpha_i} \) for some \( i \in J \). Let \( Q \) be a quotient of \( \Ind^K I E_{\chi', \chi} \) such that \( [Q : \sigma_\chi] = 0 \), then \( \Ext^1_K(\sigma, \sigma_\chi) = 0 \) for any \( \sigma \in \JH(Q) \).
Proof. Let $M$ be the kernel of $\text{Ind}^K \chi \to Q$. By Lemma 6.3.3 and the assumption, we have $[M : \sigma]\chi] = 1$. As a consequence, the natural morphism $M \to \text{Ind}^K \chi$ is surjective (as $\sigma\chi$ is the cosocle of $\text{Ind}^K \chi$), and therefore $Q$ is a quotient of $\text{Ind}^K \chi'$ by the snake lemma. By Lemma 6.2.1(i) the Jordan–Hölder factors of $\text{Ind}^K \chi'$ are of the form $F(t_\lambda + \alpha_i (-\eta_j))$ for $J \subseteq J$. It follows from Lemma 2.4.6 that the existence of $\sigma \in \text{JH}(Q)$ such that $\text{Ext}^1_q(\sigma, \sigma_{\chi}) \neq 0$ implies the existence of $J \subseteq J$ and $j \in J$ such that $F(t_{\lambda + \alpha_i}(-\eta_j)) \in \text{JH}(Q)$ and $t_{\lambda + \alpha_i}(-\eta_j) = t_{\lambda}(\pm \eta_j)$.

Consider again the unique (up to a scalar) nonzero map

$$f : \text{Proj}_{\text{GL}_2(k)} \sigma_\chi \to \text{Ind}^K \chi'.$$

By Lemma 6.3.4 we have $F(t_{\lambda}(\eta_i)) \in \text{JH}(\text{Im}(f))$. However, $\sigma_\chi \in \text{JH}(M)$, thus by uniqueness of $f$, we must have $\text{Im}(f) \subseteq M$. Then the Serre weight $F(t_{\lambda}(\eta_i))$ is a subquotient of both $M$ and $Q$. This contradicts the fact that $\text{Ind}^K \chi'$ is multiplicity-free (cf. Lemma 6.3.3). □

We fix signs $\varepsilon \in \{\pm 1\}$ and define

$$D_{\lambda, \varepsilon} \overset{\text{def}}{=} I \left( F(\lambda), F(t_\lambda(\sum_{i \in J} \varepsilon_i \eta_i)) \right).$$

Its Jordan–Hölder factors are isomorphic to $F(t_\lambda(\sum_{i \in J} \varepsilon_i \eta_i))$ for $J \subseteq J$ by Lemma 6.2.1(iii).

Remark 6.3.6. Keep the previous hypotheses and setting.

(i) We have

$$\text{Ind}^K \chi' \cong D_{\lambda, -1},$$

as follows from Lemma 6.2.1(i)

(ii) Let $\mathfrak{p}$ be a 2-dimensional semisimple Galois representation which is 2-generic (see Definition 2.3.4). Then the $\text{GL}_2(k)$-representation $D_0(\mathfrak{p})$ attached to $\mathfrak{p}$ as in [BP12, §14] is a direct sum of such $D_{\lambda, \varepsilon}$; see Theorem 14.8 in loc. cit.

We want to understand the structure of $D_{\lambda, \varepsilon} \otimes_{\mathbb{F}} F(\alpha_j)$.

Lemma 6.3.7. Suppose that $\chi = \chi_\lambda$ with $2 < (\lambda, \alpha_i') < p - 3$ for all $0 \leq i \leq f - 1$. The Jordan–Hölder factors of $D_{\lambda, \varepsilon} \otimes_{\mathbb{F}} F(\alpha_j)$ have multiplicity one and are given by $F(t_\lambda(2\varepsilon' \eta_j + \sum_{i \in J} \varepsilon_i \eta_i))$ for $J \subseteq J$ and $\varepsilon' \in \{-1, 0, 1\}$.

Proof. First note that we have $F(\lambda) \otimes_{\mathbb{F}} F(\alpha_j) \cong \otimes_{i=-1}^{1} F(\lambda + i \alpha_i)$ by [BP12, Prop. 5.4] or [LMS, Prop. 3.3(1)]. We then obtain the Jordan–Hölder factors using Remark 2.4.3(ii). The multiplicity one property then follows from the injectivity of $t_\lambda$. Namely if $2\varepsilon'_1 \eta_j + \sum_{i \in J_1} \varepsilon_i \eta_i = 2\varepsilon'_2 \eta_j + \sum_{i \in J_2} \varepsilon_i \eta_i$, then $J_1 = J_2$ by passing to $\Lambda_W/2\Lambda_W$, so $\varepsilon'_1 = \varepsilon'_2$. □

Lemma 6.3.8. Suppose that $\chi = \chi_\lambda$ with $2 < (\lambda, \alpha_i') < p - 3$ for all $0 \leq i \leq f - 1$. We have

$$\text{soc}_{\text{GL}_2(k)}(D_{\lambda, \varepsilon} \otimes_{\mathbb{F}} F(\alpha_j)) \cong \bigoplus_{\varepsilon' \in \{-1, 0, 1\}} F(t_\lambda(2\varepsilon' \eta_j)),

\text{cosoc}_{\text{GL}_2(k)}(D_{\lambda, \varepsilon} \otimes_{\mathbb{F}} F(\alpha_j)) \cong \bigoplus_{\varepsilon' \in \{-1, 0, 1\}} F(t_\lambda(2\varepsilon' \eta_j + \sum_{i \in J} \varepsilon_i \eta_i)).$$
Lemma 6.3.9. Suppose that \( \chi = \chi_{\lambda} \), where \( \lambda \) is 4-deep in \( C_0 \), i.e. \( 3 \leq \langle \lambda, \alpha_i' \rangle \leq p - 5 \) for all \( 0 \leq i \leq f - 1 \). Let \( \varepsilon \in \{-1, 1\} \) and write \( V \) for the unique extension of \( F(t_{\lambda}(\varepsilon \eta_j)) \) by \( F(\lambda) \):
\[
0 \to F(\lambda) \to V \to F(t_{\lambda}(\varepsilon \eta_j)) \to 0.
\]
Then \( V \otimes_{\mathbb{F}} F(\alpha_j) \) has a 3-step increasing filtration whose successive graded pieces are \( V_1, V_2, V_3 \), where

- \( V_1 \) is a nontrivial extension of \( F(t_{\lambda}(3\varepsilon \eta_j)) \) by \( F(t_{\lambda}(2\varepsilon \eta_j)) \),
- \( V_2 \) is a nontrivial extension of \( F(t_{\lambda}(\varepsilon \eta_j)) \) by \( F(\lambda) \) (i.e. \( V_2 \cong V \)), and
- \( V_3 \) is a nontrivial extension of \( F(t_{\lambda}(-\varepsilon \eta_j)) \) by \( F(t_{\lambda}(-2\varepsilon \eta_j)) \).

As a consequence, \( F(t_{\lambda}(\varepsilon \eta_j)) \) is not contained in the socle of \( (V \otimes_{\mathbb{F}} F(\alpha_j))/F(t_{\lambda}(2\varepsilon \eta_j)) \).

Moreover, the corresponding extensions of \( V_2 \) by \( V_1 \), and \( V_3 \) by \( V_2 \), are nonsplit.

The structure of \( V \otimes_{\mathbb{F}} F(\alpha_j) \) can be illustrated by the extension graph
\[
\begin{array}{ccc}
F(t_{\lambda}(3\varepsilon \eta_j)) & F(t_{\lambda}(\varepsilon \eta_j)) & F(t_{\lambda}(-\varepsilon \eta_j)) \\
\uparrow & \uparrow & \uparrow \\
F(t_{\lambda}(2\varepsilon \eta_j)) & F(\lambda) & F(t_{\lambda}(-2\varepsilon \eta_j))
\end{array}
\]
where the bottom (resp. top) row corresponds to the socle (resp. cosocle) of \( V \otimes_{\mathbb{F}} F(\alpha_j) \).

Proof. By Lemma 6.3.8 the socle of \( V \otimes_{\mathbb{F}} F(\alpha_j) \) is the direct sum of the \( F(t_{\lambda}(2\varepsilon \eta_j)) \) for \( \varepsilon' \in \{-1, 0, 1\} \) and (by duality) its cosocle is the direct sum of the \( F(t_{\lambda}((2\varepsilon' + \varepsilon)\eta_j)) \) (recall that \( \alpha_j = 2\eta_j \) in \( \Lambda_W \)).

Let us begin with the case where \( \varepsilon = -1 \). We define \( V_1 \) as the image of the unique (up to scalar) nonzero map \( \text{Proj}_{GL_2(k)} F(t_{\lambda}(-3\eta_j)) \to V \otimes_{\mathbb{F}} F(\alpha_j) \). Comparing Jordan–Hölder factors of \( V \otimes_{\mathbb{F}} F(\alpha_j) \) and \( \text{Proj}_{GL_2(k)} F(t_{\lambda}(-3\eta_j)) \) (e.g. by means of Lemmas 6.2.4(ii) and 2.4.4) and by the first sentence of the proof, we find that \( V_1 \) has length two with socle \( F(t_{\lambda}(-2\eta_j)) \) and cosocle \( F(t_{\lambda}(-3\eta_j)) \). We define \( V_2 \subseteq (V \otimes_{\mathbb{F}} F(\alpha_j))/V_1 \) as the image of a nonzero map \( \text{Proj}_{GL_2(k)} F(t_{\lambda}(-\eta_j)) \to (V \otimes_{\mathbb{F}} F(\alpha_j))/V_1 \), and \( V_3 \) as the quotient of \( (V \otimes_{\mathbb{F}} F(\alpha_j))/V_1 \) by \( V_2 \).

Using the fact that \( \varepsilon = -1 \) and Lemma 6.2.4(i) and (iii) we know that \( V \) is a subrepresentation of the principal series \( \text{Ind}^K_\mathbb{F} \chi \) with \( \chi = \chi_{\lambda} \). Therefore, \( V \otimes_{\mathbb{F}} F(\alpha_j) \) is a subrepresentation of
\[
(\text{Ind}^K \chi) \otimes_{\mathbb{F}} F(\alpha_j) \cong \text{Ind}^K (\chi \otimes_{\mathbb{F}} F(\alpha_j))|_I.
\]
We deduce from the exactness of induction that $\text{Ind}^k \chi \alpha_j$, $\text{Ind}^k \chi$, $\text{Ind}^k \chi \alpha_j^{-1}$.

We claim that
\[
\text{JH}(V_1) = \text{JH}(V \otimes F(\alpha_j)) \cap \text{JH}(\text{Ind}^k \chi \alpha_j).
\]

Indeed, recalling $\chi = \chi^\alpha$, the Jordan–Hölder factors of $\text{Ind}^k \chi \alpha_j = \text{Ind}^k (\chi \alpha_j)^s$ are of the form $F(t_{\chi - \alpha_j}(-\eta_j)) = F(t_{\lambda}(-2\eta_j - \eta_j))$ for $J \subseteq J$, and the claim is checked as in the proof of Lemma 6.3.4. Since $(\text{Ind}^k \chi) \otimes F(\alpha_j)$ is multiplicity-free by Lemma 6.3.7, we deduce that
\[
(\text{Ind}^k \chi) \otimes F(\alpha_j)|_t / (\text{Ind}^k \chi \alpha_j)
\]
and hence an embedding
\[
(V \otimes F(\alpha_j)|_t / V_1) \hookrightarrow (\text{Ind}^k \chi) \otimes F(\alpha_j)|_t / (\text{Ind}^k \chi \alpha_j)
\]
where the isomorphism holds because $(\chi \otimes F(\alpha_j)|_t) / \chi \alpha_j$ is isomorphic to $E_{X, \chi \alpha_j^{-1}}$ as $I$-representation.

As in the proof of Lemma 6.3.4, the $K$-socle of $\text{Ind}^k E_{X, \chi \alpha_j^{-1}}$ is equal to $F(\lambda) + F(t_{\lambda}(-2\eta_j))$, so by multiplicity-freeness we are reduced to prove that $E_{X, \chi \alpha_j^{-1}}$ admits a subquotient isomorphic to the (unique) nonsplit extension $\mathcal{E}$ of $F(t_{\lambda}(-\eta_j))$ by $F(t_{\lambda}(-2\eta_j))$. By (45) and (46), we see that $R$ embeds in $\text{Ind}^k E_{X, \chi \alpha_j}$, so by multiplicity-freeness we are reduced to prove that $\text{Ind}^k E_{X, \chi \alpha_j}$ admits a subquotient isomorphic to $\mathcal{E}$. It follows from the proof of Lemma 6.3.4 that $I(F(t_{\lambda}(-\eta_j)), \sigma_\chi)$ is isomorphic to a subquotient of $\text{Ind}^k E_{X, \chi \alpha_j}$. Note that $\sigma_\chi \cong F(t_{\lambda}(-\eta_j))$ by Lemma 6.2.1[iii], so $F(t_{\lambda}(-\eta_j))$ is a Jordan–Hölder factor of $I(F(t_{\lambda}(-2\eta_j)), F(t_{\lambda}(-\eta_j)))$ by 6.2.1[iii]. This finishes the proof in the case $\varepsilon = -1$.

Now we prove the last assertion (still when $\varepsilon = -1$). We only prove that the extension of $V_2$ by $V_1$, denoted by $R$, is nontrivial, the other case being analogous. It suffices to prove that $R$ admits a subquotient isomorphic to the (unique) nonsplit extension $\mathcal{E}$ of $F(t_{\lambda}(-\eta_j))$ by $F(t_{\lambda}(-2\eta_j))$. By (45) and (46), we see that $R$ embeds in $\text{Ind}^k E_{X, \chi \alpha_j}$, so by multiplicity-freeness we are reduced to prove that $\text{Ind}^k E_{X, \chi \alpha_j}$ admits a subquotient isomorphic to $\mathcal{E}$. It follows from the proof of Lemma 6.3.4 that $I(F(t_{\lambda}(-\eta_j)), \sigma_\chi)$ is isomorphic to a subquotient of $\text{Ind}^k E_{X, \chi \alpha_j}$. Note that $\sigma_\chi \cong F(t_{\lambda}(-\eta_j))$ by Lemma 6.2.1[iii], so $F(t_{\lambda}(-\eta_j))$ is a Jordan–Hölder factor of $I(F(t_{\lambda}(-2\eta_j)), F(t_{\lambda}(-\eta_j)))$ by 6.2.1[iii]. This finishes the proof in the case $\varepsilon = -1$.

To deal with the case $\varepsilon = +1$, we begin by constructing the quotient $V_3$, then $V_2$ and finally $V_1$. We define $V_3$ as the image of the unique nonzero map $V \otimes F(\alpha_j) \to \text{Inj}_{\text{GL}_2(k)} F(t_{\lambda}(-2\eta_j))$ extending the inclusion $F(t_{\lambda}(-2\eta_j)) \to \text{Inj}_{\text{GL}_2(k)} F(t_{\lambda}(-2\eta_j))$ (and using the fact that $F(t_{\lambda}(-2\eta_j)) \to V \otimes F(\alpha_j)$). Comparing Jordan–Hölder factors and using again the first sentence of the proof, $V_3$ has length 2 with cosocle $F(t_{\lambda}(-\eta_j))$. Let $R$ be the kernel of $V \otimes F(\alpha_j) \to V_3$. We define $V_2$ as the image of $R \to \text{Inj}_{\text{GL}_2(k)} F(\lambda)$ and $V_1$ as the kernel. Assume first $f \geq 2$. Using the fact that $\varepsilon = +1$, we know that $V$ is a quotient of $\text{Ind}^k \chi$, where $\mu = t_{\lambda}(-\eta_j)$ (use Lemma 2.4.4 and note that $\lambda = t_{\mu}(\eta_j)$). Therefore we can use a similar argument as in the case $\varepsilon = -1$. The case
Suppose that on the sign whose successive graded pieces are: which contradicts the case of the other assertions. Indeed, if the extension of $W$ is the only constituent of this socle which is also a constituent of $V \otimes_F F(\alpha_j)$. We define $V_3$ to be the image of the composite map

$$V \otimes_F F(\alpha_j) \hookrightarrow (\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(\lambda)) \otimes_F F(\alpha_j) \hookrightarrow \text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(\lambda)(2\eta_j).$$

Comparing Jordan–Hölder factors, it is easy to see that $V_3$ is equal to either $F(t_\lambda(2\eta_j))$ or a nonsplit extension of $F(t_{\lambda}(-\eta_j))$ by $F(t_{\lambda}(-2\eta_j))$. However, if we had $V_3 = F(t_{\lambda}(-2\eta_j))$, then $V' \otimes_F F(\alpha_j)$ would admit a quotient isomorphic to $(\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(t_{\lambda}(2\eta_j)))/F(t_{\lambda}(-2\eta_j))$ by the snake lemma, which contradicts the case $\varepsilon = -1$. We can continue in this way to define $V_2$ and $V_1$, and show that the corresponding extensions of $V_2$ by $V_1$ and $V_3$ by $V_2$ are nonsplit. As an example, we show that the extension of $V_2$ by $V_3$ is nonsplit, and leave to the reader the proofs of the other assertions. Indeed, if the extension of $V_3$ by $V_2$ were split, then $V \otimes_F F(\alpha_j)$ would contain a subrepresentation isomorphic to $V_3$, and the image of the composite map

$$V_3 \hookrightarrow V \otimes_F F(\alpha_j) \hookrightarrow (\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(\lambda)) \otimes_F F(\alpha_j)$$

would be contained in the summand $\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(t_{\lambda}(2\eta_j))$. Moreover, comparing Jordan–Hölder factors, we must have

$$(V \otimes_F F(\alpha_j)) \cap \text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(t_{\lambda}(2\eta_j)) = V_3,$$

the intersection being taken inside $\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(t_{\lambda}(2\eta_j))$. We then deduce an embedding

$$\left(\text{Inj}_{\text{GL}_2(\mathbb{F}_p)} F(t_{\lambda}(2\eta_j))\right)/V_3 \hookrightarrow V' \otimes_F F(\alpha_j)$$

which contradicts the case $\varepsilon = -1$. □

**Proposition 6.3.10.** Suppose that $\chi = \chi_{\lambda}$, where $\lambda$ is $4$-deep in $\mathcal{C}_0$, i.e. $4 \leq \langle \lambda, \alpha_i^\vee \rangle \leq p - 6$ for all $0 \leq i \leq f - 1$. Let $0 \leq j \leq f - 1$. There is an increasing $3$-step filtration of $D_{\lambda, \xi} F(\alpha_j)$ whose successive graded pieces are:

$$D_{\lambda + \varepsilon_j \alpha_j, \xi}, \quad D_{\lambda, \xi}, \quad D_{\lambda - \varepsilon_j \alpha_j, \xi}.$$

As a consequence, there is an embedding $D_{\lambda + \varepsilon_j \alpha_j, \xi} \hookrightarrow D_{\lambda, \xi} \otimes_F F(\alpha_j)$ whose cokernel has socle $F(\lambda) \oplus F(t_{\lambda}(2\varepsilon_j \eta_j))$.

**Proof.** By Lemma 6.3.8, we know what are the socle and cosocle of $D_{\lambda, \xi} \otimes_F F(\alpha_j)$.

During this proof, we will use the notation $\eta_{f_j}^J \overset{\text{def}}{=} \sum_{j \in J} \varepsilon_j \eta_{f_j}$ if $J \subseteq J$ (note that $\eta_{f_j}^J$ does depend on the sign $\xi$). We recall that $t_{\lambda}(\eta_{f_j}^J + 2\varepsilon_j \eta_j) = t_{\lambda + \varepsilon_j \alpha_j}(\eta_{f_j}^J)$ by Remark 2.4.3 [1]. By Lemma 6.3.7 there exists a unique (up to scalar) nonzero map

$$\text{Proj}_{\text{GL}_2(k)} F(t_{\lambda}(2\varepsilon_j \eta_j + \eta_{f_j}^J)) \rightarrow D_{\lambda, \xi} \otimes_F F(\alpha_j);$$

let $W_1$ be its image. The socle of $W_1$ is contained in the socle of $D_{\lambda, \xi} \otimes_F F(\alpha_j)$. But $F(t_{\lambda}(2\varepsilon_j \eta_j))$ is the only constituent of this socle which is also a constituent of $\text{Proj}_{\text{GL}_2(k)} F(t_{\lambda}(2\varepsilon_j \eta_j + \eta_{f_j}^J))$, cf. Lemmas 6.2.1 [1] and 2.4.4. This implies that $W_1$ is a quotient of $\text{Proj}_{\text{GL}_2(k)} F(t_{\lambda}(2\varepsilon_j \eta_j + \eta_{f_j}^J))$ with socle $F(t_{\lambda}(2\varepsilon_j \eta_j))$ and such that $[W_1 : F(t_{\lambda}(2\varepsilon_j \eta_j))] = 1$. We conclude that $W_1$ is isomorphic to $D_{\lambda + \varepsilon_j \alpha_j, \xi}$. Let $Q$ be the quotient of $D_{\lambda, \xi} \otimes_F F(\alpha_j)$ by $W_1$. Then $Q$ has socle isomorphic
to the direct sum of \( F(t_1(\pi'_f)) \) and \( F(t_1(-2\varepsilon_j \eta_j + \pi'_f)) \). Let \( W_2 \) be the image in \( Q \) of the unique nonzero map \( \text{Proj}_{GL_2(k)} F(t_1(\pi'_f)) \to Q \) and let \( W_3 \overset{\text{def}}{=} Q/W_2 \). Then \( W_3 \) is a quotient of \( \text{Proj}_{GL_2(k)} F(t_1(-2\varepsilon_j \eta_j + \pi'_f)). \)

We claim that \( F(\lambda) \) is in the socle of \( W_2 \). Let's assume it for now. As \( W_2 \) is multiplicity-free, it has a unique quotient with socle \( F(\lambda) \), namely \( W_2 \) has a quotient isomorphic to \( D_{\lambda,\varepsilon}. \)

We can check that the Serre weight \( F(t_1(-2\varepsilon_j \eta_j)) \) is not a subquotient of \( \text{Proj}_{GL_2(k)} F(t_1(\pi'_f)) \) (again, by Lemmas 6.2.1[iv] and 2.4.4) so that \( F(t_1(-2\varepsilon_j \eta_j)) \) is a constituent of the socle of \( W_2 \). As above, we can conclude that \( W_3 \) has a quotient isomorphic to \( D_{\lambda-\varepsilon,\alpha_j,\varepsilon}. \) It follows from length considerations that we must have \( W_2 \cong D_{\lambda,\varepsilon} \) and \( W_3 \cong D_{\lambda-\varepsilon,\alpha_j,\varepsilon}. \)

We still have to prove that \( F(\lambda) \) is contained in the socle of \( W_2 \) or equivalently that \( F(\lambda) \) is a subquotient of \( W_2 \). Assume it is not the case. Let \( W_3 \) be the image in \( D_{\lambda,\varepsilon} \otimes_F F(\alpha_j) \) of the unique nonzero map \( \text{Proj}_{GL_2(k)} F(t_1(\pi'_f)) \to D_{\lambda,\varepsilon} \otimes_F F(\alpha_j). \) Then \( W_3 \) is a quotient of \( W_2 \) and the kernel of \( W_2 \to W_2 \) is contained in \( W_3. \) Thus \( F(\lambda) \) is not a subquotient of \( W_2 \). The socle of \( W_2 \) is contained in the socle of \( D_{\lambda,\varepsilon} \otimes_F F(\alpha_j), \) which itself is equal to \( F(t_1(2\varepsilon_j \eta_j)) \oplus F(\lambda) \oplus F(t_1(-2\varepsilon_j \eta_j)) \) by Lemma 6.3.8. However, \( F(\lambda) \) does not appear in the socle of \( W_2 \) by hypothesis, neither does \( F(t_1(-2\varepsilon_j \eta_j)) \) since it is not a subquotient of \( \text{Proj}_{GL_2(k)} F(t_1(\pi'_f)). \) The socle of \( W_2 \) is then equal to \( F(t_1(2\varepsilon_j \eta_j)). \) By multiplicity-freeness, we have \( W_2 \cong I(F(t_1(2\varepsilon_j \eta_j)), F(t_1(\pi'_f))). \) Consequently \( W_2/F(t_1(2\varepsilon_j \eta_j)) \) contains \( F(\lambda) \) in its socle by Lemma 6.2.1[iii]. This contradicts Lemma 6.3.9. Namely if \( V \) is the unique extension of \( F(t_1(\varepsilon_j \eta_j)) \) by \( F(\lambda), \) then \( V \subseteq D_{\lambda,\varepsilon} \) and \( V \otimes_F F(\alpha_j) \subseteq D_{\lambda,\varepsilon} \otimes_F F(\alpha_j) \) and Lemma 6.3.9 shows that \( F(\lambda) \) is not contained in the socle of \( (V \otimes_F F(\alpha_j))/F(t_1(2\varepsilon_j \eta_j)). \)

The last assertion of the proposition is a consequence of the fact that the representation \( F(t_1(-2\varepsilon_j \eta_j)) \) has no extension with the subquotients of \( D_{\lambda,\varepsilon}, \) which itself is a consequence of Lemma 2.4.6. \( \square \)

**Theorem 6.3.11.** Fix \( \lambda \in X_1(T) \) which is \( 7 \)-deep in \( C_0 \) and \( \varepsilon \in \{ \pm 1 \}^J. \) We set

\[
W_{-\varepsilon} \overset{\text{def}}{=} \{ F(t_\lambda(-\sum_{j \in J} \varepsilon_j \eta_j)) : J \subseteq J \}.
\]

There exists a largest subrepresentation \( W \) of \( (\text{Inj}_{K/Z_1} F(\lambda))[m_{K_1}] \) satisfying \( [W : \tau] = \delta_{F(\lambda),\tau} \) for \( \tau \in W_{-\varepsilon}. \) Moreover it has the following properties:

(i) \( W^{K_1} = D_{\lambda,\varepsilon}; \)

(ii) the representation \( W \) is an extension of \( \bigoplus_{0 \leq i \leq f-1} D_{\lambda+\varepsilon_ia_i,\varepsilon} \) by \( D_{\lambda,\varepsilon}; \)

(iii) the representation \( W \) is multiplicity-free;

(iv) the cosocle of \( W \) is isomorphic to \( \bigoplus_{0 \leq i \leq f-1} F(t_1(2\varepsilon_j \eta_j + \sum_{0 \leq i \leq f-1} \varepsilon_i \eta_i)); \)

(v) its submodule structure is determined by: for \( 0 \leq a_i \leq 3 \) such that \( a_2 = F(t_1(\sum_{i \leq i \leq f-1} a_i \eta_i) \)

is a subquotient of \( W \), the unique subrepresentation of \( W \) with cosocle \( \sigma_2 \) has constituents \( \sigma_b \) for all \( b \) such that \( 0 \leq b_i \leq a_i \) for all \( i. \)

**Remark 6.3.12.** The proof shows that \( \lambda \) only needs to be \( 4 \)-deep in \( C_0 \) for \( W \) to exist and for part [i] to hold. In particular, in this case \( W^{K_1} = D_{\lambda,\varepsilon} \) is the largest subrepresentation of \( (\text{Inj}_{K/Z_1} F(\lambda))[m_{K_1}] = \text{Inj}_{GL_2(k)} F(\lambda) \) satisfying \( [W^{K_1} : \tau] = \delta_{F(\lambda),\tau} \) for \( \tau \in W_{-\varepsilon}. \)
Proof. Let \( I_\lambda \overset{\text{def}}{=} \text{Inj}_{GL_2(\mathbb{k})} F(\lambda) \) and let \( \tilde{I}_\lambda \overset{\text{def}}{=} (\text{Inj}_{K/I_1} F(\lambda))[m_{K_1}^2] \), which is finite-dimensional by dualising and using Nakayama’s lemma. We have \( I_\lambda = \tilde{I}_\lambda [m_{K_1}] \).

The existence of a largest subrepresentation \( W \subseteq \tilde{I}_\lambda \) satisfying the desired hypothesis follows exactly as in [BP12] Prop. 13.1. As the representation \( D_{\lambda, \xi} \) satisfies \( |W : \tau| = \delta_{F(\lambda), \tau} \) for \( \tau \in W_{-\xi} \) by Lemma 6.2.1(iii) we have \( D_{\lambda, \xi} \subseteq W_{K_1} \). Conversely, note that \( W_{K_1} \) is a subrepresentation of \( \tilde{I}_{\lambda} \cong \text{Inj}_{GL_2(\mathbb{k})} F(\lambda) \). As \( |W_{K_1} : F(\lambda)| = 1 \) it follows by [BP12] Prop. 3.6 & Cor. 3.11 that \( W_{K_1} \) is multiplicity-free. By Lemma 6.2.1(iii) and our hypothesis on multiplicities, \( JH(W_{K_1}) \subseteq JH(D_{\lambda, \xi}) \). Hence \( W_{K_1} = D_{\lambda, \xi} \), proving \([i]\).

Consider the short exact sequence:

\[
0 \to D_{\lambda, \xi} \to W \to W/D_{\lambda, \xi} \to 0.
\]

The long exact sequence of \( K_1/Z_1 \)-invariants gives an injection

\[
W/D_{\lambda, \xi} = (W/D_{\lambda, \xi})_{K_1} \to H^1(K_1/Z_1, D_{\lambda, \xi}) \cong D_{\lambda, \xi} \otimes_{\mathbb{F}} H^1(K_1/Z_1, \mathbb{F}),
\]

where the last isomorphism holds because \( K_1 \) acts trivially on \( D_{\lambda, \xi} \). Using the isomorphism

\[
H^1(K_1/Z_1, \mathbb{F}) \cong \bigoplus_{j=0}^{f-1} F(\alpha_j) \quad \text{(see [BP12] Prop. 5.1)},
\]

we have:

\[
W/D_{\lambda, \xi} \to \bigoplus_{j=0}^{f-1} (D_{\lambda, \xi} \otimes_{\mathbb{F}} F(\alpha_j)).
\]

For each \( 0 \leq j \leq f-1 \), we have a decomposition:

\[
0 \to D_{\lambda, \xi}, \xi \to D_{\lambda, \xi} \otimes_{\mathbb{F}} F(\alpha_j) \to Q_j \to 0
\]

with \( \text{soc}_{GL_2(\mathbb{k})} Q_j = F(\lambda) \oplus F(t_1(-2\xi_j \eta_j)) \) by Proposition 6.3.10.

The assumption \( |W : F(\lambda)| = 1 \) implies that

\[
\text{soc}_K(W/D_{\lambda, \xi}) = \text{soc}_K(W/W_{K_1}) \to \bigoplus_i F(t_1(\pm \xi_j \eta_j)).
\]

For \( 0 \leq j \leq f-1 \), Lemma 2.4.6 implies that the representation \( F(t_1(-2\xi_j \eta_j)) \) has no extension with Jordan–Hölder factors of \( D_{\lambda, \xi} \), consequently the Serre weights \( F(t_1(\pm \xi_j \eta_j)) \) are not in the socle of \( W/D_{\lambda, \xi} \). We conclude that the image of \( W/D_{\lambda, \xi} \) in \( Q_j \) is zero and that \( W/D_{\lambda, \xi} \subsetneq \bigoplus_{j=0}^{f-1} D_{\lambda, \xi}, \xi \).

Let \( V \) be the representation of \( K \) constructed in Proposition 6.2.2. Note that the deepness assumption on \( \lambda \) allows us to apply it with \( B_i = 4 \) if \( \varepsilon_i = 1 \) and \( B_i = 3 \) if \( \varepsilon_i = -1 \). Let \( W' = V[m_{K_1}^2] \). By Proposition 6.2.2 we have \( |W' : \tau| = \delta_{F(\lambda), \tau} \) for \( \tau \in W_{-\xi} \) so that \( W' \subseteq W \) by maximality of \( W' \). It follows from Proposition 6.2.4 with \( n = 2 \) and \( n = 1 \) that

\[
\text{cosoc}_K(W') = \bigoplus_{0 \leq j \leq f-1} F(t_1(2\xi_j \eta_j + \sum_i \varepsilon_i \eta_i))
\]

and \( W'_{K_1} = D_{\lambda, \xi} = W_{K_1} \). By what precedes we have an inclusion

\[
W'/W'_{K_1} \subseteq W/W_{K_1} \subseteq \bigoplus_{j=0}^{f-1} D_{\lambda, \xi}, \xi.
\]

However, the outside terms have the same cosocle, so these inclusions are equalities. From \( W'_{K_1} = W'_{K_1} \) and \( W'/W'_{K_1} = W/W_{K_1} \) we deduce that \( W' = W \). This also proves that \( W/D_{\lambda, \xi} \) is
isomorphic to $\bigoplus_{j=0}^{l-1} D_{\lambda+j\alpha_j} \zeta$ and gives (ii). We then deduce properties (iii) to (v) from the properties of $V$ given by Proposition 6.2.2. □

**Corollary 6.3.13.** Let $\overline{\tau} : \overline{G}_L \to \overline{GL}_2(\mathbb{F})$ be a tame Galois representation such that $\overline{\tau}|_{\tilde{\tau}_0} \cong \tau(s,\mu)$ such that $\mu - \eta$ is $8$-deep in $\mathcal{C}_0$.

(i) Let $\tau$ be a finite-dimensional semisimple representation of $K$ over $\mathbb{F}$ of the form $\tau \cong \bigoplus_{\sigma \in W(\overline{\tau})} \sigma^m_\sigma$, with $m_\sigma \geq 1$ for all $\sigma$. Then there exists a largest $K$-subrepresentation $V$ inside $(\text{Inj}_{K/Z_1} \tau)[m^2_{K_1}]$ with $\text{soc}_K V = \tau$ such that for all $\sigma' \in W(\overline{\tau})$,

$$[V : \sigma] = [\tau : \sigma] = m_\sigma.$$

Moreover $V \cong \bigoplus_{\sigma \in W(\overline{\tau})} V^m_\sigma$, where $V_\sigma \subseteq (\text{Inj}_{K/Z_1} \sigma)[m^2_{K_1}]$ is the largest $K$-subrepresentation of $\text{Inj}_{K/Z_1} \sigma$ such that $[V_\sigma : \sigma'] = \delta_{\sigma,\sigma'}$ for all $\sigma' \in W(\overline{\tau})$.

(ii) Fix $\sigma \in W(\overline{\tau})$ and choose $\lambda \in X_1^+(\mathcal{T})$ such that $\sigma \cong F(\lambda)$. There exists $\varepsilon = (\varepsilon_i) \in \{\pm 1\}^J$ such that $W(\overline{\tau}) = \{F(t_{\lambda}(-\sum_{i \in J} \varepsilon_i \pi_i)) : J \subseteq J \}$. Then $V_\sigma$ is multiplicity-free and $V^m_{\lambda_1} \cong D_{\lambda_1} \geq 0$. Moreover the Jordan–Hölder constituents of $V_\sigma$ are the $\sigma_{1,2} = F(t_{\lambda}(\sum \varepsilon_i a_i \pi_i))$, where $a_i \geq 0$ and $\sum_i |a_i/2| \leq 1$, with submodule structure determined as follows: the unique subrepresentation of $V_\sigma$ with cosocle $\sigma_{1,2}$ has constituents $\sigma_{1,2}$ for all $b$ such that $0 \leq b_i \leq a_i$ for all $i$.

(iii) If $\sigma$ and $\sigma'$ are both in $W(\overline{\tau})$ and nonisomorphic, the sets $JH(V_\sigma)$ and $JH(V_{\sigma'})$ are disjoint.

**Remark 6.3.14.** In Corollary 6.3.13(ii) the condition $a_i \geq 0$ and $\sum_i |a_i/2| \leq 1$ means exactly that $a_i \in \{0,1,2,3\}$ and that at most one of them is $\geq 2$.

**Proof.** Part (i) follows by the same argument as in the proof of [BP12] Prop. 13.1. For the existence of $V$ we have to prove that, if $V_1$ and $V_2$ are two subrepresentations of $(\text{Inj}_{K/Z_1} \tau)[m^2_{K_1}]$ such that $\text{Hom}_K(\sigma, V_1) \cong \text{Hom}_K(\text{Proj}_K \sigma, V_1)$ for all $\sigma \in W(\overline{\tau})$, then $V_1 + V_2$ has the same property. This follows from the exactness of the sequence

$$0 \to \text{Hom}_K(\text{Proj}_K[\sigma, V_1 \cap V_2] \to \text{Hom}_K(\text{Proj}_K[\sigma, V_1) \cong \text{Hom}_K(\text{Proj}_K[\sigma, V_2) \to \text{Hom}_K(\text{Proj}_K[\sigma, V_1 + V_2) \to 0.$$

By assumption, we have

$$\dim_{\mathbb{F}} \left( \text{Hom}_K(\text{Proj}_K[\sigma, V_1) \right) = \dim_{\mathbb{F}} \left( \text{Hom}_K(\text{Proj}_K[\sigma, V_1 \cap V_2) \right) = m_\sigma$$

so that

$$\dim_{\mathbb{F}} \left( \text{Hom}_K(\text{Proj}_K[\sigma, V_1 + V_2) \right) = m_\sigma = \dim_{\mathbb{F}} \left( \text{Hom}_K(\sigma, V_1 + V_2) \right).$$

As $\tau \cong \bigoplus_{\sigma \in W(\overline{\tau})} \sigma^m_\sigma$, there is a $K$-equivariant inclusion

$$V \hookrightarrow \bigoplus_{\sigma \in W(\overline{\tau})} (\text{Inj}_{K/Z_1} \sigma)^m_\sigma [m^2_{K_1}]$$

and, by maximality of $V$, we have

$$\bigoplus_{\sigma \in W(\overline{\tau})} V^m_\sigma \subseteq V \subseteq \bigoplus_{\sigma \in W(\overline{\tau})} (\text{Inj}_{K/Z_1} \sigma)^m_\sigma [m^2_{K_1}]$$

By definition of $V_\sigma$, the socle of $(\text{Inj}_{K/Z_1} \sigma)[m^2_{K_1}]/V_\sigma$ contains only Serre weights of $W(\overline{\tau})$. Hence the socle of $V/(\bigoplus_{\sigma \in W(\overline{\tau})} V^m_\sigma)$ has the same property. However it follows from the exactness of
\[
\text{Hom}_K(\text{Proj}_{K/Z_1} \sigma, -) \text{ that we have for all } \sigma \in W(\overline{p})
\]
\[
\text{Hom}_K \left( \text{Proj}_{K/Z_1} \sigma, V/( \bigoplus_{\sigma \in W(\overline{p})} V_{\sigma}^{m_{\sigma}}) \right) = 0,
\]
so that \( \text{soc}_K(V/(\bigoplus_{\sigma \in W(\overline{p})} V_{\sigma}^{m_{\sigma}})) = 0 \) and
\[
V = \bigoplus_{\sigma \in W(\overline{p})} V_{\sigma}^{m_{\sigma}}.
\]

Now we prove part \((ii)\). By Proposition \ref{prop:2.4.2} the elements of \( W(\overline{p}) \) are of the form \( F(t_{\mu, \eta}(s\overline{\eta}_{J'})) \) for \( J' \subseteq J \) and we let \( J \subseteq J \) be such that \( \sigma \cong F(t_\lambda(0)) \cong F(t_{\mu, \eta}(s\overline{\eta}_J)) \). In particular, all elements of \( W(\overline{p}) \) are 7-deep in \( C_0 \) (for example, by Remark \ref{rem:2.4.7}(iv)). By Remark \ref{rem:2.4.7} there exists \( \epsilon = (\epsilon_i) \in \{\pm 1\}^J \) such that \( W(\overline{p}) = \{F(t_\lambda(-\sum_{i \in J} \epsilon_i \overline{\eta}_i)) : J' \subseteq J \} \). The properties of \( V_\sigma \) are then immediate consequences of Theorem \ref{thm:6.3.14}(i), (iii), and (v).

For part \((iii)\) let \( \lambda, \lambda' \in X_1(T) \) be such that \( \sigma \cong F(\lambda), \sigma' \cong F(\lambda') \) and \( \epsilon \) such that
\[
W(\overline{p}) = \{F(t_\lambda(-\sum_{i \in J} \epsilon_i \overline{\eta}_i)) : J \subseteq J \}.
\]

Then
\[
JH(V_\sigma) = \{F(t_\lambda(\sum_i \epsilon_i a_i \overline{\eta}_i)) : a_i \geq 0, \sum_i |a_i/2| \leq 1 \}.
\]
Choose \( J \subseteq J \) such that \( F(\lambda') \cong F(t_\lambda(-\sum_{i \in J} \epsilon_i \overline{\eta}_i)) \). Then by part \((ii)\) and Remark \ref{rem:2.4.7} we see that
\[
JH(V_{\sigma'}) = \{F(t_\lambda(-\sum_{i \in J} \epsilon_i (b_i + 1) \overline{\eta}_i + \sum_{j \not\subset J} \epsilon_i b_i \overline{\eta}_i)) : b_i \geq 0, \sum_i |b_i/2| \leq 1 \}.
\]
(Note that \( W(\overline{p}) \) is obtained by putting \( -1 \leq b_i \leq 0 \).) If \( JH(V_\sigma) \) and \( JH(V_{\sigma'}) \) are not disjoint, then \( J = \emptyset \) (as \( b_j + 1 > 0 \)), contradicting \( \sigma \not\cong \sigma' \).

**Corollary 6.3.15.** Let \( p, m_\sigma \) and \( V \) be as in Corollary \ref{cor:6.3.13}. Then
\[
V[m_\kappa] = \bigoplus_{\sigma \in W(\overline{p})} D_{0, \sigma}(\overline{p})^{m_{\sigma}},
\]
where \( D_{0, \sigma}(\overline{p}) \) is the representation of \( \text{GL}_2(k) \) constructed in \cite{BP12} \S 13.

**Proof.** This follows from Corollary \ref{cor:6.3.13}(i) and (ii) as well as Remark \ref{rem:6.3.12}.

---

### 6.4. Multiplicity one result for the pro-\( p \)-Iwahori

The aim of this subsection is to prove that some multiplicity one assumption on the first two layers of the \( K_1 \)-socle filtration implies a multiplicity one result on the first three layers of the \( I_1 \)-socle filtration of an admissible smooth representation of \( \text{GL}_2(L) \).

**Proposition 6.4.1.** Suppose that \( \chi = \chi_\lambda \) with \( 2 < \langle \lambda, \alpha_i^\vee \rangle < p - 3 \) for all \( 0 \leq i \leq f - 1 \). Let \( W \) be a smooth and finite length representation of \( I \) over \( \mathbb{F} \) satisfying the following conditions:

- both the socle and cosocle of \( W \) are irreducible and isomorphic to \( \chi \);
- we have \( \text{soc}_I(W) \subseteq \text{rad}_I(W) \) and \( \text{rad}_I(W)/\text{soc}_I(W) \) is semisimple; in other words, the Loewy length of \( W \) is equal to 3.
Let $Q$ be a nonzero quotient of $\text{Ind}^K_I W$ such that $[Q : \sigma_\chi] = 1$. Then the composition
\[ \chi = \text{soc}_I(W) \hookrightarrow W \xrightarrow{f} Q|_I \]
is zero, where $f$ is induced by Frobenius reciprocity.

Proof. Assume that $f|_{\text{soc}_I(W)}$ is nonzero, or equivalently $f$ is injective, for a contradiction. Then the image of $\text{Ind}^K_I \text{soc}_I(W) \to Q$ is nonzero and has cosocle $\sigma_\chi$ (recall that $\sigma_\chi$ is the cosocle of $\text{Ind}^K_I \chi$). Since $[Q : \sigma_\chi] = 1$ by assumption, we may replace $Q$ by the image of the unique (up to scalar) nonzero morphism $Q \to \text{Inj}_{K/Z_1} \sigma_\chi$, and therefore assume $\text{soc}_K(Q) \cong \sigma_\chi$. Indeed, letting $Q'$ be this image, we have $[\text{Ker}(Q \to Q') : \sigma_\chi] = 0$. Since $\sigma_\chi$ is a Jordan–Hölder factor of the image of $\text{Ind}^K_I \text{soc}_I(W)$ in $Q$, the map from $\text{Ind}^K_I \text{soc}_I(W)$ to $Q'$ is nonzero and hence the composite $\text{soc}_I(W) \to Q \to Q'$ is nonzero. From now on we suppose that $\text{soc}_K(Q) \cong \sigma_\chi$. Note that, the image of the map
\[ \text{Ind}^K_I \text{soc}_I(W) \to Q \]
is then exactly $\text{soc}_K Q = \sigma_\chi$.

Using Lemma 6.1.1, we deduce that $\text{rad}_I(W)/\text{soc}_I(W)$ is isomorphic to a direct sum of characters of the form $\chi\alpha_i^{-1}$, each appearing at most once. Let $S_+$ (resp. $S_-$) be the set of characters appearing in $\text{rad}_I(W)/\text{soc}_I(W)$ and of the form $\chi\alpha_i$ (resp. $\chi\alpha_i^{-1}$). Also let $W' \subseteq W$ be the subrepresentation defined by
\[ 0 \to \chi \to W' \to \bigoplus_{\chi' \in S_-} \chi' \to 0, \]
and $W'' = W/W'$ so that
\[ 0 \to \bigoplus_{\chi' \in S_+} \chi' \to W'' \to \chi \to 0. \]
Note that both $W'$ and $W''$ are fixed by $K_1$, see Lemma 6.1.1(iii).

We claim that $f(W')$ is contained in $\sigma_\chi$. This is equivalent to showing that the morphism $\text{Ind}^K_I W' \to Q$ (induced from $f$ by Frobenius reciprocity) has image contained in (and hence equal to) $\sigma_\chi$. Let $Q'$ denote the image of $\text{Ind}^K_I W'$. Clearly, $Q'$ is contained in $Q^{K_1}$, which itself is a subrepresentation of $\text{Inj}_{GL_2(k)} \sigma_\chi$. If $\sigma_\chi \subseteq Q'$, then, as $f(\text{soc}_I W) \subseteq \sigma_\chi$, we would obtain a nonzero morphism $\text{Ind}^K_I (W'/\chi) \to Q'/\sigma_\chi \hookrightarrow (\text{Inj}_{GL_2(k)} \sigma_\chi)/\sigma_\chi$. However, one checks that no Jordan–Hölder factors of $\text{Ind}^K_I \chi'$ for $\chi' \in S_-$ can appear in $\text{Inj}_{GL_2(k)} \sigma_\chi$, using Lemma 6.2.1. Hence we have $Q' = \sigma_\chi$.

We obtain a surjective morphism
\[ \text{Ind}^K_I W'' \twoheadrightarrow Q'' \overset{\text{def}}{=} Q/\sigma_\chi. \]
Since $[Q'' : \sigma_\chi] = 0$, Lemma 6.3.5 implies that no Jordan–Hölder factors of $Q''$ have nontrivial extensions with $\sigma_\chi$. However, as $Q$ has irreducible socle $\sigma_\chi$ we obtain a contradiction. \qed

Definition 6.4.2. Let $V$ be a semisimple smooth representation of $I$ over $\mathbb{F}$. We say $V$ is connected if the following condition is satisfied: for any two smooth characters $\chi \neq \chi''$ of $I$ occurring in $V$ such that $\chi'' \in \text{soc}_I(W_{\chi,3})$, there exists a character $\chi'$ occurring in $V$ such that $\text{Ext}^1_{I/Z_1}(\chi', \chi'') \neq 0$ and $\text{Ext}^1_{I/Z_1}(\chi, \chi') \neq 0$.

The motivation of the above definition comes from the following result.
Lemma 6.4.3. Let $\mathfrak{p} : G_L \to \text{GL}_2(\mathbb{F})$ be a 6-generic representation, not necessarily semisimple. Let $D_0(\mathfrak{p})$ be the $\text{GL}_2(k)$-representation constructed in [BP12, §13]. Then $D_1(\mathfrak{p}) \overset{\text{def}}{=} D_0(\mathfrak{p})^{11}$ is connected in the sense of Definition 6.4.2. As a consequence, if $V$ is a semisimple representation of $I$ such that $\text{JH}(V) = \text{JH}(D_1(\mathfrak{p}))$ up to multiplicity, then $V$ is connected.

Proof. We first note the general fact that up to multiplicity

$$\text{JH}(D_0(\mathfrak{p})) = \text{JH} \left( \bigoplus_{\sigma \in \mathcal{W}(\mathfrak{p})} \text{Inj}_{\text{GL}_2(k)} \sigma \right)$$

Indeed, the inclusion “$\subseteq$” is trivial and “$\supseteq$” follows from [BP12] Lemma 12.8, Prop. 13.4]. As a consequence, we have

$$\text{JH}(D_0(\mathfrak{p})) \subseteq \text{JH}(D_0(\mathfrak{p}^{ss})).$$

We write $\mathfrak{p}^{ss}\mid I_L \cong \mathfrak{p}(s, \mu)$ such that $\mu - \eta$ is 6-deep in $\mathcal{C}_n$. As in the proof of Corollary [6.3.13 ii] we know that $\text{W}(\mathfrak{p}^{ss}) = \{ F(r_{\mu-\eta}(\sum_i \varepsilon_i \eta_i)) : J \subseteq J \}$ for some choice of $\varepsilon_i \in \{\pm 1\}$. By using Remarks 6.3.12 and 2.4.7 we see that $\text{JH}(D_0(\mathfrak{p}^{ss})) = \{ F(r_{\mu-\eta}(\sum_i \varepsilon_i a_i \eta_i)) : -1 \leq a_i \leq 2 \}$.

Suppose $\chi$ and $\chi''$ are as in Definition 6.4.2 for $V = D_1(\mathfrak{p})$. By Lemma 6.1.1, $\chi''$ has the form

$$\chi' \alpha_{i_1}^{\pm 1} \alpha_{i_2}^{\pm 1}$$

for some $0 \leq i_1, i_2 \leq f - 1$. Say $\chi = \chi' \chi''$ and $\chi'' = (\chi'')^f$ for some $\sigma, \sigma'' \in \text{JH}(D_0(\mathfrak{p}))$. By the discussion in last paragraph, we may write $\sigma \cong F(t_{\mu-\eta}(\sum \varepsilon_i a_i \eta_i))$ and $\sigma'' \cong F(t_{\mu-\eta}(\sum \varepsilon_i a_i'^{\prime} \eta_i))$ for some $-1 \leq a_i, a_i'' \leq 2$.

First suppose that $i_1 = i_2$. Recalling that $F(\lambda)^{11} = \chi_\lambda$ and $t_{\lambda \pm 2\alpha_i}(\omega) = t_{\lambda}(\omega \pm 4\eta_i)$ we see that

$$\sum_i \varepsilon_i a_i^{''} \eta_i = \sum_i \varepsilon_i a_i \eta_i \pm 4\eta_i$$

for some $-1 \leq a_i, a_i'' \leq 2$; contradiction. (The 6-deepness of $\mu - \eta$ guarantees that we are staying inside $\Lambda^{11 \mu - \eta}$.)

Now suppose $i_1 \neq i_2$. As in the previous case we know that $|a_i - a_i''| = 2$ if $i \in \{i_1, i_2\}$ and $a_i = a_i''$ otherwise. We let $a_i' \overset{\text{def}}{=} a_i$ for $i \neq i_1$, $a_i' \overset{\text{def}}{=} a_i''$, $\sigma' \overset{\text{def}}{=} F(t_{\mu-\eta}(\sum \varepsilon_i a_i \eta_i))$, and $\chi' \overset{\text{def}}{=} (\sigma')^{f_1}$. We claim that $\chi' \in D_1(\mathfrak{p})^{11}$. Equivalently we need to show that the unique principal series with cosocle $\sigma'$ contains an element of $\text{W}(\mathfrak{p})$ as constituent (then the principal series admits a quotient that contains precisely one element of $\text{W}(\mathfrak{p})$ and that as its socle). By Lemma 6.2.1[i] and Remark 2.4.7 the principal series with cosocle $\sigma$ has constituents $F(t_{\mu-\eta}(\sum \varepsilon_i a_i \eta_i))$ for certain signs $\varepsilon_i' \in \{\pm 1\}$. By Remark 2.4.7 the same is true for the principal series with cosocle $\sigma'$ (resp. $\sigma''$), by replacing $a_i$ by $a_i'$ (resp. $a_i''$). The claim follows, since the condition of containing a weight of $\text{W}(\mathfrak{p})$ is checked separately for each embedding. (Use 2.4.2 if $\mathfrak{p}$ is semisimple and [Le19, Prop. 3.2], as well as [LMS, Def. 3.5], otherwise.)

The last assertion immediately follows from the first one, because by definition the connectedness of $V$ depends only on $\text{JH}(V)$ up to multiplicity. \hfill \Box

We now consider an admissible smooth $G$-representation $\pi$ satisfying the following properties:

(a) $\pi|_{\text{GL}_2(K)}|_K$ is isomorphic to a subrepresentation of a direct sum

$$\bigoplus_{\sigma \in \mathcal{W}} \tilde{D}_{\sigma} \oplus m_\sigma$$

for some set of Serre weights $\mathcal{W}$, some $K$-representations $\tilde{D}_{\sigma}$ with $\text{soc}_K \tilde{D}_{\sigma} \cong \sigma$, and some integers $m_\sigma \geq 1$;
(b) the $K$-representation
\[ \tilde{D} \overset{\text{def}}{=} \bigoplus_{\sigma \in \mathcal{W}} \tilde{D}_\sigma \]
is multiplicity-free and for each Jordan–Hölder factor $\sigma'$ of $\tilde{D}$ we have $\chi_{\sigma'} \neq \chi_{\sigma''}$ (equivalently, $1 < \dim \mathbb{F} \sigma' < q$).

In our application below we will have $\mathcal{W} = W(p)$ for some tame mod $p$ Galois representation $\overline{\rho}$. Note that if $\chi \in \tilde{D}^{I_1}$, then Frobenius reciprocity induces a nonzero morphism $\text{Ind}^K_I \chi \to \tilde{D}^{K_1}$. By condition [b], $\text{Ind}^K_I \chi$ has irreducible cosocle $\sigma_\chi$, so there is a unique $\sigma \in \mathcal{W}$ such that $\sigma_\chi$ occurs in $\tilde{D}^{K_1}_\sigma$ (or equivalently, such that $\chi$ occurs in $\tilde{D}^{I_1}_\sigma$). In particular, $\sigma_\chi$ does not occur as a subquotient of $\tilde{D}/\tilde{D}^{K_1}$.

We also note that $\tilde{D}^{I_1}$ is multiplicity-free: for a character $\chi$ of $I$ we have $\text{Hom}_I(\chi, \tilde{D}^{I_1}) \cong \text{Hom}_K(\text{Ind}^K_I \chi, \tilde{D})$. By condition [b] we know that $\chi \neq \chi'$, so $\text{Ind}^K_I \chi$ has an irreducible cosocle.

As moreover $\tilde{D}$ is multiplicity-free, we deduce that $\text{Hom}_K(\text{Ind}^K_I \chi, \tilde{D})$ is one-dimensional.

**Lemma 6.4.4.** Let $\pi$ and $\tilde{D}$ be as above satisfying the conditions [a] [b]. Suppose $\chi \in \pi^{I_1}$ is of the form $\chi_\lambda$ with $2 < \langle \lambda, \alpha_i^- \rangle < p - 3$ for all $0 \leq i \leq f - 1$. Then the natural quotient morphism $W_{\chi,2} \to \chi$ induces an isomorphism
\[ \text{Hom}_I(\chi, \pi) \cong \text{Hom}_I(W_{\chi,2}, \pi). \]

**Proof.** Since $W_{\chi,2}$ is killed by $m_I^{\pm 1}$, any morphism $W_{\chi,2} \to \pi|_I$ has image contained in $\pi|m_I^{\pm 1}] \subseteq \pi|m_I^{\pm 1}]$.

Let $f : W_{\chi,2} \to \pi|_I$ be an $I$-equivariant morphism. For $\sigma \in \mathcal{W}$, consider the map $f_\sigma : W_{\chi,2} \to \tilde{D}^{m_\sigma}_{m_\sigma}|_I$ obtained by composing $f$ with the projection to the corresponding direct factor in condition [a].

Let $\chi'$ be a character in $\text{soc}_I(W_{\chi,2})$. By Lemma 6.1.1, there exists $i \in \mathcal{J}$ such that $\chi' = \chi\alpha_i^{\pm 1}$ and the $\chi'$-isotypic subspace is 1-dimensional.

We first consider the case where $\chi'$ is of the form $\chi_\lambda \alpha_i^{-1}$ for some $i \in \mathcal{J}$. Assume for contradiction that $f$ is nonzero on the (one-dimensional) $\chi'$-isotypic space of $W_{\chi,2}$. Then there exists at least one $\sigma \in \mathcal{W}$ such that $f_\sigma$ is nonzero on the $\chi'$-isotypic subspace of $W_{\chi,2}$.

As a consequence of Lemma 6.3.3 (and Frobenius reciprocity), no character $\psi$ of $\text{soc}_I(W_{\chi,2})$ other than $\chi'$ can occur in $\tilde{D}^{I_1}_\sigma$, otherwise $\sigma$ would be a common irreducible subquotient of both $\text{Ind}_I^K \chi'$ and $\text{Ind}_I^K \psi$. Hence, the map $f_\sigma$ factors through the quotient $E_{\chi',\chi}$ of $W_{\chi,2}$ and induces an embedding $E_{\chi',\chi} \to \tilde{D}^{m_\sigma}_{m_\sigma}|_I$. Let
\[ \tilde{f}_\sigma : \text{Ind}_I^K E_{\chi',\chi} \to \tilde{D}^{m_\sigma}_{m_\sigma} \]
be the induced morphism by Frobenius reciprocity. Lemma 6.3.1 implies that the cosocle of $\text{Ind}_I^K E_{\chi',\chi}$ is equal to that of $\text{Ind}_I^K \chi$, i.e. $\sigma_\chi$, hence so is the cosocle of $\text{Im}(\tilde{f}_\sigma)$. Since $E_{\chi',\chi}$ is not $K_1$-invariant, neither is $\text{Im}(\tilde{f}_\sigma)$ because the morphism $E_{\chi',\chi} \to \text{Im}(\tilde{f}_\sigma)|_I$ is injective. We deduce that $\sigma_\chi$ occurs in $\tilde{D}_\sigma/\tilde{D}^{K_1}_\sigma$. This contradicts [b] as remarked just before this lemma.

We conclude that the map $f$ is zero on all $\chi'$-isotypic subspaces of $W_{\chi,2}$ for $\chi' = \chi\alpha_i^{-1}, i \in \mathcal{J}$.
The general case can be reduced to the above case, using the fact that $\pi$ carries an action of $t \overset{\text{def}}{=} (0, 1, p, 0)$. Namely let $f'$ be the map from $W'_{\lambda, 2}$ (conjugate representation by $t$) to $\pi$ defined by $t \circ f$. As $f$ is $I$-equivariant, the map $f'$ is $I$-equivariant. As $W'_{\lambda, 2} \cong W'_{2, 2}$ and as the $\chi'$-isotypic subspace of $W'_{\lambda, 2}$ coincides with the $\chi'$-isotypic subspace of $W'_{2, 2}$, it follows from the first case that $t \circ f$, and hence $f$, is zero on the $\chi'$-isotypic subspace of $W'_{\lambda, 2}$ for $\chi' = \chi\alpha_i$ with $i \in \mathcal{J}$. As a consequence, $f$ is zero on $\text{soc}_I(W'_{\lambda, 2})$. \hfill $\square$

We will not use the following Corollary of Lemma 6.4.4 but we state it since the result can be useful.

**Corollary 6.4.5.** Let $\pi$ and $\tilde{D}$ be as above satisfying the conditions [a] [b]. Suppose $\chi \in \pi^{f_1}$ is of the form $\chi_\lambda$ with $2 < \langle \lambda, \alpha_i' \rangle < p - 3$ for all $0 \leq i \leq f - 1$. Then for any character $\chi' \in \pi^{f_1}$ such that $\text{Ext}_{I/\mathbb{Z}}^1(\chi, \chi') \neq 0$ there exists no $I$-equivariant embedding

$$E_{\chi', \chi} \hookrightarrow \pi|_I.$$

We now make an additional assumption on $\pi$:

(c) $\pi^{f_1}$ is connected (cf. Definition 6.4.2).

**Proposition 6.4.6.** Let $\pi$ and $\tilde{D}$ be as above satisfying the conditions [a] [b] [c]. Suppose $\chi \in \pi^{f_1}$ is of the form $\chi_\lambda$ with $2 < \langle \lambda, \alpha_i' \rangle < p - 3$ for all $0 \leq i \leq f - 1$. Then the natural quotient morphism $W_{\lambda, 3} \rightarrow \chi$ induces an isomorphism

$$\text{Hom}_I(\chi, \pi) \overset{\sim}{\rightarrow} \text{Hom}_I(W_{\lambda, 3}, \pi).$$

**Proof.** Let $f : W_{\lambda, 3} \rightarrow \pi|_I$ be a nonzero $I$-equivariant morphism. It suffices to prove that $f$ factors through the cosocle $W_{\lambda, 3} \rightarrow \chi$. Let’s assume this is not the case and derive a contradiction. Note that this implies that $f|_{\text{soc}_I(W_{\lambda, 3})}$ is nonzero by Lemma 6.4.4.

Step 1. We first show that $f$ is zero when restricted to $X'' \overset{\text{def}}{=} \oplus \chi''$, where the direct sum is taken over all characters $\chi''$ in $\text{soc}_I(W_{\lambda, 3})$ which are different from $\chi$ (recall that $[W_{\lambda, 3} : \chi''] = 1$ for such a $\chi''$). Indeed, if there exists such a $\chi''$ such that $f$ is nonzero when restricted to $\chi''$, then in particular $\chi'' \in \pi^{f_1}$. Since $\pi^{f_1}$ is assumed to be connected by [c], we can find $\chi' \in \pi^{f_1}$ as in Definition 6.4.2. By construction, $\chi'$ occurs in the second layer of the socle filtration of $W_{\lambda, 3}$ and Lemma 6.1.2 shows that $\chi''$ occurs in the socle of the image of any nonzero morphism

$$W_{\chi', 2} \rightarrow W_{\chi, 3}.$$  

But, the composition $W_{\chi', 2} \rightarrow W_{\chi, 3} \rightarrow \pi$ gives a morphism that does not factor through its cosocle $\chi'$, which contradicts Lemma 6.4.4. As a consequence, $f$ factors through the quotient $W_{\lambda, 3}/X''$. Note that $W_{\lambda, 3}/X''$ is killed by $m_{K_1}^2$, because we may define a suitable subrepresentation $W'$ of $W_{\lambda, 3}/X''$, with quotient $W''$, such that both $W'$ and $W''$ are killed by $m_{K_1}$ (cf. the proof of Proposition 6.4.1). Hence, $\text{Im}(f)$ is contained in $\pi[m_{K_1}^2]$.

Step 2. Since $f|_{\text{soc}_I(W_{\lambda, 3})}$ is nonzero, combining with Step 1, we deduce that $\chi$ occurs in the socle of $\text{Im}(f)$. By [a] $\pi[m_{K_1}^2] \subseteq \bigoplus_{\sigma \in \mathbb{W}} \tilde{D}_\sigma m_{\sigma}$, so there exists a projection $pr : \bigoplus_{\sigma \in \mathbb{W}} \tilde{D}_\sigma m_{\sigma} \rightarrow \tilde{D}_\sigma$
such that \( \text{pr} \circ f \) remains nonzero on the \( \chi \)-isotypic part of \( \text{soc}_I(W_{\chi,3}) \). By Frobenius reciprocity \( \sigma_\chi \) occurs as a subquotient in \( \tilde{D}_\sigma[m_{K_1}] \). Consider the composite morphism

\[
f_\sigma : W_{\chi,3} \xrightarrow{f} \pi[m_{K_1}] \xrightarrow{\text{pr}} \tilde{D}_\sigma[I].
\]

Let \( W \overset{\text{def}}{=} \text{Im}(f_\sigma) \) and \( Q \) be the image of the induced morphism \( \text{Ind}_I^K W_{\chi,3} \to \tilde{D}_\sigma \). By Lemma 6.3.3 any \( \chi' \) with \( \text{Ext}^1_{/Z_1}(\chi, \chi') \neq 0 \) cannot occur in \( \tilde{D}_\sigma^{I_1} \), otherwise \( \sigma \) would be a common Jordan–Hölder factor of both \( \text{Ind}_I^K \chi \) and \( \text{Ind}_I^K \chi' \). Combining with Step 1, we deduce that \( \text{soc}_I(W) \) is \( \chi \)-isotypic (being a subrepresentation of \( \tilde{D}_\sigma[I_1] \)). Since \( \tilde{D}_\sigma[I_1] \) is multiplicity-free by (b) (as observed above), we must have \( \text{soc}_I(W) = \chi \). Since \([Q : \sigma_\chi] = 1 \) (as \( \tilde{D}_\sigma \) is multiplicity-free by (b)), Proposition 6.4.1 provides the desired contradiction.

We can now prove the main theorem of this section. Let \( \overline{\rho} : G_L \to \text{GL}_2(\mathbb{F}) \) be a tame Galois representation such that \( \overline{\rho}|_{I_L} \cong \tau(s, \mu) \) (cf. Definition 2.3.1) with \( \mu - \eta \) being 8-deep in \( \mathbb{C}_0 \) (§2.1).

**Theorem 6.4.7.** Let \( \pi \) be an admissible smooth \( \text{GL}_2(L) \)-representation over \( \mathbb{F} \) with a central character. Assume that:

1. we have \( \text{JH}(\text{soc}_K(\pi)) = W(\overline{\rho}) \) (up to multiplicity);
2. for all \( \sigma \in W(\overline{\rho}) \), we have \([\pi[m_{K_1}] : \sigma] = [\text{soc}_K(\pi) : \sigma] \);
3. we have \( \text{JH}(\pi^{I_1}) = \text{JH}(\text{D}_1(\overline{\rho})) \) (up to multiplicity).

Then \( \dim_{\text{GL}_2(L)}(\pi) \leq f \).

**Proof.** As \( \pi \) has a central character, the group \( Z_1 \) acts trivially on \( \pi \). Therefore, by Corollary 6.3.13 Corollary 6.3.15 and Lemma 6.4.3 the representation \( \pi \) satisfies hypotheses (a), (b), (c) above. Then Proposition 6.4.6 shows that \( \text{Hom}_I(\chi, \pi) \cong \text{Hom}_I(W_{\chi,3}, \pi) \) for all characters \( \chi \) occurring in \( \pi^{I_1} \). We can then apply Corollary 5.3.5 to conclude that \( \dim_I(\pi|_I) \leq f \) and thus that \( \dim_{\text{GL}_2(L)}(\pi) \leq f \) (since \( I \) is open in \( \text{GL}_2(L) \)). \( \square \)
In this section we construct a GL₂(𝓞_L)-stable lattice with simple cosocle in some particular locally algebraic representation of GL₂(L).

We keep the notation of section 6. Hence, L is a finite unramified extension of ℚ_p of degree f, ring of integers 𝓞_L, residue field k. Recall that we have set \( K \overset{\text{def}}{=} \text{GL}_2(𝓞_L), K_1 \overset{\text{def}}{=} 1 + pM_2(𝓞_L) \) and \( Z_1 \overset{\text{def}}{=} Z(𝓞_L) \cap K_1 \).

Let \( \sigma \) be a Serre weight for \( G_0 \times _{Z_p} \mathbb{F}_p \). We write \( P_\sigma \overset{\text{def}}{=} \text{Proj}[\text{GL}_2(k)]\sigma \) for the projective envelope of \( \sigma \) in the category of \( \mathbb{F}[\text{GL}_2(k)] \)-modules and we let \( \tilde{P}_\sigma \) be the projective \( \mathcal{O}[\text{GL}_2(k)] \)-module lifting \( P_\sigma \). Then \( \tilde{P}_\sigma \otimes \mathcal{O} E \) is a (semisimple) finite-dimenisonal representation of \( \text{GL}_2(k) \) over \( E \). By inflation, we view it as \( K \)-representation on which the subgroup \( K_1 \) acts trivially.

The space \( \mathfrak{sl}_{2,L} \) of \( 2 \times 2 \) matrices of trace zero with coefficients in \( L \) is endowed with the adjoint action of \( \text{GL}_2/L \), which is isomorphic to \( V(\alpha)/L \cong \text{Sym}^2(L^2) \otimes \det^{-1} \). In particular it has an action of \( K \). The goal of this section is to show the existence of a \( K \)-stable lattice \( V^o \) in the locally \( \mathbb{Q}_p \)-algebraic representation \( \mathfrak{sl}_{2,L} \otimes \mathbb{Q}_p \tilde{P}_\sigma \) such that \( (V^o/pV^o)_{K_1} \) is isomorphic to \( P_\sigma \) (and hence such that \( \sigma \) is the \( K \)-cosocle of \( V^o \)) under some mild genericity assumption on \( \sigma \).

As \( \tilde{P}_\sigma \) is defined over \( W(F) \), and since \( \text{Hom}_{\mathbb{Q}_p\text{-alg}}(L,W(F)[1/p]) \) has \([L : \mathbb{Q}_p]\) elements, we may assume that \( E \) is unramified over \( \mathbb{Q}_p \).

Throughout this section, \( E \) is assumed to be unramified over \( \mathbb{Q}_p \). We recall that, as before, we assume \( p > 2 \).

7.1. **Locally algebraic lattices.** Let \( V^o \) be some \( K \)-stable \( \mathcal{O} \)-lattice in some continuous finite-dimensional representation \( (V, \rho) \) of \( K/Z_1 \) over \( E \). We assume that the group \( K_1 \) acts trivially on \( V^o/pV^o \).

As \( p > 2 \), the map \( x \mapsto \exp(px) \) induces a bijection \( \mathfrak{sl}_{2,O_L} \overset{\sim}{\rightarrow} K_1/Z_1 \) (note that since \( p > 2 \), the map \( K_1 \cap \text{SL}_2(L) \rightarrow K_1/Z_1 \) is an isomorphism) and a group isomorphism

\[
\mathfrak{sl}_{2,O_L}/p\mathfrak{sl}_{2,O_L} \overset{\sim}{\rightarrow} (K_1/Z_1)/(K_1/Z_1)^p.
\] (See [Laz65, III.1.1.4, III.1.1.5, III.1.1.8].)

By assumption, we have \( \rho(k) \in \text{Id}_{V^o} + p\text{End}_\mathcal{O}(V^o) \) for \( k \in K \). For \( x \in \mathfrak{sl}_{2,k} \) and \( v \in V^o/pV^o \), we choose lifts \( \tilde{x} \in \mathfrak{sl}_{2,O_L} \) of \( x \) and \( \tilde{v} \in V^o \) of \( v \) and we define:

\[
\beta_{V^o}(x,v) \overset{\text{def}}{=} p^{-1}(\rho(\exp(px))\tilde{v} - \tilde{v}) \mod pV^o.
\]

Note that \( \beta_{V^o}(x,v) \) does not depend on the choices of \( \tilde{x} \) and \( \tilde{v} \) and is \( \mathbb{F}_p \)-linear in \( x \) and \( \mathbb{F} \)-linear in \( v \). The independence and linearity in \( x \) is a consequence of (49) and of the fact that if \( g \in K_1 \), we have \( [g^p] - 1 \in \mathfrak{m}_{K_1}^2 \) in \( \mathbb{F}[K_1] \).

Therefore there exists a unique \( \mathbb{F} \)-linear map

\[
\beta_{V^o} : \mathfrak{sl}_{2,k} \otimes \mathbb{F}_p (V^o/pV^o) \rightarrow V^o/pV^o
\]
such that $\beta_{V^o}(x \otimes v) = \beta'_{V^o}(x, v)$ for $x \in \mathfrak{sl}_2, k$ and $v \in V^o/pV^o$. (Alternatively, one can verify that the natural Lie algebra action of $\mathfrak{sl}_2, O_L$ on $V$ preserves $V^o$ and gives rise to $\beta_{V^o}$ upon reduction modulo $p$.)

The map $\beta_{V^o}$ measures the defect of exactness of the functor $(-)_{K_1}$ on finite quotients of $V^o$. It is a particular case of a Bockstein homomorphism in some homology long exact sequence. More precisely, we have the following lemma.

**Lemma 7.1.1.** The following sequence is exact:

$$\mathfrak{sl}_2, k \otimes_{\mathbb{F}_p} (V^o/pV^o) \xrightarrow{\beta_{V^o}} V^o/pV^o \xrightarrow{p} (V^o/p^2V^o)_{K_1} \rightarrow V^o/pV^o \rightarrow 0,$$

where the last map is the reduction mod $p$ (recall that $(V^o/pV^o)_{K_1} = V^o/pV^o$).

**Proof.** As the functor of $K_1$-coinvariants is right exact and since $(V^o/pV^o)_{K_1} = V^o/pV^o$, it is sufficient to check that the kernel of the second map coincides with the image of $\beta_{V^o}$.

Let $x \in \mathfrak{sl}_2, k$ and $v \in V^o/pV^o$ and choose $\tilde{x} \in \mathfrak{sl}_2, O_L$ and $\tilde{v} \in V^o$ lifting $x$ and $v$. By definition we have:

$$p\beta_{V^o}(x \otimes v) = \rho(\exp(p\tilde{x}))\tilde{v} - \tilde{v} \mod p^2V^o \in \ker((V^o/p^2V^o) \rightarrow (V^o/p^2V^o)_{K_1}).$$

This implies that the composite $p\beta_{V^o}$ is zero.

Conversely let $v \in V^o/pV^o$ be such that $pv$ is zero in $(V^o/p^2V^o)_{K_1}$. This implies that there exist $k_1, \ldots, k_r, \tilde{v}_1, \ldots, \tilde{v}_r$ in $V^o$ such that

$$pv = \sum_{i=1}^r (\rho(k_i) - 1)\tilde{v}_i \mod p^2V^o.$$

Then there exist $\tilde{x}_1, \ldots, \tilde{x}_r$ in $\mathfrak{sl}_2, O_L$ such that $k_i = \exp(p\tilde{x}_i)$ and we have $\beta_{V^o}(\sum_i x_i \otimes v_i) = v$ in $V^o/pV^o$, where $x_i \in \mathfrak{sl}_2, k$, $v_i \in V^o/pV^o$ are the images of $\tilde{x}_i, \tilde{v}_i$. \[\square\]

Recall that the group $K$ acts by the adjoint action on $\mathfrak{sl}_2, L$ and induces a $\mathbb{Q}_p$-algebraic $E$-linear representation of $K$ on $\mathfrak{sl}_2, L \otimes_{\mathbb{Q}_p} E$. There is a decomposition

$$\mathfrak{sl}_2, L \otimes_{\mathbb{Q}_p} E \cong \bigoplus_{i=0}^{f-1} \mathfrak{sl}_2, E,$$

where $K$ acts on the $i$-th summand by the adjoint action via the embedding $K \hookrightarrow \text{GL}_2(E)$ given by $\sigma_i : L \hookrightarrow E$ on the coefficients. The sub-$O$-module $\mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}_p} O$ is a $K$-stable lattice and the action of $K$ on $(\mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}_p} O)/p(\mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}_p} O) \cong \mathfrak{sl}_2, k \otimes_{\mathbb{F}_p} \mathbb{F}$ factors through $\text{GL}_2(k)$ so that $K_1$ acts trivially on this quotient.

Now we compute $\beta_{V^o}$ in the case where $V^o$ is the lattice $\mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}_p} O$ in the locally algebraic representation $\mathfrak{sl}_2, L \otimes_{\mathbb{Q}_p} E$.

**Lemma 7.1.2.** Assume that $V^o = \mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}_p} O$. Then $V^o/pV^o \cong \mathfrak{sl}_2, k \otimes_{\mathbb{F}_p} \mathbb{F}$ and the map $\beta_{V^o}$ is given explicitly by

$$\beta_{V^o}(x \otimes y \otimes z) = [x, y] \otimes z$$

for $x, y \in \mathfrak{sl}_2, k$ and $z \in \mathbb{F}$. 
Proof. Let \( \tilde{x} \) and \( \tilde{y} \) in \( \mathfrak{sl}_2, O_L \) lifting \( x \) and \( y \). We have:

\[
\exp(p\tilde{x})\tilde{y}\exp(p\tilde{x})^{-1} - \tilde{y} \equiv p\tilde{x}\tilde{y} - p\tilde{y}\tilde{x} \pmod{p^2\mathfrak{sl}_2, O_L}
\]

so that \( \beta_{\mathfrak{sl}_2, O_L \otimes_{\mathbb{Z}} \mathbb{O}}(x \otimes y \otimes 1) = [x, y] \) and we conclude by \( \mathbb{F} \)-linearity.

\[\square\]

**Remark 7.1.3.** By construction of \( \beta_{V^p} \) we can check that \( \beta_{V^p \otimes V^p} = \beta_{V^p} \oplus \beta_{V^p} \) and, if \( W^p \) is another lattice on which \( K_1 \) acts trivially, \( \beta_{V^p \otimes_{\mathbb{O}} W^p} = \beta_{V^p} \otimes \text{Id}_{W^p / pW^p} \).

We leave to the reader the task to verify the following lemma along the lines of the proof of Lemma 7.1.1.

**Lemma 7.1.4.** Let \( W \subseteq V^p / pV^p \) be a sub-\( \mathbb{F} \)-vector space stable under \( K \) and let \( V^p_1 \subseteq V^p \) be the inverse image of \( W \) in \( V^p \). We have a commutative diagram with exact lines:

\[
\begin{array}{ccccccccc}
\mathfrak{sl}_2 \otimes_{\mathbb{F}_p} W & \xrightarrow{\beta_{V^p \otimes \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} W}} & V^p / pV^p & \xrightarrow{p} & (V^p_1 / pV^p)_{K_1} & \xrightarrow{} & W & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{sl}_2 \otimes_{\mathbb{F}_p} (V^p / pV^p) & \xrightarrow{\beta_{V^p}} & V^p / pV^p & \xrightarrow{p} & (V^p_1 / pV^p)_{K_1} & \xrightarrow{} & V^p / pV^p & \xrightarrow{} & 0.
\end{array}
\]

### 7.2. Preliminary computations

In this technical subsection, we make some explicit computations with \( \mathfrak{sl}_2, \mathbb{F} \)-representations and deduce that a certain endomorphism of a direct sum of Serre weights is actually an automorphism.

If \( G \) is an algebraic group over \( \mathbb{F} \), we use the notion of \( G \)-module \( M \) as defined in [Jan03 I.2.7]. Such an object has an underlying structure of an \( \mathbb{F} \)-vector space. It has moreover a natural structure of a module over the Lie algebra \( \text{Lie}(G) \) such that the structure map \( \text{Lie}(G) \otimes M \to M \) is a morphism of \( G \)-modules, where \( \text{Lie}(G) \) is considered as a \( G \)-module for the adjoint action (Jan03 I.7.11 & I.7.18(1))

Given \( \lambda \in X^*(T) \) (resp. \( \lambda \in X^*(T) \)), as in §2.2 we let \( L(\lambda) / \mathbb{F} \) be the irreducible algebraic representation of \( GL_2, \mathbb{F} \) (resp. of \( G \)) of highest weight \( \lambda \). We write \( L(\lambda) \) instead of \( L(\lambda) / \mathbb{F} \) in order not to overload notation.

If \( \lambda = (\lambda_i)_{0 \leq i \leq f - 1} \) with \( \lambda_i \in X_1(T) \), we have

\[
L(\lambda) \cong \bigotimes_{i=0}^{f-1} L(\lambda_i)^{(i)},
\]

where \( L(\lambda_i)^{(i)} \) is the inflation of the \( GL_2, \mathbb{F} \)-module \( L(\lambda_i) \) to \( G \) via the map \( G \cong \prod_{\mathbb{F}} GL_2 \xrightarrow{\pi} GL_2 \) corresponding to the \( i \)-th projection.

Moreover \( L(\lambda) \) inherits an action of the group \( G(\mathbb{F}) = GL_2(k \otimes_{\mathbb{F}} \mathbb{F}) \) and \( F(\lambda) = L(\lambda)|_{GL_2(k)} \) via the inclusion \( GL_2(k) \hookrightarrow G(\mathbb{F}) = \text{GL}_2(k \otimes_{\mathbb{F}} \mathbb{F}) \) corresponding to the ring homomorphism \( k \to k \otimes_{\mathbb{F}} \mathbb{F}, a \to a \otimes 1 \) (see §2.2).

We fix the following \( \mathbb{F} \)-basis \((e, h, f)\) of \( \mathfrak{sl}_2, \mathbb{F} \):

\[
e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Recall that the space \( \mathfrak{sl}_2, \mathbb{F} \) is a \( GL_2, \mathbb{F} \)-module for the adjoint action and if \( p > 2 \) we have \( \alpha \in X_1(T) \) and \( \mathfrak{sl}_2, \mathbb{F} \) is isomorphic to \( L(\alpha) \).
Let $\lambda \in X_1(T)$. We recall that $L(\lambda)$ has a structure of $\mathfrak{sl}_2\mathbb{F}$-module. Let $v_\lambda$ be a highest weight vector of $L(\lambda)$. Then the $\mathbb{F}$-vector space $L(\lambda)$ has a basis given by $(f^i v_\lambda)_{0 \leq i \leq r}$ with $r \overset{\text{def}}{=} \langle \lambda, \alpha^\vee \rangle$ and the action of $GL_2(\mathbb{F})$ is given, for $v \in L(\lambda)$, by

$$
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix} v = \sum_{n \geq 0} a^n \frac{e^n}{n!} v, \quad \begin{pmatrix}
1 & 0 \\
-1 & 1
\end{pmatrix} v = \sum_{n \geq 0} a^n \frac{f^n}{n!} v.
$$

(See [Jan03, II.1.19(6)] and note that here the sum over $0 \leq n \leq p - 1$ suffices.)

Assume from now on that $\lambda$ is 2-deep in the lowest alcove, i.e. $2 \leq r \leq p - 4$. Then we have an isomorphism of $GL_2/\mathbb{F}$-modules (see [Hum89, Lemma]):

$$
\mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda) \cong L(\alpha) \otimes \mathbb{F} L(\lambda) \cong L(\lambda) \oplus L(\lambda + \alpha) \oplus L(\lambda - \alpha),
$$

noting that the weights $\lambda + \alpha$ and $\lambda - \alpha$ are $p$-restricted. We note that the vector $2(e \otimes f v_\lambda) + r(h \otimes v_\lambda)$ is annihilated by $e$ and is a weight vector of weight $\lambda$, it therefore generates the submodule isomorphic to $L(\lambda)$ in $\mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda)$. The vector $e \otimes v_\lambda$ (resp. $e \otimes f^2 v_\lambda + (r - 1) h \otimes f v_\lambda - r(r - 1) f \otimes v_\lambda$) is annihilated by $e$ and is a weight vector of weight $\lambda + \alpha$ (resp. $\lambda - \alpha$) and generates the submodule isomorphic to $L(\lambda + \alpha)$ (resp. $L(\lambda - \alpha)$).

We denote by $d_\lambda$ the unique map of $GL_2/\mathbb{F}$-modules $L(\lambda) \hookrightarrow \mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda)$ sending $v_\lambda$ onto $2(e \otimes f v_\lambda) + r(h \otimes v_\lambda)$. Note that this is the unique (up to scalar) nonzero map between these $GL_2/\mathbb{F}$-modules.

**Lemma 7.2.1.** The composite map of $GL_2/\mathbb{F}$-modules

$$
\psi_\lambda : \mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda) \xrightarrow{\text{Id}_{\mathfrak{sl}_2} \otimes d_\lambda} \mathfrak{sl}_2 \otimes \mathbb{F} \mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda) \xrightarrow{[-,-] \otimes \text{Id}_{L(\lambda)}} \mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda)
$$

is an isomorphism.

**Proof.** As both sides have the same dimension, it is sufficient to prove that this map is injective. As a $GL_2/\mathbb{F}$-module, $\mathfrak{sl}_2 \otimes \mathbb{F} L(\lambda)$ is a direct sum of simple modules by ([50]), it is therefore sufficient to prove that the map $\psi_\lambda$ is nonzero on some well chosen vector of each direct summand. We will check this for each of these modules.

The submodule isomorphic to $L(\lambda + \alpha)$ contains the vector $e \otimes v_\lambda$. We have

$$
\psi_\lambda(e \otimes v_\lambda) = ([-, -] \otimes \text{Id}_{L(\lambda)})(e \otimes (2(e \otimes f v_\lambda) + r(h \otimes v_\lambda)))
= 2[e, e] \otimes f v_\lambda + r[e, h] \otimes v_\lambda
= -2re \otimes v_\lambda \neq 0
$$

since $2r \neq 0$ in $\mathbb{F}$.

The submodule isomorphic to $L(\lambda)$ contains the vector $d_\lambda(v_\lambda) = 2(e \otimes f v_\lambda) + r(h \otimes v_\lambda)$. Note that

$$
d_\lambda(f v_\lambda) = f(2e \otimes f v_\lambda + rh \otimes v_\lambda)
= 2[f, e] \otimes f v_\lambda + 2e \otimes f^2 v_\lambda + r[f, h] \otimes v_\lambda + rh \otimes f v_\lambda
= -2h \otimes f v_\lambda + 2e \otimes f^2 v_\lambda + 2rf \otimes v_\lambda + rh \otimes f v_\lambda
= 2e \otimes f^2 v_\lambda + (r - 2)h \otimes f v_\lambda + 2rf \otimes v_\lambda.
$$
We have
\[ \psi_{\lambda}(d_{\lambda}(v_{\lambda})) = ((-,-) \otimes \text{Id}_{L(\lambda)})(2e \otimes d_{\lambda}(f_{\lambda}v_{\lambda}) + rh \otimes d_{\lambda}(v_{\lambda})) \]
\[ = 4[e, e] \otimes f^2v_{\lambda} + 2(r - 2)[e, h] \otimes f_{\lambda}v_{\lambda} + 4r[e, f] \otimes v_{\lambda} = 4r[h, e] \otimes f_{\lambda}v_{\lambda} + r^2[h, h] \otimes v_{\lambda} \]
\[ = -4(r - 2)e \otimes f_{\lambda}v_{\lambda} + 4rh \otimes v_{\lambda} + 4re \otimes f_{\lambda}v_{\lambda} \]
\[ = 8e \otimes f_{\lambda}v_{\lambda} + 4rh \otimes v_{\lambda} \neq 0 \]
since, for example, 8 \neq 0 in \mathbb{F}.

The submodule isomorphic to \( L(\lambda - \alpha) \) contains the vector \( e \otimes f^2v_{\lambda} + (r - 1)h \otimes f_{\lambda}v_{\lambda} - (r - 1)f \otimes v_{\lambda} \).

We first check that
\[ d_{\lambda}(f^2v_{\lambda}) = 2e \otimes f^2v_{\lambda} + (r - 4)h \otimes f^2v_{\lambda} + 4(r - 1)f \otimes f_{\lambda}v_{\lambda}. \]
Then we have
\[ \psi_{\lambda}(e \otimes f^2v_{\lambda} + (r - 1)h \otimes f_{\lambda}v_{\lambda} - (r - 1)f \otimes v_{\lambda}) \]
\[ = 2(r + 2)e \otimes f^2v_{\lambda} + 2(r - 1)(r + 2)h \otimes f_{\lambda}v_{\lambda} - 2r(r - 1)(r + 2)f \otimes v_{\lambda} \]
and this is nonzero, since \( 2 \leq r \leq p - 4 \). This proves the lemma.

Let \( \sigma \) be a Serre weight for \( G_0 \times \mathbb{F}_p \). It is an absolutely irreducible representation of \( G_0(\mathbb{F}_p) = \text{GL}_2(k) \). There exists a \( p \)-restricted weight \( \lambda \in X_1(\mathbb{T}) \) such that \( \sigma \cong F(\lambda) = L(\lambda) \mid_{\text{GL}_2(k)} \cong \bigotimes_{i=0}^{f-1} L(\lambda_i) \) (see 2.2).

Assume from now on that \( \lambda \) is 2-deep in \( G_0 \). Then the weights \( \lambda, \lambda \pm \alpha_i \) are \( p \)-restricted, hence we have an isomorphism of \( \text{GL}_2(k) \)-representations
\[ \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, f(\lambda) \cong F(\lambda) \oplus F(\lambda + \alpha_i) \oplus F(\lambda - \alpha_i), \]
where the summands on the right-hand side are irreducible and pairwise nonisomorphic. For each \( i \), we choose a nonzero map \( d_{\sigma,i} \in \text{Hom}_{G_0(\mathbb{F}_p)}(\sigma, \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma) \). By comparing with (50) it follows that the map \( d_{\sigma,i} \) is a nonzero multiple of the map \( \text{Id}_{\bigotimes_{j \neq i} L(\lambda_j)} \otimes d_{\lambda_i}^{(i)} \), and we define
\[ d_{\sigma} \overset{\text{def}}{=} (d_{\sigma,i}) \]
which is a \( \text{GL}_2(k) \)-equivariant map from \( \sigma \) to \( \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma \cong \bigoplus_i (\mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma) \). (Note that \( \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma \) is isomorphic to the \( \text{GL}_2(k) \)-restriction of \( (\mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} L(\lambda_i)) \otimes _{j \neq i} L(\lambda_j) \) or, equivalently, of \( L(\alpha_i) \otimes _{\mathbb{F}_p} L(\lambda) \).)

**Proposition 7.2.2.** Assume that \( \lambda \) is 2-deep in \( G_0 \). Then the map of \( \text{GL}_2(k) \)-representations
\[ \Psi : \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma \xrightarrow{\text{Id}_{\mathfrak{sl}_{2,k}} \otimes d_{\sigma}} \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma \xrightarrow{[-,-] \otimes \text{Id}_{\sigma}} \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma \]
is an isomorphism.

**Proof.** As the map \( [-,-] \) is \( k \)-bilinear, the map \( [-,-] \otimes \text{Id}_{\sigma} \) factors through
\[ \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma \to \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \mathfrak{sl}_{2,k} \otimes _{\mathbb{F}_p} \sigma. \]
Therefore, the map \( \Psi \) is the direct sum of the maps \( \Psi_i \), where \( \Psi_i \) is the \( \mathbb{F} \)-linear composite map
\[ \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma \xrightarrow{\text{Id}_{\mathfrak{sl}_{2,k}} \otimes d_{\sigma,i}} \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma \xrightarrow{[-,-] \otimes \text{Id}_{\sigma}} \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \sigma. \]
First of all we remark that all the modules involved in the statement are actually restrictions to \( \text{GL}_2(k) \) of \( G \)-modules. Namely, \( \sigma = L(\lambda) \mid_{\text{GL}_2(k)} \) and the action of \( \text{GL}_2(k) \) on \( \mathfrak{sl}_{2,k} \otimes _{k, \sigma}, \mathbb{F} \) is the
restriction to \( \text{GL}_2(k) \) of the action of \( G \) on \( \mathfrak{sl}_2^k \). Moreover the maps \( d_{\sigma,i} \) and \( [-,-] \) are maps of \( G \)-modules. As \( L(\lambda) \cong \bigotimes \, L(\lambda_j)^{i_j} \) and \( \mathfrak{sl}_2 \otimes_{k,\sigma} \cong \bigotimes_{j \neq i} \, L(\lambda_j)^{j_j} \otimes \mathfrak{g}(\sigma_i \mathfrak{g}) \), we have \( \Psi_i = \bigotimes \, \psi_{i,j} \), where \( \psi_{i,j} \) is the identity of \( L(\lambda_j) \) when \( j \neq i \) and \( \psi_{i,i} \) is a nonzero scalar multiple of the endomorphism \( \psi_{\lambda_i} \) (where \( \psi_{\lambda_i} \) is defined in Lemma 7.2.1). By Lemma 7.2.1 the map \( \Psi_i \) is an isomorphism, hence so is \( \Psi \).

### 7.3. Construction of the lattice

Let \( \sigma \) be a Serre weight. We recall that we denote by \( P_\sigma \) the projective envelope of \( \sigma \) in the category of \( F[\text{GL}_2(k)] \)-modules and \( \tilde{P}_\sigma \) the projective \( \mathcal{O}[\text{GL}_2(k)] \)-module lifting \( P_\sigma \). Then \( \tilde{P}_\sigma \mathcal{O} E \) is a (semisimple) finite-dimensional representation of \( \text{GL}_2(k) \) over \( E \). By inflation, we view it as a \( K \)-representation on which the subgroup \( K_1 \) acts trivially.

We set \( R_1 = \tilde{P}_\sigma \) and we recall that we have the \( \mathbb{Q}_p \)-algebraic action of the group \( K \) on \( \mathfrak{sl}_2 \otimes_{\mathbb{Q}_p} E \) by the adjoint action. The sub-\( \mathcal{O} \)-module \( R_2 \) is a \( K \)-stable lattice such that \( K_1 \) acts trivially on \( R_2 / pR_2 \). As the group \( K_1 \) acts trivially on \( \tilde{P}_\sigma \), Remark 7.1.3 implies that \( \beta_{R_2} = \beta_{\mathfrak{sl}_2 \mathcal{O}_E \otimes \mathfrak{g}} \). From Lemma 7.1.2 we deduce that

\[
\beta_{R_2} = [\cdot,\cdot] \otimes \text{Id}_{P_\sigma} : \mathfrak{sl}_2 \otimes_{p} \mathfrak{g} \mathfrak{sl}_2 \otimes_{p} P_\sigma \to \mathfrak{sl}_2 \otimes_{p} P_\sigma.
\]

Let \( R_{2,i} = \mathfrak{sl}_2 \mathcal{O}_E \otimes \mathfrak{g} \mathfrak{sl}_2 \), \( \tilde{P}_\sigma \) so that \( R_2 = \bigoplus \, R_{2,i} \). Let \( \lambda \in X_1(T) \) be such that \( \sigma \cong F(\lambda) \) and assume that \( \lambda \in X_1(T) \). For \( 0 \leq i < f - 1 \), it is well known that there exists an isomorphism of \( K \)-representations (see for example [LMS Prop. 3.3(2)]):

\[
R_{2,i} / pR_{2,i} \cong \mathfrak{sl}_2 \mathcal{O}_E \otimes \mathfrak{g} \mathfrak{sl}_2 \mathcal{O}_E \otimes \mathfrak{g} \mathfrak{sl}_2 \mathcal{O}_E \otimes \mathfrak{g} P_\sigma \oplus \sigma_{2,i} \oplus \sigma_{2,i}.
\]

where \( \sigma_{1,i} = F(\lambda - \alpha_i) \) and \( \sigma_{2,i} = F(\lambda + \alpha_i) \). We fix such an isomorphism and use it to define a \( K \)-equivariant injection \( \iota_i : P_\sigma \to R_{2,i} / pR_{2,i} \). We let \( \tau \) denote the “diagonal” embedding of \( P_\sigma \):

\[
\tau : x \mapsto (\iota_i(x))_i \in R_2 / pR_2 \cong \bigoplus_i R_{2,i} / pR_{2,i}.
\]

As a first step, we consider a modification of the lattice \( R_2 \). We define a new lattice in \( R_2[1/p] \) as follows:

\[
R_2[1/p] = \{ x \in R_2 : (x \mod pR_2) \in \iota(P_\sigma) \}.
\]

Note that \( pR_2 \subseteq R'_2 \). As \( K_1 \) acts trivially on \( P_\sigma \), the map \( R'_2 / pR'_2 \to P_\sigma \) sending \( x \) to \( \iota^{-1}(x \mod p) \) factors through \( R'_2 / pR'_2 \to (R'_2 / pR'_2)K_1 \) and gives rise to a \( K \)-equivariant surjective map \( (R'_2 / pR'_2)K_1 \to P_\sigma \).

### Proposition 7.3.1

For \( x \in R_2 \), we can find elements \( k_1, \ldots, k_r \in K_1 \) and \( x_1, \ldots, x_r \in R_2[1/p] \) such that

\[
\sum_{i=1}^r (k_i - 1)x_i \equiv px \quad (\text{mod } p^2 R_2).
\]

Hence the \( K \)-equivariant map \( (R'_2 / pR'_2)K_1 \to P_\sigma \) is an isomorphism.

### Proof

By Lemmas 7.1.1 and 7.1.4 we have a commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
\mathfrak{sl}_2 \otimes_{\mathbb{Q}_p} P_\sigma & \to & R_2 / pR_2 & \to & (R'_2 / p^2 R'_2)K_1 & \to & P_\sigma & \to & 0 \\
\text{Id} \otimes_{\mathbb{Q}_p} & \downarrow & & \downarrow & & \downarrow & \downarrow & & \\
\mathfrak{sl}_2 \otimes_{\mathbb{Q}_p} (R_2 / pR_2) & \to & R_2 / pR_2 & \to & (R'_2 / p^2 R'_2)K_1 & \to & R_2 / pR_2 & \to & 0.
\end{array}
\]
We will prove that the diagonal map is an isomorphism. This implies immediately the desired results.

As \( R_2/pR_2 \cong \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \) and \( \beta_{R_2} = [-, -] \otimes \text{Id}_{P_\sigma} \), we need to prove that the composite map \( ([-, -] \otimes \text{Id}_{P_\sigma}) \circ (\text{Id}_{\mathfrak{sl}_2} \otimes \iota) \) is surjective:

\[
\mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \xrightarrow{\text{Id}_{\mathfrak{sl}_2} \otimes \iota} \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \xrightarrow{[-,-] \otimes \text{Id}_{P_\sigma}} \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma.
\]

For dimension reasons, it is equivalent to prove that it is injective. This can be checked on the socle.

The socle of \( P_\sigma \) is isomorphic to \( \sigma \) and the nonzero map (unique up to scalar) \( \sigma \mapsto P_\sigma \) induces a \( K \)-equivariant map \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \sigma \mapsto \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \) whose image is the socle of \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \) (see Lemma 7.3.2 below).

To summarize, we have a commutative diagram

\[
\begin{array}{ccc}
\mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma & \xrightarrow{\text{Id}_{\mathfrak{sl}_2} \otimes \iota} & \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \\
\Downarrow & & \Downarrow \\
\mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma & \xrightarrow{[-,-] \otimes \text{Id}_{P_\sigma}} & \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma.
\end{array}
\]

We need to prove that the composition of the maps of the top row is injective and we will be done.

In the decomposition \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \sigma \cong \bigoplus_{i=0}^{f-1} (\mathfrak{sl}_2 \otimes_{k, \sigma_i} \sigma_i) \), the map \( \iota|_\sigma \) corresponds to \( (\iota|_\sigma)|_{0 \leq i \leq f-1} \). As \( \iota_i \) is injective and \( \sigma \) is the socle of \( P_\sigma \), we have that \( \iota_i|_\sigma \) is nonzero. We can apply Proposition 7.2.2 to conclude that the composite map in the top row of the diagram above is an isomorphism.

**Lemma 7.3.2.** The \( \text{GL}_2(k) \)-equivariant map \( \sigma \mapsto P_\sigma \) (resp. \( P_\sigma \to \sigma \)) induces a \( \text{GL}_2(k) \)-equivariant map \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \sigma \mapsto \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \) (resp. \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \to \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \sigma \)) whose image is the socle (resp. cosocle) of \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \).

**Proof.** As the map \( \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} \sigma \mapsto \mathfrak{sl}_2 \otimes_{\mathbb{F}_p} P_\sigma \) is \( k \otimes \mathbb{F} \)-linear, it can be decomposed as the direct sum of the maps \( \mathfrak{sl}_2 \otimes_{k, \sigma_i} \sigma \to \mathfrak{sl}_2 \otimes_{k, \sigma_i} P_\sigma \). Therefore it is sufficient to prove that the image of the map \( \mathfrak{sl}_2 \otimes_{k, \sigma_i} \sigma \to \mathfrak{sl}_2 \otimes_{k, \sigma_i} P_\sigma \) is the socle of the right-hand side for each \( 0 \leq i \leq f-1 \). We observe that the left-hand side is semisimple (by (50)), the map is injective and the socle of the right-hand side has the same dimension as the left-hand side (by (51)). This implies the result. The case of the cosocle is similar.

Using Proposition 7.3.1, we identify \( (R'_2/pR'_2)_K \) with \( P_\sigma \) and we define the lattice \( R \) by “gluing” \( R_1 \) and \( R'_2 \) along \( P_\sigma \):

\[
R \overset{\text{def}}{=} \{(x_1, x_2) \in R_1 \oplus R'_2 : (x_1 \mod p) = (\text{image of } x_2 \mod p) \text{ in } P_\sigma \overset{\text{def}}{=} (R'_2/pR'_2)_K \} \\
= \{(x_1, x_2) \in R_1 \oplus R'_2 : (x_2 \mod p) = \iota(x_1 \mod p) \in R_2/pR_2 \}
\]

(equivalently, \( R \cong R_1 \times_{P_\sigma} R'_2 \)). This is a \( K \)-stable lattice in \( R_1[1/p] \oplus R_2[1/p] \). We define \( r \) to be the map \( R \to P_\sigma \) sending \((x_1, x_2)\) to \((x_1 \mod p)\).
Theorem 7.3.3. There exists a short exact sequence of $K$-representations
\[(52) \quad 0 \rightarrow R_2/pR_2 \rightarrow R/pR \rightarrow P_2 \rightarrow 0.\]
Moreover the map $r : R/pR \rightarrow P_2$ induces an isomorphism $(R/pR)_{K_1} \cong P_2$.

Proof. As $pR_2 \subseteq \ker(r) \subseteq R$ we have $p^2R_2 \subseteq pR$ and the inclusion of $pR_2$ in $\ker(r)$ induces a map $pR_2/p^2R_2 \rightarrow \ker(r)/pR$. This map is actually a $K$-equivariant isomorphism $pR_2/p^2R_2 \cong \ker(r)/pR$.

Namely these two representations are finite-dimensional over $F$ and have the same dimension. It is therefore sufficient to prove that $pR \cap pR_2 = p^2R_2$. The right-hand side is clearly included in the left-hand side. Conversely let $(px_1, px_2)$ be some element in the left-hand side. We have $r(x_1 \mod p) = (x_2 \mod p)$ in $R_2/pR_2$. As $x_1 = 0$, we have $x_2 \in pR_2$, which proves the assertion. This gives us the short exact sequence $(52)$.

Now we prove the second assertion. We define $\tau : R/pR \rightarrow P_2$ as the factorization of $r$ by $R/pR$. As $K_1$ acts trivially on $P_2$ and $r$ is $K$-equivariant, the map $\tau$ factors as $(R/pR)_{K_1} \rightarrow P_2$. We need to prove that the kernel of $\tau$ is contained in the kernel of $R/pR \rightarrow (R/pR)_{K_1}$, i.e. that each element of $\ker(\tau)$ can be written as a finite sum $\sum (k_j - 1)y_j$ with $k_j \in K_1$ and $y_j \in R/pR$.

Let $x \in \ker(\tau)$. By what precedes, there exists $y \in R_2$ such that $py$ reduces to $x$ modulo $pR$. By Proposition 7.3.1 we can find $k_1, \ldots, k_r$ in $K_1$ and $x_1, \ldots, x_r$ in $R_2$ such that
\[py \equiv \sum_{j=1}^r (k_j - 1)x_j \pmod{p^2R_2}.\]

Let $z_1, \ldots, z_r$ in $R_1$ be such that $\nu(z_j \mod p) = (x_j \mod p)$ for all $1 \leq j \leq r$. Then $(z_j, x_j) \in R$ for all $1 \leq j \leq r$. Since $K_1$ acts trivially on $R_1$, we have $(k_j - 1)(z_j, x_j) = (0, (k_j - 1)x_j)$ so that
\[(53) \quad \sum_{j=1}^r (k_j - 1)(z_j, x_j) = (0, py + p^2u)\]
for some $u \in R_2$. Let $y_j$ be the image of $(z_j, x_j) \in R$ in $R/pR$. Reducing $(53)$ modulo $pR$, we obtain
\[\sum_{j=1}^r (k_j - 1)y_j = x,\]
proving that $\tau$ induces an isomorphism $(R/pR)_{K_1} \cong P_2$. \hfill $\square$

Corollary 7.3.4. The $K$-cosocle of $R/pR$ is isomorphic to $\sigma$. Moreover the $K$-representations $(\Proj_{K/Z_1}(\sigma))_{m^2_K}$ and $(\Proj_{K/Z_1}(\sigma))_{m^2_{K_1}}$ are isomorphic.

Proof. As $K_1$ is a normal pro-$p$-subgroup of $K$, the group $K_1$ acts trivially on every semisimple representation of $K$. Therefore the $K$-cosocle of $R/pR$ is the $\GL_2(k)(\sigma)$-cosocle of $(R/pR)_{K_1}$. As $(R/pR)_{K_1}$ is isomorphic to $P_2$ by Theorem 7.3.3 we obtain
\[\cosoc_K(R/pR) \cong \cosoc_{\GL_2(k)}(P_2) \cong \cosoc_{\GL_2(k)}(P_2) \cong \sigma.\]

Note that $Z_1$ acts trivially on $R_1$ and $R_2$, and hence also on $R$. This implies that there exists a $K$-equivariant map $\theta : \Proj_{K/Z_1}(\sigma) \rightarrow R/pR$ which is surjective on cosocles and is hence surjective. Note that $R_2/pR_2$ is killed by $m_K$, so that Theorem 7.3.3 implies that $R/pR$ is killed by $m^2_K$. The
map $\theta$ factors through the quotient $(\text{Proj}_{K/Z_1} \sigma)/m_{K_1}^2(\text{Proj}_{K/Z_1} \sigma)$ and gives rise to a surjective map

$$(\text{Proj}_{K/Z_1} \sigma)/m_{K_1}^2(\text{Proj}_{K/Z_1} \sigma) \twoheadrightarrow R/pR.$$  

We now prove that this map is an isomorphism. Namely, since $R$ is a lattice of $\tilde{P}_\sigma[1/p] \oplus \bigoplus_{i=0}^{f-1}(\mathfrak{sl}_2, L \otimes_{O_{L,\sigma}} \tilde{P}_\sigma)$, we have

$$\dim_F(R/pR) = \dim_E \left( \tilde{P}_\sigma[1/p] \oplus \bigoplus_{i=0}^{f-1}(\mathfrak{sl}_2, L \otimes_{O_{L,\sigma}} \tilde{P}_\sigma) \right)$$

$$= (3f + 1) \dim_E(\tilde{P}_\sigma[1/p]) = (3f + 1) \dim_F(P_\sigma).$$

On the other hand, the isomorphism $(\text{Proj}_{K/Z_1} \sigma)/m_{K_1}(\text{Proj}_{K/Z_1} \sigma) \cong P_\sigma$ induces an exact sequence

$$0 \rightarrow (m_{K_1/Z_1}/m_{K_1/Z_1}^2) \otimes_F P_\sigma \rightarrow (\text{Proj}_{K/Z_1} \sigma)/m_{K_1}^2(\text{Proj}_{K/Z_1} \sigma) \rightarrow P_\sigma \rightarrow 0.$$  

(Note that $\text{Proj}_{K/Z_1} \sigma$ is projective in the category of pseudocompact $K_1/Z_1$-modules, since $K_1$ is an open subgroup of $K$. As the group $K_1/Z_1$ is uniform of dimension $3f$, we deduce

$$\dim_F((\text{Proj}_{K/Z_1} \sigma)/m_{K_1}^2(\text{Proj}_{K/Z_1} \sigma)) = (3f + 1) \dim_F(P_\sigma).$$

This implies that $\dim_F((\text{Proj}_{K/Z_1} \sigma)/m_{K_1}^2(\text{Proj}_{K/Z_1} \sigma)) = \dim_F(R/pR)$, so the map $\theta$ is an isomorphism. \qed
8. Global applications

We prove our main global results: Theorem 8.3.10 Theorem 8.4.1 Theorem 8.4.2 Corollary 8.4.3 and Corollary 8.4.5

8.1. Patching functors. We introduce the global background and the patching functors that we will use (following [EGS15 §6.2]). We assume $p > 5$ (for the main theorem, we will in fact need $p > 19$) and $E$ unramified, i.e. $\mathcal{O} = W(F)$. We use the notation and conventions of §2.

We fix $F$ a totally real number field, and denote by $O_F$ its ring of integers and $S_p$ the set of places of $F$ above $p$. We assume $F$ is unramified at each place in $S_p$. For each place $w$ of $F$ we denote by $F_w$ the completion of $F$ at $w$, $O_{F_w}$ its ring of integers and $\text{Frob}_w$ a geometric Frobenius element at $w$. We denote by $A_F^\infty$ the finite adèles of $F$. For any finite place $w$ of $F$, let $q_w$ denote the cardinality of the residue field of $F_w$.

We fix $D/F$ a quaternion algebra of center $F$ which is split at all places in $S_p$ and at no more than one infinite place of $F$ (in the sequel we call the two cases the “indefinite case” and the “definite case”). In the indefinite case we assume $(F, D) \neq (\mathbb{Q}, \text{GL}_2)$ (our main result is already known in the case $(F, D) = (\mathbb{Q}, \text{GL}_2)$). We denote by $S_D$ the set of finite places where $D$ ramifies. We fix a maximal order $O_D$ in $D$ and isomorphisms $(O_D)_w \cong M_2(O_{F_w})$ for $w \notin S_D$, where $(O_D)_w \overset{def}{=} O_D \otimes_{O_F} O_{F_w}$.

We fix $\tau : G_F \rightarrow \text{GL}_2(F)$ a continuous representation and set $\tau_w \overset{def}{=} \tau|_{G_{F_w}}$ for each finite place $w$ of $F$. We assume that $\tau|_{G_F \cap T}$ is absolutely irreducible and $\tau_w$ is generic in the sense of [BP12 Def. 11.7] (or [EGS15 Def. 2.1.1]) for $w \in S_p$. We let $S_T$ be the set of (finite) places where $\tau$ is ramified (hence $S_p \subseteq S_T$ by the previous genericity) and we moreover assume that the universal framed deformation ring $R_{\tau_w}$ of $\tau_w$ over $W(F)$ is formally smooth over $W(F)$ if $w \in (S_D \cup S_T) \setminus S_p$ (see Remark 8.1.1 below). We let $\psi : G_F \rightarrow W(F)^\times$ be the Teichmüller lift of $\omega \text{det} \tau$ and set $\psi_w \overset{def}{=} \psi|_{G_{F_w}}$.

Assume first that we are in the indefinite case. For a compact open subgroup $V$ of $(D \otimes_F A_F^\infty)^\times$ let $X_V$ be the associated smooth projective algebraic Shimura curve over $F$ (see e.g. [BD14 §3.1] and the references therein). We assume that there exists $V$ such that

$$\text{Hom}_{G_F}(\tau, H^1_{et}(X_V \times_F \overline{F}, \mathcal{F})) \neq 0.$$ (54)

Then one can always take $V$ of the following form: $V = \prod V_w$ with $V_w \subseteq (O_D)_w^\times$ for all $w$, $V_w = (O_D)_w^\times$ for $w \notin S_D \cup S_T$ and $V_w = 1 + pM_2(O_{F_w})$ for $w \in S_p$ (see e.g. [BD14 Thm. 3.2.2] or the proof of [BD14 Cor. 3.2.3]). For Serre weights $(\sigma_w)_{w \in S_p}$ and any $V = \prod V_w$ such that (54) holds and $V_w \subseteq 1 + pM_2(O_{F_w})$ is normal in $(O_D)_w^\times$ for $w \in S_p$ we have by [GK14 §5.5]:

$$\text{Hom}_{\text{GL}_2(O_F \otimes \mathbb{Z}_p)}(\otimes_{F_w} \sigma_w, \text{Hom}_{G_F}(\tau, H^1_{et}(X_V \times_F \overline{F}, \mathcal{F}))) \neq 0 \iff \sigma_w \in W(\tau_w^\vee) \forall w \in S_p,$$

where we recall that $W(\tau_w^\vee)$ is defined as in [BDJI0 §3] (with $\rho$ there being $\tau_w^\vee$), cf. §2.2.

We now fix

(i) a finite place $v \in S_p$ such that $\tau_v$ is semisimple of one of the following forms up to twist:

(a) $\tau_v|_{I_{F_v}} \cong \left(\begin{array}{cc} \omega^f (r_0 + 1) + \ldots + p^{-1}(r_f - 1) + 1 & 0 \\ 0 & 1 \end{array} \right)$, $9 \leq r_i \leq p - 12$,
For instance, if Norm \( w \) is not congruent to 1 mod \( p \), then \( R_{\tau_w} \) (or equivalently \( R_{\tau_w} \), the two rings are isomorphic by duality) is always formally smooth, except when \( \tau_w \cong \left( \begin{array}{c} \omega \\ 0 \\ 0 \\ 1 \end{array} \right) \) up to twist.

The following lemma due to Hamann [Ham75 Thm. 4] will be convenient below.

**Lemma 8.1.2.** Suppose that \( R, S \) are local rings. If \( R[[x]] \cong S[[x]] \), then \( R \cong S \).

For each \( w \in S_p \setminus \{ v \} \) we fix a tame inertial type \( \tau_w \) such that \( JH(\sigma(\tau_w)^\vee) = JH(\sigma(\tau_w^\vee)) \) contains exactly one Serre weight in \( W(\tau_w^\vee) \) ([EGS15 Prop. 3.5.1]) and we fix a \( GL_2(\mathcal{O}_{F_w}) \)-invariant lattice \( \sigma^0(\tau_w^\vee) \) in \( \sigma(\tau_w^\vee) \) (so, increasing \( F \) if necessary, \( \sigma^0(\tau_w) \) is a free \( W(\mathcal{O}_w) \)-module, see the last statement in [EGS15 Lemma 3.1.1]). As any Serre weight in \( W(\tau_w^\vee) \) has central character \( \omega^{-1} \det \tau_w^\vee = \psi_{I_{F_w}}^{-1} \) and \( \tau_w \) is tame, the central character of \( \sigma^0(\tau_w) \) is \( \psi_{I_{F_w}}^{-1} \) and \( \det \tau_w = \psi_{I_{F_w}} \).

We define a representation \( \sigma_w \) of \( \prod_{w \in S_p \setminus \{ v \}} U_w \) over \( W(\mathcal{F}) \) by

\[
\sigma_w \overset{def}{=} \otimes_{w \in S_p \setminus \{ v \}} \sigma^0(\tau_w^\vee),
\]

with \( \prod_{w \in S_p \setminus \{ v \}} U_w \) acting via \( \prod_{w \in S_p \setminus \{ v \}} U_w \rightarrow \prod_{w \in S_p \setminus \{ v \}} U_w = \prod_{w \in S_p \setminus \{ v \}} GL_2(\mathcal{O}_{F_w}) \). As in [EGS15 §§6.2, 6.4] using \( K = U \), we then define a patching functor (depending on \( \sigma_p^w \))

\[
M^w_{\infty} : \sigma_w \rightarrow M_{\infty}(\sigma_p^w \otimes_{W(\mathcal{F})} \sigma_v)
\]

from the category of continuous representations \( \sigma_v \) of \( GL_2(\mathcal{O}_v) \) on finite type \( W(\mathcal{F}) \)-modules with central character \( \psi_{I_{F_v}}^{-1} \) to the category of finite type \( R_{\infty} \)-modules, where (see [GK14 §5.4.1])

\[
R_{\infty} \overset{def}{=} R^{loc} \mathbb{[}X_1, \cdots, X_q - \mathbb{Q}[1/\mathbb{Z}] - 1].
\]
Here \( q \) is an integer \( \geq [F : \mathbb{Q}] \) and

\[
R^{\text{loc}} = \left( \bigotimes_{w \in S_p \setminus \{v\}} R^{\psi_w}_{\tau_w} \right) \otimes_{W(F)} \left( \bigotimes_{w \in S_p \setminus \{v\}} R^{(0, -1), \tau_w, \psi_w}_{\tau_w} \right) \otimes_{W(F)} R^{\psi_w}_{\tau_w},
\]

where the exponent \( \psi_w \) means framed deformations of \( \tau_w \) with fixed determinant \( \varepsilon^{-1} \psi \) and where \( R^{(0, -1), \tau_w, \psi_w}_{\tau_w} \) is the reduced \( p \)-torsion free quotient of \( R^{\psi_w}_{\tau_w} \) parametrizing those deformations which have parallel Hodge–Tate weights \( (0, -1) \) and inertial \( \tau_w \) (by local-global compatibility and the inertial Langlands correspondence, for \( w \in S_p \setminus \{v\} \) the action of \( R^{\psi_w}_{\tau_w} \) on \( M_\infty(\sigma_p^w \otimes W(F) \sigma_v) \) factors through this quotient). By assumption \( \text{iii) b)} \) above (with \([\text{GK14}] \) Rk. 5.2.2) and Lemma 8.1.2 we have \( R^{\psi_w}_{\tau_w} \cong W(F)[X_1, X_2, X_3] \) for \( w \in S \setminus S_p \), and by genericity of \( \tau \) we have \( R^{\psi_w}_{\tau_w} \cong W(F)[X_1, \ldots, X_{3 + 3[\mathbb{F}_p : \mathbb{Q}_p]}] \). Taking the duals of representations induces a canonical isomorphism \( R^{\psi_w}_{\tau_w} \cong R^{(1, 0), \tau_w, \psi_w^{-1}}_{\tau_w} \), where the ring on the right-hand side is the more familiar quotient of \( R_{\tau_w} \) parametrizing potentially Barsotti–Tate deformations of \( \tau_w \) with inertial type \( \tau_w \) and determinant \( \varepsilon^{-1} \psi \). By \([\text{EGS15}] \) Thm. 7.2.1(2) (with \([\text{GK14}] \) Rk. 5.2.2) and Lemma 8.1.2 we have \( R^{(1, 0), \tau_w, \psi_w^{-1}}_{\tau_w} \cong W(F)[X_1, \ldots, X_{3 + [\mathbb{F}_p : \mathbb{Q}_p]}] \), so that we finally get

\[
R_\infty \cong R^{\psi_w}_{\tau_w} \left[ X_1, \ldots, X_{4(|S| - 1) + q - [\mathbb{F}_p : \mathbb{Q}_p]} \right] \cong W(F)[X_1, \ldots, X_{4|S| + q + 1 - 2[\mathbb{F}_p : \mathbb{Q}_p]}].
\]

**Remark 8.1.3.** Here are several remarks on the definition of \( M_\infty(\sigma_p^w \otimes W(F) \sigma_v) \) in \([\text{EGS15}] \).

(i) One needs to extend the action of \( U \) on \( \sigma_p^w \otimes W(F) \sigma_v \) (which acts via \( U \to \prod_{w \in S_p \setminus \{w\}} U_w \)) to an action of \( U(\mathbb{A}_F) \times \) with \( (\mathbb{A}_F) \times \) acting via

\[
(\mathbb{A}_F) \times \to (\mathbb{A}_F) \times /F \times \text{Artin} \to \text{Gal}(F^{ab}/F) \to W(F) \times.
\]

Note that we believe this action of \( (\mathbb{A}_F) \times \) in \([\text{EGS15}] \) §6.2 should also be via \( \psi^{-1} \), not \( \psi \) (as it is there), otherwise there is a contradiction with (at least) \( \det \tau = \psi|_{\mathbb{F}_p} \) in \([\text{EGS15}] \) §7.1, since the normalization of \( \sigma(\tau) \) in \([\text{EGS15}] \) §1.9 is dual to the one in \([\text{BM02}] \) §2.1.1.

(ii) Accordingly, we need to modify the maximal ideal \( m \) associated to \( \tau \) in \([\text{EGS15}] \) §6.2 as follows: \( m \) is the maximal ideal generated by \( T_w - S_m \text{tr}(\text{Frob}_w) \), \( \text{Norm}(w) - S_m \text{det}(\text{Frob}_w) \) for \( w \not\in S \cup \{w\} \) (this is the maximal ideal of \([\text{BD10}] \) §4).

(iii) For any \( V \subseteq U \) the finite group \( V(\mathbb{A}_F) \times /F \times \) acts on \( X_1 \) without fixing any geometric point (see e.g. part (iv) of the proof of \([\text{BD14}] \) Lemma 3.6.2), replacing \( w_0 \) there by \( w_1 \).

In the definition of \( S(\sigma) \) in \([\text{EGS15}] \) §6.2 in the indefinite case, one should replace the Shimura curve by its quotient by this finite action (which is still a smooth projective curve over \( F \)), analogously to the definite case of loc. cit., where \( S(\sigma) \) is defined as functions \( f : D \times \mathbb{A}_F \times \sigma(\theta) \times \to \sigma(\theta) \times \) such that \( f(gd) = d^{-1}f(g) \) for \( d \in U(\mathbb{A}_F) \times \). Note that replacing \( X_1 \) by its quotient does not change \( \text{Hom}_{G_F}(\tau, H^1_{et}(X_1 \times_F \mathbb{F}, F)) \) (arguing as in the proof of \([\text{BD14}] \) Thm. 3.7.1)).

Denote by \( m_\infty \) the maximal ideal of \( R_\infty \) and for \( w \in S_p \setminus \{v\} \) let \( \sigma_w \) be the unique Serre weight in \( W(\tau_{w}^{\vee}) \) that appears in \( JH(\sigma_{w}^{\vee}) \). By a standard Hochschild–Serre spectral sequence (see e.g. the proof of \([\text{BD10}] \) Lemma 4.11) or of \([\text{BD14}] \) Lemma 3.6.2) we have isomorphisms of finite-dimensional \( F \)-vector spaces for any representation \( \sigma_v \) of \( \text{GL}_2(\mathcal{O}_F) \) over \( W(F) \) as above (see also
Lemma 8.2.1. Let $A$ be a commutative ring and $N \subseteq M$ two $A$-modules. We assume there is an integer $r \geq 1$ such that

(i) $N$ and $M/N$ are free of rank $r$ over their respective scheme-theoretic supports;

(ii) $M$ can be generated as an $A$-module by $r$ elements;

(iii) there is an isomorphism of $A$-modules $\text{Ann}_A(M/N)/\text{Ann}_A(M) \cong A/\text{Ann}_A(N)$.

Then $M$ is free of rank $r$ over its scheme-theoretic support.

Proof. Replacing $A$ by $A/\text{Ann}_A(M)$, we can assume $\text{Ann}_A(M) = 0$. Let $I \overset{\text{def}}{=} \text{Ann}_A(M/N)$ and $f : A' \rightarrow M$ an $A$-linear surjection by $\text{(iii)}$. Then the composition of $f$ with $M \rightarrow M/N$ factors through $(A/I)^r$ and we deduce a commutative diagram of $A$-modules

\[
\begin{array}{cccccc}
0 & \longrightarrow & I^r & \longrightarrow & A' & \longrightarrow & (A/I)^r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & M/N & \longrightarrow & 0.
\end{array}
\]
By [i] we have an isomorphism of $A$-modules $M/N \cong (A/I)^r$ and it follows from e.g. [Mat89 Thm. 2.4] that the surjection on the right is an isomorphism. The snake lemma then shows that the vertical map on the left is surjective. Since $I \cong A/\text{Ann}_A(N)$ by [iii] (recall $\text{Ann}_A(M) = 0$) and $N \cong (A/\text{Ann}_A(N))^r$ by [i] [Mat89 Thm. 2.4] again shows that the vertical map on the left is bijective, and hence all vertical maps are bijective. \hfill \qed

Recall that a finite type module $M$ over a noetherian local ring $A$ is called maximal CM if it is Cohen–Macaulay and if its Krull dimension (which is the Krull dimension of $A/\text{Ann}_A(M)$) is equal to the Krull dimension of $A$. In particular, $A/\text{Ann}_A(M)$ has no embedded associated prime.

Lemma 8.2.2. Let $\sigma$ be any smooth representation of $K$ on a finite length $W(\mathbb{F})$-module. Then the finite type $R_\infty$-module $M_\infty(\sigma)$ is maximal CM over its scheme-theoretic support.

Proof. We can assume $M_\infty(\sigma) \neq 0$. For each Serre weight $\sigma_v$ such that $M_\infty(\sigma_v) \neq 0$, it follows from [EGSL15 Def. 6.1.1] that the Krull dimension of $M_\infty(\sigma_v)$ does not depend on $\sigma_v$, call it $d$, and that $M_\infty(\sigma_v)$ is Cohen–Macaulay. By exactness of the functor $\text{Ann}_M$, the Krull dimension of $M_\infty(\sigma)$ is the maximum of the Krull dimensions of the $M_\infty(\sigma_v)$ for the constituents $\sigma_v$ of $\sigma$, hence is also $d$. In particular, each nonzero such $M_\infty(\sigma_v)$ is maximal CM over $R_\infty/\text{Ann}_{R_\infty}(M_\infty(\sigma_v))$. But being maximal CM over a given noetherian local ring $A$ of residue field $\mathbb{F}$ is preserved by extensions of modules (as can be checked from the characterization of Cohen–Macaulay modules using $\text{Ext}^1_A(\mathbb{F}, -)$). Hence $M_\infty(\sigma)$ is Cohen–Macaulay. \hfill \qed

If $\tau$ is a tame inertial type and $\lambda = ((a_j, b_j))_{j \in \{0, \ldots, f-1\}}$, where $a_j > b_j$ are integers, we set

$$R_\infty^{\lambda, \tau} \overset{\text{def}}{=} R_\infty \otimes_{R_{\mathcal{F}_v}} R_{\mathcal{F}_v}^{\lambda, \tau},$$

where $R_{\mathcal{F}_v}^{\lambda, \tau}$ parametrizes (framed) deformations of $W(\mathbb{F})$-lattices $\sigma_v$ of inertial type $\tau$ and Hodge–Tate weights $(a_j, b_j)$ in the embedding $\sigma_j : F_v \to E$. Note that from the determinant condition (see (57)), one must have $a_j + b_j = 1$ for all $j$ in order for $R_\infty^{\lambda, \tau}$ to be nonzero. When $a_j = a$ and $b_j = b$ for all $j$, we write $R_\infty^{(a, b), \tau}$. We finally write $R_\infty^{\mathbb{F}} = R_\infty/(p)$ and $R_\infty^{\lambda, \tau} = R_\infty^{\lambda, \tau}/(p)$.

Proposition 8.2.3. There exists an integer $r \geq 1$ such that

(i) for all $\sigma_v \in W(\mathbb{F})$ the module $M_\infty(\sigma_v)$ is free of rank $r$ over its scheme-theoretic support, which is formally smooth over $\mathbb{F};$

(ii) for all tame inertial types $\tau$ such that $\text{JH}(\overline{\sigma(\tau)}) \cap W(\mathbb{F})$-lattices $\sigma^0(\tau)$ in $\sigma(\tau)$ with irreducible cosocle, the module $M_\infty(\sigma^0(\tau))$ is free of rank $r$ over its scheme-theoretic support, which is a domain.

Proof. Note first that the last assertions in [i] and [iii] are a consequence of [EGSL15 Def. 6.1.1], [EGSL15 Thm. 7.2.1(2), (5)], and [EGSL15 Prop. 3.5.1]. The strategy of the proof is very close to the one of [EGSL15 Thm. 10.1.1] (which proves the case $r = 1$), and we freely use some notation from loc. cit. (it would be too tedious to recall everything). By [EGSL15 §5.1] there is a set $\mathcal{P}_\tau$ of subsets of $\{0, \ldots, f-1\}$ and a unique $J \in \mathcal{P}_\tau$ such that $\sigma^0(\tau) = \sigma_J^0(\tau)$. The constituents of $\text{JH}(\sigma_J^0(\tau)) \cap W(\mathbb{F})$ are indexed by a certain subset $\mathcal{W}$ of $\mathcal{P}_\tau$, and for certain subsets $\mathcal{J} \subseteq \mathcal{W}$ called intervals (see [EGSL15 Def. 10.1.4]) there exists a subquotient $\sigma_{\mathcal{J}}$ of $\sigma_J^0(\tau)$ such that the irreducible constituents of $\sigma_{\mathcal{J}}$ are exactly the constituents of $\text{JH}(\sigma_J^0(\tau)) \cap W(\mathbb{F})$ indexed by the
elements of $\mathcal{J}$. We first prove by induction on $|\mathcal{J}|$ that the module $M_\infty(\sigma^J)$ is free of rank $r$ over its scheme-theoretic support for an integer $r$ which depends neither on $\tau$ nor on $\mathcal{J}$.

By the argument in the proof of [LLHLM20 Lemma 3.6.2], the ring $\mathcal{R}_\infty/\text{Ann}_\mathcal{R}_\infty(M_\infty(\sigma^J))$ is reduced. Indeed, it is generically reduced by dévissage, since the scheme-theoretic supports of $M_\infty(\sigma_\nu)$ for Serre weights $\sigma_\nu \in W(\tau^J)$ are reduced, irreducible, and pairwise distinct (of dimension independent of $\sigma_\nu$) and since $\sigma^J$ is multiplicity-free; it also has no embedded associated prime, since $M_\infty(\sigma^J)$ is Cohen–Macaulay by Lemma 8.2.2. Let $I_\mathcal{J}$ be the ideal of $\mathcal{R}_\infty$ defined in [EGS15 §10.1], it follows that

$$\text{Ann}_{\mathcal{R}_\infty}(M_\infty(\sigma^J)) = I_\mathcal{J}. \tag{62}$$

If $|\mathcal{J}| \leq 2$, then by [EGS15 Prop. 3.5.1], [EGS15 Prop. 10.1.11] and the very last paragraph in the proof of [EGS15 Lemma 10.1.12] there is a tame inertial type $\tau'$ and a $W(\mathcal{F})$-lattice $\sigma^0(\tau')$ in $\sigma(\tau')$ such that $\mathcal{H}(\sigma^0(\tau')) \cap W(\tau^J) = \mathcal{H}(\sigma^J)$ and $M_\infty(\sigma^0(\tau')) \cong M_\infty(\sigma^J)$. By [EGS15 Thm. 7.2.1(2)] (and [GK14 Rk. 5.2.2]) the local ring $R_{\mathcal{F}'}^{1(0),\tau',\psi^{-1}}$ is regular, and hence also $R_{\mathcal{F}'}^{1(0),\tau'}$ by (57) and (61). By [EGS15 Lemma 6.1.4] it follows that $M_\infty(\sigma^0(\tau'))$ is free of finite type over $R_{\mathcal{F}'}^{1(0),\tau'}$. Hence $M_\infty(\sigma^0(\tau')) \cong M_\infty(\sigma^J)$ is also free of finite type over $\mathcal{R}_\infty/\mathcal{I}_\mathcal{J}$.

If $|\mathcal{J}| = 2$, then $\sigma^J$ has two distinct constituents $\sigma_1$, $\sigma_2$ and the freeness of $M_\infty(\sigma^J)$ over $\mathcal{R}_\infty/I_\mathcal{J}$ (which is a power series ring over $\mathcal{F}[X_1, X_2]/(X_1X_2)$) easily implies that $M_\infty(\sigma_1)$ and $M_\infty(\sigma_2)$ have the same rank over their scheme-theoretic support (which is a power series ring over, respectively, $\mathcal{F}[X_1]$ and $\mathcal{F}[X_2]$). Using [EGS15 Prop. 10.1.11] and the fact that all Serre weights in $W(\tau^J)$ can be “connected” by nonsplit extensions (as follows e.g. from [EGS15 Prop. 3.5.2] applied to a semisimple $\mathcal{P}$), we obtain (1) for a certain integer $r \geq 1$.

If $|\mathcal{J}| > 2$ and $\mathcal{J}$ has a unique minimal element $J_0$ (for inclusion inside $\{0, \ldots, f - 1\}$), then exactly as in the analogous case of the proof of [EGS15 Thm. 10.1.1] but using [Le19 Lemma 4.5] instead of [EGS15 Lemma 10.1.13], we deduce that the $\mathcal{R}_\infty$-module $M_\infty(\sigma^J)$ is generated by $r$ elements. Then one applies Lemma 8.2.1 to $M = M_\infty(\sigma^J)$ and $N = M_\infty(\sigma^J \setminus \{J_0\})$ (the hypotheses of the lemma are satisfied, as $M/N \cong M_\infty(\sigma^J \setminus \{J_0\})$, $I_{\mathcal{J}\setminus\{J_0\}}/I_\mathcal{J} \cong \mathcal{R}_\infty/I_{\mathcal{J}}$ and using (62)) together with the induction hypothesis on $|\mathcal{J}|$ to deduce that $M_\infty(\sigma^J)$ is free of rank $r$ over $\mathcal{R}_\infty/I_\mathcal{J}$.

If $|\mathcal{J}| > 2$ and $\mathcal{J}$ has at least two distinct minimal elements $J_1$, $J_2$, let $J_i \defeq J\setminus\{J_i\}$, $i = 1, 2$. Then by the induction hypothesis $M_\infty(\sigma^J)$, $M_\infty(\sigma^{J_1})$ and $M_\infty(\sigma^{J_2})$ are all free of rank $r$ over (respectively) $\mathcal{R}_\infty/I_{\mathcal{J}_1}$, $\mathcal{R}_\infty/I_{\mathcal{J}_2}$ and $\mathcal{R}_\infty/I_{\mathcal{J}_1 \cap \mathcal{J}_2}$. Hence so is the fiber product $M_\infty(\sigma^J) \times_{M_\infty(\sigma^{J_1} \cap \mathcal{J}_2)} M_\infty(\sigma^{J_2}) \cong M_\infty(\sigma^J)$ over $\mathcal{R}_\infty/I_{\mathcal{J}_1} \times_{\mathcal{R}_\infty/I_{\mathcal{J}_1 \cap \mathcal{J}_2}} \mathcal{R}_\infty/I_{\mathcal{J}_2} \cong \mathcal{R}_\infty/I_\mathcal{J}$ (see the analogous case in the proof of [EGS15 Thm. 10.1.1]).

It remains to finish the proof of (ii). By the previous proof, $M_\infty(\sigma^{J_1}(\tau)) \cong M_\infty(\sigma^{J_2}(\tau)) \cong M_\infty(\sigma^W)$ is free of rank $r$ over $\mathcal{R}_\infty/I_\mathcal{W} \cong \mathcal{R}_\infty^{1(0),\tau}$. By Nakayama’s lemma, we deduce a surjection of $\mathcal{R}_\infty^{1(0),\tau}$-modules $f : (\mathcal{R}_\infty^{1(0),\tau})^r \twoheadrightarrow M_\infty(\sigma^{J_1}(\tau))$ which is an isomorphism modulo $p$, hence satisfies $p \ker(f) = \ker(f)$ since $M_\infty(\sigma^{J_1}(\tau))$ has no $p$-torsion. By Nakayama’s lemma again we deduce $\ker(f) = 0$, which finishes the proof. □

**Corollary 8.2.4.** Let $\sigma = \oplus_{i=1}^m \sigma_i^{n_i}$, where $m, n_i \geq 1$ and the $\sigma_i$ are pairwise nonisomorphic absolutely irreducible locally $\mathbb{Q}_p$-algebraic representations of $K$ over $\mathcal{E}$ satisfying...
the following hypothesis: $\sigma^0$ is tame smooth (i.e. the action of $K$ factors through $K \rightarrow GL_2(k)$) and $JH(\sigma) \cap W(\mathfrak{p}_v) \neq \emptyset$. Let $\sigma^0$ be any $W(\mathbb{F})$-lattice in $\sigma$ preserved by $K$. Then

(i) $M_\infty(\sigma^0)$ is maximal CM over its scheme-theoretic support $S \overset{\text{def}}{=} R_\infty/\text{Ann}_{R_\infty}(M_\infty(\sigma^0))$, which is reduced;

(ii) $M_\infty(\sigma^0) \otimes_{W(\mathbb{F})} E$ is locally free over its scheme-theoretic support $S[1/p]$, which is formally smooth over $E$.

Proof. For $i \in \{1, \ldots, m\}$ let $\sigma^0_i$ be any $K$-invariant $W(\mathbb{F})$-lattice in $\sigma_i$. It easily follows from the exactness of the functor $M_\infty$ that there is an isomorphism of $R_\infty[1/p]$-modules

$$M_\infty(\sigma^0)[1/p] \cong \bigoplus_{i=1}^m M_\infty(\sigma^0_i)[1/p]^{\oplus n_i}. \quad (63)$$

From the Taylor–Wiles–Kisin method, we know that the action of $R_\infty$ on $M_\infty(\sigma^0)$ factors through a reduced equidimensional $p$-torsion free quotient of $R_\infty$ and that the support of $M_\infty(\sigma^0)$ is a union of irreducible components of that quotient (see e.g. [CEG+16] Lemmas 4.17, 4.18). Hence the scheme-theoretic support of $M_\infty(\sigma^0)$ is also a reduced $p$-torsion free quotient $R_\infty/I_i$ of $R_\infty$. It follows from (63) that the support of $M_\infty(\sigma^0)[1/p]$ is $S[1/p] \cong (R_\infty/\bigcap_i I_i)[1/p]$ (as there is no $p$-torsion). Since the Spec $(R_\infty/I_i)[1/p]$ for $1 \leq i \leq m$ correspond to disjoint closed subschemes of Spec $R_\infty[1/p]$ (as the locally algebraic representations $\sigma_i$ are pairwise distinct), one has by the Chinese remainder theorem

$$S[1/p] = (R_\infty/\bigcap_i I_i)[1/p] \cong \bigoplus_{i=1}^m (R_\infty/I_i)[1/p], \quad (64)$$

which is thus reduced and regular by [Kis08] Thm. (3.3.4) and [Mat89] Thm. 28.7. Since $S$ has no $p$-torsion (as $S$ acts faithfully on $M_\infty(\sigma^0)$ which has no $p$-torsion by exactness of $M_\infty$), we deduce that $S$ is also reduced.

The module $M_\infty(\sigma^0)/(p) \cong M_\infty(\bar{\sigma}^0)$ is a Cohen–Macaulay-module by Lemma 8.2.2 and $p$ is a nonzero-divisor on $M_\infty(\sigma^0)$, hence $M_\infty(\sigma^0)$ is also Cohen–Macaulay, hence maximal CM over $S$. Moreover applying [Mat89] Thm. 17.3(iii) to $M_\infty(\sigma^0)$ we see that $M_\infty(\sigma^0)[1/p]$ is also Cohen–Macaulay as an $S[1/p]$-module. The Auslander–Buchsbaum formula applied to the localizations of $S[1/p]$ at prime ideals of the Cohen–Macaulay module $M_\infty(\sigma^0)[1/p]$ over the regular ring $S[1/p]$ implies $M_\infty(\sigma^0)[1/p]$ is locally free over $S[1/p]$. \hfill \Box

For any Serre weight $\sigma_v$, recall that we have defined in 8.2.4 the two $GL_2(k)$-representations $P_\sigma = \text{Proj}_{GL_2(k)} \sigma_v$ and $\bar{P}_\sigma$ over, respectively, $\mathbb{F}$ and $\mathcal{O} = W(\mathbb{F})$.

**Proposition 8.2.5.** If $\sigma_v \in W(\mathfrak{p}_v^{\sigma})$, then $M_\infty(\bar{P}_\sigma_v)$ is free of rank $r$ over $R_\infty/\mathfrak{p}_\tau$, where $\tau$ runs over the tame inertial types such that $\sigma_v \in JH(\sigma(\tau))$ and $\mathfrak{p}_\tau$ is the prime ideal ker$(R_\infty \rightarrow R_\infty^{(1,0),\tau})$.

Proof. (i) We first prove that the $R_\infty$-module $M_\infty(\bar{P}_\sigma_v)$ can be generated by $r$ elements. By Nakayama’s lemma, it is enough to prove the same statement with $M_\infty(P_\sigma_v)$, or equivalently to prove $\dim_{\mathbb{F}}(M_\infty(\bar{P}_\sigma_v)/m_\infty) \leq r$. By [58] and [59] it is enough to prove $\dim_{\mathbb{F}}(\text{Hom}_{GL_2(k)}(P_\sigma_v,V)) = r$, where $V$ is the finite-dimensional representation of $GL_2(k)$ over $\mathbb{F}$ on the right-hand side of [58] or [59]. By Proposition 8.2.3(i) we have $\dim_{\mathbb{F}}(\text{Hom}_{GL_2(k)}(\sigma_v,V)) = r$. Let $D_0(\mathfrak{p}_v^{\sigma})$ be the
representation of GL$_2(k)$ over $F$ defined in [BPT2] §13 (see also Lemma 6.4.3) and recall that by construction
\[ \text{Hom}_{\text{GL}(k)}(P_{\sigma}, D_0(\tau_{\sigma}^\vee))/\text{soc}_{\text{GL}(k)}(D_0(\tau_{\sigma}^\vee)) = 0. \]
Hence it is enough to prove that there is a $GL_2(k)$-equivariant injection
\[ V \hookrightarrow D_0(\tau_{\sigma}^\vee)^{\oplus r} \]
(which is necessarily an isomorphism on $\text{soc}_{\text{GL}(k)}(V) = (\text{soc}_{\text{GL}(k)}(D_0(\tau_{\sigma}^\vee)))^{\oplus r}$, or equivalently a $GL_2(k)$-equivariant surjection $(D_0(\tau_{\sigma}^\vee))^{\oplus r} \twoheadrightarrow V^{\vee}$. But this follows exactly as in the proofs of [LMS, Lemma 4.5] and [LMS, Prop. 4.6] (plus Proposition 8.2.3). More precisely, one replaces the integer $1$ by the integer $r$ in the statements of loc. cit., and the proofs are basically the same, replacing the surjection $\oplus_{\kappa} P_{\kappa} \twoheadrightarrow D_0^0$ by a surjection $\oplus_{\kappa} P_{\kappa}^{\oplus r} \twoheadrightarrow D_0^0$ (for [LMS, Lemma 4.5], one gets at the end of the proof $\dim(\text{Hom}_K(D_0^0, \sigma^i(\tau))) > r$ instead of $\dim(\text{Hom}_K(D_0^0, \sigma^i(\tau))) > 1$).

(ii) We now prove the proposition. Let $S = R_\infty/\text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v}))$ be the scheme-theoretic support of $M_\infty(P_{\sigma_v})$. The representation $\tilde{P}_{\sigma_v}[1/p]$ over $E$ is the direct sum of the (tame smooth) representations $\sigma(\tau)$ for all the tame inertial types $\tau$ such that $\sigma_v \in \text{JH}(\sigma(\tau))$, and each such $\sigma(\tau)$ occurs only once. It follows from [63] (with all $n_i = 1$), [64] and Proposition 8.2.3(ii) that $M_\infty(\tilde{P}_{\sigma_v})[1/p]$ is free of rank $r$ over $S[1/p]$. By (i), we have a surjection $S' \twoheadrightarrow M_\infty(P_{\sigma_v})$ which is thus an isomorphism after inverting $p$ ([Mat99, Thm. 2.4]), hence is also injective. Finally we obtain $S = R_\infty/\cap_\tau p_\tau$ from [63], from $M_\infty(\tilde{P}_{\sigma_v}) \hookrightarrow M_\infty(\tilde{P}_{\sigma_v})[1/p]$ and from the fact the rings $R_\infty^{(1,0),\tau}$ are all domains (Proposition 8.2.3(ii)).

\[ \Box \]

8.3. Freeness for projective envelopes. We prove that $M_\infty(R)$ is free over its scheme-theoretic support, where $R$ is the lattice defined in §7.3.

We keep all the notation of §8.2 and we fix a Serre weight $\sigma_v \in W(\tau_{\sigma_v}^\vee)$.

Lemma 8.3.1. Let $V'$ be a finite-dimensional $\mathbb{F}$-representation of a finite group $G$. Let $V \subseteq V'$ be a subrepresentation. Then there exists a quotient $Q$ of $V'$ such that the composite map
\[ V \hookrightarrow V' \twoheadrightarrow Q \]
induces an isomorphism $\text{cosoc}_G(V) \cong \text{soc}_G(Q)$.

Proof. Let $I$ be an injective envelope of $\text{cosoc}_G(V)$. Then the composite $V \hookrightarrow \text{cosoc}_G(V) \hookrightarrow I$ can be extended to a map $V' \twoheadrightarrow I$. Let $Q$ be the image of this map. Then $\text{soc}_G(Q) \hookrightarrow \text{soc}_G(I) = \text{cosoc}_G(V)$. Moreover the composition $V \hookrightarrow V' \twoheadrightarrow Q$ factors through an injecton $\text{cosoc}_G(V) \hookrightarrow Q$ whose image is contained in $\text{soc}_G(Q)$ since $\text{cosoc}_G(V)$ is semisimple. From what is before, this injection is thus an isomorphism.

\[ \Box \]

Define $W(\mathbb{F})$-lattices $R_2' \subseteq R_2$ and $R = \tilde{P}_{\sigma_v} \times_{P_{\sigma_v}} R_2'$ as in §7.3. By Theorem 7.3.3 we have a nonsplit extension
\[ 0 \longrightarrow P_{\sigma_v}^{\oplus f} \oplus \bigoplus_{j=0}^{f-1} (P_{\sigma_1,j} \oplus P_{\sigma_2,j}) \longrightarrow R/pR \longrightarrow R_{\sigma_v} \longrightarrow 0, \]
where $\sigma_{1,j}, \sigma_{2,j}$ are the two Serre weights appearing in [51] for $i = j$ and where the $P_{\sigma_v}$ on the right is isomorphic to the $K_1$-coinvariants of $R/pR$. We define $W$ as the cokernel of the map
Let $Q$ be a quotient of $R/pR$. The following are equivalent:

(i) the quotient map $R/pR \rightarrow Q$ factors through $R/pR \rightarrow W \rightarrow Q$;
(ii) for each quotient $Q'$ of $Q$ such that $\text{soc}_K(Q')$ is $\sigma_v$-isotypic, we have $Q^K_1 = Q'$.

**Proof.** We prove (i) implies (ii). Let $Q'$ be a quotient of $Q$ with $\sigma_v$-isotypic socle. It suffices to prove that the composite map

$$P_{\sigma_{1,j}} \oplus P_{\sigma_{2,j}} \rightarrow R/pR \rightarrow Q'$$

is zero for all $j$. If nonzero, its image has $\sigma_v$ in its socle. However we know that $P_{\sigma_{1,j}} \oplus P_{\sigma_{2,j}}$ does not contain $\sigma_v$ as a subquotient, so that the map has to be 0.

We prove (ii) implies (i). Let $V$ be the image of the composition

$$\ker(R/pR \rightarrow W) = P^f_{\sigma_v} \hookrightarrow R/pR \rightarrow Q$$

and assume that $V$ is nonzero. Then $V$ is a subrepresentation of $Q$ with a $\sigma_v$-isotypic cosocle. By Lemma 8.3.1 applied to $V \subseteq Q$ we can find $Q'$ a quotient of $Q$ such that $\text{soc}_K(Q') = \text{cosoc}_K(V)$. Since $Q^K_1 = Q'$ by (ii) the map $R/pR \rightarrow Q'$ factors through $(R/pR)_K = P_{\sigma_v}$ (see Theorem 7.3.3), which implies that the composition $P^f_{\sigma_v} \hookrightarrow R/pR \rightarrow Q'$ is zero. This is a contradiction and we have (i).

**Corollary 8.3.3.** Let $Q'$ be a subquotient of $W$ and let $f : R/pR \rightarrow Q'$ be a surjective $K$-equivariant map. Then $f$ factors as $R/pR \rightarrow W \rightarrow Q'$ for a $K$-equivariant surjection $W \rightarrow Q'$.

**Proof.** Let $Q''$ be a quotient of $Q'$ such that $\text{soc}_K(Q'')$ is $\sigma_v$-isotypic (note that $\sigma_v$ appears in $(Q')_K$, hence in $Q'$, as $(R/pR)_K = P_{\sigma_v}$). Since $Q''$ is a subquotient of $W$, it can be written as a subrepresentation of a certain quotient $Q^{(3)}$ of $W$. By Lemma 8.3.1 applied to $\text{soc}_K(Q'') \subseteq Q^{(3)}$ there exists a further quotient $Q^{(4)}$ of $Q^{(3)}$ such that $\text{soc}_K(Q^{(4)}) = \text{cosoc}_K(Q'n)$. Since $Q^{(4)}$ is a quotient of $W$, the part (ii) implies (ii) of Lemma 8.3.2 implies $(Q^{(4)})_K = Q^{(4)}$. The composite map $Q'' \rightarrow Q^{(3)} \rightarrow Q^{(4)}$ is injective since it is injective on the socle of $Q''$ by construction. Consequently we also have $(Q'')_K = Q''$. It then follows from the part (ii) implies (i) of Lemma 8.3.2 that the map $f : R/pR \rightarrow Q'$ factors as $R/pR \rightarrow W \rightarrow Q'$.

**Proposition 8.3.4.** The $R_\infty$-module $M_\infty(W)$ is generated by $r$ elements.

**Proof.** We prove by induction on the length of $Q$ (as a representation of $K$) that if $Q$ is a nonzero quotient of $W$, then $M_\infty(Q)$ is minimally generated by $r$ elements. If $\text{lg}(Q) = 1$, then $Q = \sigma_v$ (as $W_K = P_{\sigma_v}$) and $M_\infty(\sigma_v)$ is minimally generated by $r$ elements by Proposition 8.2.3(i). Now assume that the result is proved for all Serre weights $\sigma_v \in W(\tau'_v)$ and all quotients of $W$ of length $\leq n$. Returning to our fixed $\sigma_v \in W(\tau'_v)$, let $Q$ be a quotient of $W$ of length $n + 1$. If the socle of $Q$ contains a Serre weight $\sigma$ which is not in $W(\tau'_v)$, then $M_\infty(Q) = M_\infty(Q/\sigma)$ and $M_\infty(Q)$ is minimally generated by $r$ elements by induction. Hence we can assume that all the Serre weights in the socle of $Q$ are in $W(\tau'_v)$.
Write $\text{soc}_K(Q) = \sigma_1 \oplus \cdots \oplus \sigma_m$ with $\sigma_i \in W(\tau'_v)$ and use Lemma 8.3.1 to choose a quotient $Q_i$ of $Q$ for $1 \leq i \leq m$ such that the composition $\sigma_i : Q \rightarrow Q_i$ induces an isomorphism $\sigma_i \cong \text{soc}_K(Q_i)$ (and recall $\text{cosoc}_K(Q_i) = \sigma_i$).

Assume first that $[Q_i : \sigma_v] = 1$ for all $1 \leq i \leq m$ such that $\sigma_i \not\equiv \sigma_v$. Then it follows from [HW Thm. 2.22] that, for $i$ such that $\sigma_i \not\equiv \sigma_v$, the quotient $Q_i$ factors through $W \rightarrow P_{\sigma_v}$ and consequently that $Q^{K_i} = Q_i$. For $i$ such that $\sigma_i \cong \sigma_v$, we also have $Q^{K_i} = Q_i$ by the part (i) implies (ii) of Lemma 8.3.2. Now the map $Q \rightarrow \bigoplus_i Q_i$ is injective since it is injective on the socle, so that $Q^{K_1} = Q_i$. This implies that $Q$ is a quotient of $P_{\sigma_v} = (R/pR)_{K_1}$. As $M_{\infty}(P_{\sigma_v})$ is generated by $r$ elements by Proposition 8.2.5 and $M_{\infty}(Q)$ is a quotient of $M_{\infty}(P_{\sigma_v})$, it follows that $M_{\infty}(Q)$ is also generated by $r$ elements. As $M_{\infty}(\sigma_v)$ is minimally generated by $r$ elements by Proposition 8.2.3(ii) and $M_{\infty}(\sigma_v)$ is a quotient of $M_{\infty}(Q)$, we finally have that $M_{\infty}(Q)$ is minimally generated by $r$ elements.

Assume now that there exists at least one $1 \leq i \leq m$ such that $\sigma_i \not\equiv \sigma_v$ and $[Q_i : \sigma_v] \geq 2$. As $\text{cosoc}_K(Q_i)$ is irreducible ( explodes $\sigma_v$), the hypothesis $[Q_i : \sigma_v] \geq 2$ (together with the projectivity of $R/pR$, see Corollary 7.3.4) implies there exists a $K$-equivariant map $R/pR \rightarrow Q_i$ such that $\sigma_v$ occurs only once in its image, in particular which is nonzero and not surjective. Since $Q \rightarrow Q_i$, by the projectivity of $R/pR$ again, we can lift this map to $R/pR \rightarrow Q \rightarrow Q_i$, and we denote by $Q'$ the image of $R/pR \rightarrow Q$ (note that $R/pR \rightarrow Q$ is not surjective either). By Corollary 8.3.3 $Q'$ is a quotient of $W$, hence $\text{cosoc}_K(Q') = \sigma_v$. Moreover we cannot have $Q' = \text{soc}_K(Q') = \text{cosoc}_K(Q') = \sigma_v$, as $\sigma_i = \text{soc}_K(Q_i) \not\equiv \sigma_v$ occurs in $Q'$. Since $Q' \subseteq Q$, we have length($Q'$) < length($Q$) so that, by induction, the $R_{\infty}$-module $M' \overset{\text{def}}{=} M_{\infty}(Q')$ is minimally generated by $r$ elements. Let $\sigma'$ be a Serre weight in the socle of $Q'$ and let $M'' \overset{\text{def}}{=} M_{\infty}(\sigma')$ and $M' \overset{\text{def}}{=} M_{\infty}(Q)$. We have $M/M'' \cong M_{\infty}(Q/\sigma')$, which is minimally generated by $r$ elements by induction. As $\sigma_v$ occurs in $Q'/\sigma'$ (otherwise we would have $\sigma' \equiv \sigma_v = Q'$), we have that $M'/M'' = M_{\infty}(Q'/\sigma')$ is again minimally generated by $r$ elements by induction. We can then apply [Le19 Lemma 4.5] to conclude that $M = M_{\infty}(Q)$ is minimally generated by $r$ elements.

For $j \in \{0, \ldots, f-1\}$ let $V(\alpha_j) \overset{\text{def}}{=} V((1, -1))/W(\mathbb{F}) \cong (\text{Sym}^2(W(\mathbb{F})^2) \otimes \det^{-1})(j)$ be the algebraic representation of $K$ over $W(\mathbb{F})$ as defined in §2.1. As in §7.3 we define the locally algebraic representation $R_{2,j} = V(\alpha_j) \otimes_{W(\mathbb{F})} P_{\sigma_v}$ of $K$ over $W(\mathbb{F})$ (so $R_2 = \oplus_{j} R_{2,j}$). We set

$$R'_{2,j} \overset{\text{def}}{=} \{ x \in R_{2,j} : (x \mod pR_{2,j}) \in P_{\sigma_v} \}$$

using the fixed embedding $\iota_j : P_{\sigma_v} \hookrightarrow R_{2,j}/pR_{2,j}$ from §7.3. This is a $K$-invariant $W(\mathbb{F})$-lattice in $R_{2,j}[1/p]$ such that $pR_{2,j} \subseteq R'_{2,j} \subseteq R_{2,j}$. As the natural map $R'_2/pR'_2 \rightarrow R'_{2,j}/pR'_{2,j}$ is surjective (see the proof of Corollary 8.3.5 below), by Proposition 7.3.1 we have $(R'_{2,j}/pR'_{2,j})_{K_1} = P_{\sigma_v}$ (hence $\text{cosoc}_K(R'_{2,j}/pR'_{2,j}) = \sigma_v$) and a $K$-equivariant short exact sequence

$$0 \rightarrow P_{\sigma_{1,j}} \oplus P_{\sigma_{2,j}} \rightarrow R'_{2,j}/pR'_{2,j} \rightarrow P_{\sigma_v} \rightarrow 0.$$

**Corollary 8.3.5.** For all $j \in \{0, \ldots, f-1\}$ the $R_{\infty}$-module $M_{\infty}(R_{2,j})$ is generated by $r$ elements.

**Proof.** For $j \in \{0, \ldots, f-1\}$, we have $K$-equivariant surjections $R \rightarrow R'_2 \rightarrow R'_{2,j}$, where the first map is the natural projection to $R'_2$ and the second comes from the projection $x = (x')_{j'} \in \oplus_{j'} R_{2,j'} \rightarrow x_j \in R_{2,j}$ (the map $R'_2 \rightarrow R'_{2,j}$ is surjective, as all the maps $R_{2,j'} \rightarrow R_{2,j'}/pR_{2,j'} \rightarrow P_{\sigma_v}$ are surjective, see the definition of $R'_2$). The induced surjection $R/pR \rightarrow R_{2,j}/pR'_{2,j}$ clearly factors
through $R/pR \to W \to R_{2,j}'/pR_{2,j}'$. By Proposition 8.3.4 the $R_\infty$-module $M_\infty(R_{2,j}'/pR_{2,j}')$ is generated by $r$ elements. By Nakayama’s lemma, the $R_\infty$-module $M_\infty(R_{2,j}')$ is also generated by $r$ elements. \hfill $\square$

**Theorem 8.3.6.** Let $j \in \{0, \ldots, f - 1\}$. Then $M_\infty(R_{2,j}')$ is free of rank $r$ over $R_\infty/\mathfrak{p}_r$, where $\mathfrak{p}$ runs over the tame inertial types such that $\sigma_\mathfrak{p} \in JH(\bar{\sigma}(\tau))$ and $\mathfrak{p}_\tau$ is the prime ideal $\ker(R_\infty \to R_{(2,-1),\tau})$, where $(2,-1)$ is in the embedding $\sigma_j : F_v \to E$ and $(1,0)$ elsewhere.

**Proof.** By Corollary 8.3.5 the $R_\infty$-module $M_\infty(R_{2,j}')$ is generated by $r$ elements, i.e. there is a surjection $f : S^\tau \to M_\infty(R_{2,j}')$, where $S \overset{\text{def}}{=} R_\infty/\text{Ann}_R_\infty(M_\infty(R_{2,j}'))$. The representation $R_{2,j}'[1/p] = R_{2,j}/pR_{2,j}$ is a direct sum of distinct absolutely irreducible locally algebraic representations of $K$ as in Corollary 8.2.4 where for all $i$ we have $\sigma_i^{\text{tr}} = V(\alpha_j)$ and $n_i = 1$. As at the end of the proof of Proposition 8.2.5, it follows from [63] and the fact that all the rings $R_{(2,-1),\tau}$ for $\tau$ such that $\sigma_\mathfrak{p} \in JH(\bar{\sigma}(\tau))$ are domains (apply Proposition [4.2.1] and [GK14] Rk. 5.2.2] to $\mathfrak{p} = \mathfrak{p}_\tau$ after a suitable twist) that $S = R_\infty/\mathfrak{p}_r$ for $\mathfrak{p}_r$ as in the statement. Moreover, since the $S[1/p]$-module $M_\infty(R_{2,j}')[1/p]$ is locally free of rank $r$ by Corollary 8.2.4(ii), the localization of $M_\infty(R_{2,j}')[1/p]$ at each prime ideal of $S[1/p]$ is free of rank $r$. Hence (using again [Mat89] Thm. 2.4), we see that $(\ker(f)[1/p])_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of $S[1/p]$, which implies $\ker(f)[1/p] = 0$, and hence $\ker(f) = 0$ since $S$ has no $\mathfrak{p}$-torsion. This finishes the proof. \hfill $\square$

**Remark 8.3.7.** To prove freeness over $S$ in the second half of the proof of Theorem 8.3.6 we could also have argued as in the proof of Theorem 8.3.6 using Corollary 8.2.4(ii) instead of Proposition [8.2.3(ii)]

Set $L_{-1} \overset{\text{def}}{=} \tilde{P}_{\sigma_v}$ and for $j \in \{0, \ldots, f - 1\}$ define a $K$-equivariant $W(\mathbb{F})$-lattice $L_j$ in

$$\tilde{P}_{\sigma_v}[1/p] \oplus (\oplus_{j'=0}^j V(\alpha_j) \otimes_{W(\mathbb{F})} \tilde{P}_{\sigma_v})[1/p] = \tilde{P}_{\sigma_v}[1/p] \oplus (\oplus_{j'=0}^j R_{2,j'})[1/p]$$

by induction by

$$L_j \overset{\text{def}}{=} L_{j-1} \times_{P_{\sigma_v}} R_{2,j'},$$

or equivalently

$$L_j = \{(x_1, (x_{2,j'})_{0 \leq j' \leq j}) \in \tilde{P}_{\sigma_v} \oplus (\oplus_{j'=0}^j R_{2,j'}) : (x_{2,j'} \mod pR_{2,j'}) = (x_1 \mod p\tilde{P}_{\sigma_v})\}$$

in $P_{\sigma_v} \to R_{2,j'}/pR_{2,j'} \forall j' \in \{0, \ldots, j\}$.

Note that $L_{f-1} = R$ (see [7.3]).

**Theorem 8.3.8.** Let $j \in \{-1, \ldots, f - 1\}$. Then $M_\infty(L_j)$ is free of rank $r$ over $R_\infty/\mathfrak{p}_{\lambda,\tau}$, where $\mathfrak{p}_{\lambda,\tau}$ is the prime ideal $\ker(R_\infty \to R_{(2,-1)})$ with $\tau$ running over the tame inertial types such that $\alpha_\mathfrak{p} \in JH(\bar{\sigma}(\tau))$ and $\lambda = (\lambda_{j'})_{0 \leq j' \leq f - 1}$ running over the Hodge–Tate weights such that $\lambda_{j'} \in \{(1,0), (2,-1)\}$ if $0 \leq j' \leq j$ and $\lambda_{j'} = (1,0)$ if $j + 1 \leq j' \leq f - 1$.

**Proof.** Twisting all the Galois deformations by $\varepsilon$, we can replace $\tau_\mathfrak{p}'$ by $\tau_\mathfrak{p}'(1), \{(1,0), (2,-1)\}$ by $\{(2,1), (3,0)\}$ and $\sigma_\mathfrak{p} \in JH(\bar{\sigma}(\tau))$ by $\sigma_\mathfrak{p} \otimes (N_{k/\mathbb{F}_p} \circ \text{det}^{-1})$ in $JH(\bar{\sigma}(\tau))$ (all the deformations now have determinant $\varepsilon^3 \psi^{-1}$). Note first that all the rings $R_{(2,-1)}$ are domains by Proposition 4.2.1 (and [GK14] Rk. 5.2.2] applied to a suitable twist of $\tau_\mathfrak{p}'$ to get $\mathfrak{p} = \tau_\mathfrak{p}'$ as in [4.1]. The proof is
by induction on \(j \geq -1\). If \(j = -1\), this is Proposition [8.2.5]. Assume the statement is true for \(M_\infty(L_{j-1})\) and let us prove it for \(M_\infty(L_j)\). From the exactness of \(M_\infty\) and (60) we deduce
\[
M_\infty(L_j) = M_\infty(L_{j-1}) \times_{M_\infty(R_{2,j}^0)} M_\infty(R_{2,j}^1).
\]

By Theorem [8.3.6] Proposition [8.2.5] and the induction hypothesis, and using (66)
\[
A/I \cap J \sim A/I \times_{A/(I+J)} A/J
\]
(where \(I, J\) are ideals in a commutative ring \(A\)), it is not difficult to see that it is enough to prove
\[
\text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v})) = \text{Ann}_{R_\infty}(M_\infty(L_{j-1})) + \text{Ann}_{R_\infty}(M_\infty(R_{2,j}^1)).
\]

Since the maps \(M_\infty(L_{j-1}) \to M_\infty(P_{\sigma_v}), M_\infty(R_{2,j}^1) \to M_\infty(P_{\sigma_v})\) are surjective, we have inclusions \(\text{Ann}_{R_\infty}(M_\infty(L_{j-1})) \subseteq \text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v}))\) and \(\text{Ann}_{R_\infty}(M_\infty(R_{2,j}^1)) \subseteq \text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v}))\), hence it is enough to prove
\[
\text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v})) \subseteq \text{Ann}_{R_\infty}(M_\infty(L_{j-1})) + \text{Ann}_{R_\infty}(M_\infty(R_{2,j}^1)).
\]

Consider the ring
\[
R_{\infty}^{\leq(3,0),\sigma_v} \overset{\text{def}}{=} R_\infty \otimes_{R_\infty^e} R_{\infty}^{\leq(3,0),\sigma_v} \simeq R_\infty/\cap_{\lambda,\tau} p_{\lambda,\tau},
\]
where \(R_{\infty}^{\leq(3,0),\sigma_v}\) is as in Proposition [4.3.1] and where \(p_{\lambda,\tau} = \ker(R_\infty \to R_\infty^\lambda\tau)\) with \(\tau\) running over the tame inertial types such that \(\sigma_v \otimes (N_{k/\mathbb{F}_p} \circ \det^{-1}) \in \text{JH}(\overline{\sigma(\tau)})\) and \(\lambda = (\lambda')_0 \leq \jmath \leq f - 1\) running over \(\{(2,1), (3,0)\}\). By Proposition [4.3.1] and (57), and increasing \(q\) if necessary, we have for some integer \(h \geq 1\) and certain explicit rings \(R^{(j')}\) that
\[
R_{\infty}^{\leq(3,0),\sigma_v} \simeq \left( \bigotimes_{W(\mathbb{F}_q), 0 \leq \jmath' \leq f - 1} R^{(j')} \right)[X_1, \ldots, X_h]
\]
(again in [GK14] Rk. 5.2.2 and Lemma [8.1.2] as we have conditions on the determinant here). For each \(j'\) in Proposition [4.3.1] four prime ideals of \(R^{(j')}\) are defined that we denote here simply by \(q_1^{(j')}, (2,1), q_2^{(j')}, (3,0)\), \(q_3^{(j')}, (2,1)\), \(q_4^{(j')}, (3,0)\) and that we consider as ideals of \(R_{\infty}^{\leq(3,0),\sigma_v}\). Moreover there is a bijection
\[
\iota : \{\tau : \sigma_v \otimes (N_{k/\mathbb{F}_p} \circ \det) \in \text{JH}(\overline{\sigma(\tau)})\} \rightarrow \{(2,1)^f\}
\]
\[
\tau \mapsto (\iota(\tau))_0 \leq \jmath \leq f - 1
\]
such that
\[
p_{\lambda,\tau} = q_1^{(\iota(\tau))_0} + q_2^{(\iota(\tau))_1} + \ldots + q_4^{(\iota(\tau))_{f-1}}.
\]
For \(\lambda_{j'} \in \{(2,1), (3,0)\}\) set \(I^{(j')} = q_1^{(j')} \cap q_2^{(j')} \lambda_{j'}\). Then by Proposition [8.2.5] Proposition [8.3.6] and Lemma [8.3.2] we deduce
\[
\text{Ann}_{R_\infty}(M_\infty(P_{\sigma_v})) = (p) + I^{(0),(2,1)} + I^{(1),(2,1)} + \ldots + I^{(f-1),(2,1)}
\]
\[
\text{Ann}_{R_\infty}(M_\infty(R_{2,j'}^1)) = I^{(0),(2,1)} + \ldots + I^{(j'-1),(2,1)} + I^{(j'),(3,0)} + I^{(j'+1),(2,1)} + \ldots + I^{(f-1),(2,1)}.
\]
From the definition of \( L_{j-1} \) as an iterated fiber product we have using (66) that

\[
\text{Ann}_{R_{\infty}}(M_{\infty}(L_{j-1})) = \text{Ann}_{R_{\infty}}(M_{\infty}(P_{\sigma})) \cap \left( \bigcap_{0 \leq j' < j - 1} \text{Ann}_{R_{\infty}}(M_{\infty}(R_{2,j}')) \right) \\
= \left( I^{(0),(2,1)} + \ldots + I^{(f-1),(2,1)} \right) \cap \left( \bigcap_{0 \leq j' < j - 1} I^{(0),(2,1)} + \ldots + I^{(j'),(3,0)} + \ldots + I^{(f-1),(2,1)} \right).
\]

In particular, we see that

\[
\sum_{j \leq j' \leq f-1} I^{(j'),(2,1)} \subseteq \text{Ann}_{R_{\infty}}(M_{\infty}(L_{j-1})),
\]

\[
\sum_{0 \leq j' < j - 1} I^{(j'),(2,1)} \subseteq \text{Ann}_{R_{\infty}}(M_{\infty}(R_{2,j})),
\]

to prove (67) it is enough to prove that \( p \in \text{Ann}_{R_{\infty}}(M_{\infty}(L_{j-1})) + \text{Ann}_{R_{\infty}}(M_{\infty}(R_{2,j})) \). But we also have \( I^{(0),(2,1)} + I^{(j),(3,0)} \subseteq \text{Ann}_{R_{\infty}}(M_{\infty}(L_{j-1})) + \text{Ann}_{R_{\infty}}(M_{\infty}(R_{2,j})) \), hence it is enough to prove that \( p \in I^{(j),(2,1)} + I^{(j),(3,0)} \), which is precisely the content of Proposition 4.3.2.

\[\square\]

**Corollary 8.3.9.** The module \( M_{\infty}(R) \) is free of rank \( r \) over \( R_{\infty}/(\cap_{\lambda,\tau} p_{\lambda,\tau}) \), where \( p_{\lambda,\tau} \) is the prime ideal \( \ker(R_{\infty} \to R_{\infty}^{|\tau|}) \) with \( \tau \) running over the tame inertial types such that \( \sigma_v \in \text{JH}(\sigma(\tau)) \) and \( \lambda = (\lambda_j)_{0 \leq j \leq f-1} \) running over the Hodge–Tate weights such that \( \lambda_j \in \{(1,0),(2,-1)\} \) for all \( j \). In particular, \( \dim R M_{\infty}(R)/m_{\infty} = r \).

Recall that we have defined the \( K \)-representation \( (\text{Proj}_{K/Z} \sigma_v)/m_{K_1}^2 \) with cosocle \( \sigma_v \) (see e.g. §7.3). From Corollary 8.3.9 Proposition 8.2.3(i) and the isomorphism \( R/pR \cong (\text{Proj}_{K/Z} \sigma_v)/m_{K_1}^2 \) of Corollary 7.3.4 we deduce the following result.

**Theorem 8.3.10.** The surjection

\[
(\text{Proj}_{K/Z} \sigma_v)/m_{K_1}^2 \to \sigma_v
\]

induces an isomorphism of (nonzero finite-dimensional) \( \mathbb{F} \)-vector spaces

\[
M_{\infty}((\text{Proj}_{K/Z} \sigma_v)/m_{K_1}^2)/m_{\infty} \cong M_{\infty}(\sigma_v)/m_{\infty}.
\]

**Remark 8.3.11.** The exactness of the functor \( M_{\infty} \) shows that the isomorphism in Theorem 8.3.10 is of course totally wrong without quotienting by \( m_{\infty} \).

### 8.4. Gelfand–Kirillov dimensions

We keep all our previous notation. We recall our assumptions: \( F \) is a totally real number field unramified at \( p \), \( D \) is a quaternion algebra of center \( F \) split above \( p \) and at not more than one infinite place, \( v \) is a fixed place of \( F \) above \( p \) and \( \tau : G_F \to \text{GL}_2(\mathbb{F}) \) is a continuous representation satisfying the following conditions: \( \tau|_{G_F(\mathbb{A}^F)} \) is absolutely irreducible, \( \tau_w \) is generic in the sense of [BP12] Def. 11.7] if \( w \mid p \), \( v \neq w \), \( \tau_v \) is semisimple generic in the sense of §8.1 (the latter implies \( p > 19 \)) and \( R_{\tau_w} \) is formally smooth over \( W(\mathbb{F}) \) if \( w \in (SD \cup S_v) \setminus S_p \).

We choose \( w_1 \), \( S \) and \( U = \prod U_w \) as in §8.1. For Serre weights \( \sigma_w \in W(\mathbb{F}) \), \( w \in S_p \setminus \{v\} \), we consider the following admissible smooth representation \( \pi \) of \( \text{GL}_2(F_v) \) over \( \mathbb{F} \) with central
The Hecke operators are applicable to...the GL(2) group. Thus, we can apply Theorem 6.4.7 to GL(2) modules. The...case, we use Theorem 6.4.7 (for $\pi = \pi_v'$). It follows...$\pi = \pi_v'$ and Theorem 8.3.10 (choosing $M_\infty = M_\infty^{\infty}$ as in §8.1 for $\sigma_p^v$ as in (56)) with $JH(\sigma(\pi_v')) \cap W(\pi_v') = \{\sigma_w\}$ for all $\sigma_w \in W(\pi_v')$ we have

$$\pi|_{m_2}^K : \sigma_v = [\text{soc}_K(\pi) : \sigma_v],$$

so that $\pi$ satisfies assumption (ii) in Theorem 6.4.7. Finally, we prove that $JH(\pi^I_1) = JH(D_I(\pi_v'))$ (up to multiplicity), and so by Lemma 6.4.3 $\pi$ satisfies assumption (iii) in Theorem 6.4.7. We only give the proof in the definite case, the indefinite case can be treated similarly (see e.g. [70]) below. The $K$-equivariant embedding $\bigoplus_{\sigma_v \in \pi_v'} \sigma_v^{m_\sigma v} \rightarrow \pi$, where $m_\sigma v = [\text{soc}_K(\pi) : \sigma_v]$, induces a $K \times (U^v/V^v)$-equivariant morphism

$$\bigoplus_{\sigma_v \in \pi_v'} \sigma_v^{m_\sigma v} \otimes \mathbb{F} \bigotimes_{\sigma_v \in \pi_v'} \sigma_v \rightarrow \lim_{V_v} S(V^vV_v,F)[m],$$

which is injective because $\bigotimes_{\sigma_v \in \pi_v'} \sigma_v$ is irreducible. By [Bre14] Lemma 9.2, the last embedding extends to an embedding

$$\bigoplus_{\sigma_v \in \pi_v'} D_0,\sigma_v(\mathbb{F})^{m_\sigma v} \otimes \mathbb{F} \bigotimes_{\sigma_v \in \pi_v'} \sigma_v \rightarrow \lim_{V_v} S(V^vV_v,F)[m]$$

and gives in turn an embedding

$$\bigoplus_{\sigma_v \in \pi_v'} D_0,\sigma_v(\mathbb{F})^{m_\sigma v} \rightarrow \pi.$$
and $\mathcal{M}_\infty$ is endowed with its natural profinite topology. It follows from [GN, Lemma A.16], Lemma 5.1.2 and [57] that we have (where the grade $j_A$ is as in §5.1)

\[(68) \quad j_{R,\infty}[K_1/Z_1](\mathcal{M}_\infty) \geq j_F[K_1/Z_1](\mathcal{M}_\infty/\mathfrak{m}_\infty) = \dim(K_1/Z_1) - \dim_{GL_2(F_v)}(\pi) = 3[F_v : \mathbb{Q}_p] - \dim_{GL_2(F_v)}(\pi).
\]

Since $\mathcal{M}_\infty$ is free of finite type over $S_\infty[K_1/Z_1]$, we have $j_{S_\infty[K_1/Z_1]}(\mathcal{M}_\infty) = 0$. It then follows from [GN, Lemma A.19] (together with [GN, Def. A.2] and [GN, Prop. A.4(1)]) that

\[(69) \quad j_{R,\infty}[K_1/Z_1](\mathcal{M}_\infty) = (\dim(R_\infty) + \dim(K_1/Z_1)) - (\dim(S_\infty) + \dim(K_1/Z_1)) = 2[F_v : \mathbb{Q}_p],
\]

where the last equality follows from (57) and the definition of $S_\infty$. Combining (68) and (69), we deduce $2[F_v : \mathbb{Q}_p] \geq 3[F_v : \mathbb{Q}_p] - \dim_{GL_2(F_v)}(\pi)$, i.e. $\dim_{GL_2(F_v)}(\pi) \geq [F_v : \mathbb{Q}_p]$, which finishes the proof.

Recall that for any Serre weight $\sigma_v$ we have defined in §6 the injective envelope $\text{Inj}_{K/Z_1} \sigma_v$ with socle $\sigma_v$.

**Theorem 8.4.2.** There is an integer $r \geq 1$ such that $\pi[\mathfrak{m}_{K_1}^2] \cong (\oplus_{V, \tau \in W(\mathcal{F}_v)} \tilde{D}_{\sigma_v})^{\oplus r}$, where $\tilde{D}_{\sigma_v}$ is the largest subrepresentation of $(\text{Inj}_{K/Z_1} \sigma_v)[\mathfrak{m}_{K_1}^2]$ containing $\sigma_v$ with multiplicity 1 (= its socle) and no other Serre weights of $W(\mathcal{F}_v)$. In particular, each irreducible constituent of $\pi[\mathfrak{m}_{K_1}^2]$ has multiplicity $r$.

**Proof.** The existence of $\tilde{D}_{\sigma_v}$ is proven in Corollary 6.3.13(1). It follows from its construction in [DL] §6.2 and [CEG+16] that $\mathcal{M}_\infty$ (see part (ii) of the proof of Theorem 8.4.1) is projective of finite type over $S_\infty[K_1/Z_1]$, where $S_\infty[K_1/Z_1]$ is the largest quotient of $S_\infty[K_1]$ on which the center of $K = GL_2(O_{F_1})$ acts by $\psi|_{F_1}$. In particular, $\mathcal{M}_\infty/(p, x_1, \ldots, x_{4|S|+q-1})$ is finite projective over $F[K_1]$. As $\text{Hom}_{F[K_1]}^{cont}(\mathcal{M}_\infty, \sigma_v^\vee) \neq 0$ if and only if $\sigma_v \in W(\mathcal{F}_v)$, we deduce

\[\mathcal{M}_\infty/(p, x_1, \ldots, x_{4|S|+q-1}) \cong \oplus_{\sigma_v \in W(\mathcal{F}_v)} (\text{Proj}_{K/Z_1} \sigma_v^\vee)^{\oplus m_{\sigma_v}}\]

for some integers $m_{\sigma_v} \geq 1$ (in fact $m_{\sigma_v} \geq r$, where $r \geq 1$ is as in Proposition 8.2.3(1)). This implies by the definition of $\tilde{D}_{\sigma_v}$

\[\text{Hom}_{F[K_1]}^{cont} \left( \mathcal{M}_\infty/(p, x_1, \ldots, x_{4|S|+q-1}), \tilde{D}_{\sigma_v}^\vee \right) \sim \text{Hom}_{F[K_1]}^{cont} \left( \mathcal{M}_\infty/(p, x_1, \ldots, x_{4|S|+q-1}), \sigma_v^\vee \right)\]

and hence taking on both sides the subspaces where $\mathfrak{m}_\infty$ acts by 0 ($\mathfrak{m}_\infty$ acts through the action of $R_\infty$ on $\mathcal{M}_\infty/(p, x_1, \ldots, x_{4|S|+q-1})$) we get

\[\text{Hom}_{F[K_1]}^{cont} \left( \mathcal{M}_\infty/\mathfrak{m}_\infty, \tilde{D}_{\sigma_v}^\vee \right) \sim \text{Hom}_{F[K_1]}^{cont} \left( \mathcal{M}_\infty/\mathfrak{m}_\infty, \sigma_v^\vee \right).
\]

Using $\mathcal{M}_\infty/\mathfrak{m}_\infty \cong \pi^\vee$ this last isomorphism can be rewritten

\[\text{Hom}_K(\tilde{D}_{\sigma_v}, \pi) = \text{Hom}_K(\tilde{D}_{\sigma_v}, \pi[\mathfrak{m}_{K_1}^2]) \sim \text{Hom}_K(\sigma_v, \pi) = \text{Hom}_K(\sigma_v, \text{soc}_K \pi).
\]

Since $\text{soc}_K \pi = (\oplus_{\sigma_v \in W(\mathcal{F}_v)} \sigma_v)^{\oplus r}$ by Proposition 8.2.3(1), we deduce an inclusion

\[(70) \quad (\oplus_{\sigma_v \in W(\mathcal{F}_v)} \tilde{D}_{\sigma_v})^{\oplus r} \subseteq \pi[\mathfrak{m}_{K_1}^2].\]

But it follows from Corollary 6.3.13(1) and Theorem 8.3.10 that $\pi[\mathfrak{m}_{K_1}^2]$ cannot be (strictly) larger, whence the isomorphism of the statement. The last sentence in the statement then follows from Corollary 6.3.13(1) and (iii).

$\square$
Corollary 8.4.3. Let \( x : R_\infty \to \mathcal{O}' \) be any homomorphism of local \( W(\mathbb{F}) \)-algebras, where \( \mathcal{O}' \) is the ring of integers of a finite extension \( E' \) of \( E \), and set

\[
V(x) \overset{\text{def}}{=} \text{Hom}_{\mathcal{O}'}^\text{cont}(M_\infty \otimes_{R_\infty,x} \mathcal{O}', \mathcal{O}').
\]

Then \( V(x) \) is a nonzero admissible unitary Banach representation of \( GL_2(F_v) \) over \( E' \) with a \( GL_2(F_v) \)-invariant unit ball (given by \( \text{Hom}_{\mathcal{O}'}^\text{cont}(M_\infty \otimes_{R_\infty,x} \mathcal{O}', \mathcal{O}') \)) lifting \( \pi \otimes_F \mathbb{F}' \), where \( \mathbb{F}' \) is the residue field of \( \mathcal{O}' \).

Proof. The fact that \( V(x) \) is an admissible unitary Banach representation of \( GL_2(F_v) \) follows from [CEG+16] Prop. 2.13. We need to prove \( V(x) \neq 0 \) (note that we know \( M_\infty \otimes_{R_\infty,x} \mathcal{O}' \neq 0 \), as \( M_\infty/m_\infty \neq 0 \), but it could be \( p \)-power torsion). However, since \( M_\infty \) is free of finite type over \( S_\infty[[K_1/Z_1]] \), it follows from [GN] Cor. A.29 applied to \( M = M_\infty \), \( A = S_\infty[[K_1/Z_1]] \) and \( B = R_\infty[[K_1/Z_1]] \) (using (67)) that \( M_\infty \) is a Cohen–Macaulay \( R_\infty[[K_1/Z_1]] \)-module. By Theorem 8.4.1 (68), and [69] we have

\[
j_{R_\infty[[K_1/Z_1]]}(M_\infty) = j_{\mathbb{F}'[[K_1/Z_1]]}(M_\infty/m_\infty) = 2[F_v : \mathbb{Q}_p],
\]

and it then follows from [GN] Cor. A.30 ("Miracle Flatness") that \( M_\infty \) is flat over \( R_\infty \). Hence \( M_\infty \otimes_{R_\infty,x} \mathcal{O}' \) is flat over \( \mathcal{O}' \) by base change, and the result easily follows by [ST02] Thm. 1.2.

Remark 8.4.4. Under slightly more general hypotheses on \( \tau \), one can prove Theorem 8.4.1 Theorem 8.4.2 and Corollary 8.4.3 with \( \pi \) replaced by the "minimal local factor" of [BDJ14 §3.3] and [EGS15 §6.5]. The strategy is completely similar using Theorem 6.4.7, the patching functor \( M^\text{min}_\infty \) of [EGS15 §6.5] (and the "big" minimal patched module of [DL §6]), and the variant of Corollary 8.3.9 with \( M^\text{min}_\infty \) instead of \( M_\infty = M^\tau_\infty \), where we now have \( r = 1 \). Details are left to the reader.

Corollary 8.4.5. For any compact open subgroup

\[
V^v = \prod_{w \notin S_D \cup S_\sigma} (\mathcal{O}_D)_w \times \prod_{w \in (S_D \cup S_\sigma) \setminus \{v\}} V_w \subseteq \prod_{w \neq v} (\mathcal{O}_D)_w
\]

such that \( V_w \) is a subgroup of \( 1 + pM_2(\mathcal{O}_{F_w}) \) for \( w \in S_p \setminus \{v\} \) and such that \( \pi \neq 0 \), where

\[
\pi \overset{\text{def}}{=} \lim_{\overrightarrow{V_v}} \text{Hom}_{G_F}(\tau, H^1_{dR}(X_{V^{\text{def}}V_v} \times_F F, \mathbb{F}))
\]

in the indefinite case,

\[
\pi \overset{\text{def}}{=} \lim_{\overrightarrow{V_v}} S(V^{\text{def}}V_v, F)[m]
\]

in the definite case,

we have \( \dim_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p] \).

Proof. Note that the ideal \( m \) in the definite case is as in Remark 8.1(ii) for \( S \) big enough (the resulting eigenspace does not depend on \( S \) by [BDJ10 Lemma 4.6(a)]). We prove the indefinite case only, the definite case being similar. We can and do choose a place \( w_1 \) as in §8.1.

(i) We first prove \( \dim_{GL_2(F_v)}(\pi) \leq [F_v : \mathbb{Q}_p] \). Since the Gelfand–Kirillov dimension of a subspace is at most as big as the one of the space, it is enough to prove this upper bound for a smaller \( V \). In particular, we can assume that \( V_{w_1} \) is a subgroup of the group of matrices that are upper-triangular unipotent mod \( w_1 \) and that \( V_w \) is a subgroup of \( 1 + pM_2(\mathcal{O}_{F_w}) \) which is normal in
GL\(_2(\mathcal{O}_{F_w})\) for \(w \in S_p\{v\}\). Let \(S \overset{\text{def}}{=} S_D \cup S_T\) and \(U = \prod_w U_w\) with \(U_w \overset{\text{def}}{=} V_w\) if \(w \notin S_p\) and \(U_w \overset{\text{def}}{=} (\mathcal{O}_D)_{w}^{*} \cong GL_2(\mathcal{O}_{F_w})\) if \(w \in S_p\), then \(S\) and \(U\) satisfy all the conditions in §8.1 and we have

\[
\pi \cong \lim_{\overset{\longrightarrow}{V_w}} \text{Hom}_{U'/V'} \left( \otimes_{w \in S_p\{v\}} (\text{Ind}_{V_w}^{GL_2(\mathcal{O}_{F_w})} 1)_{Z} \right) \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})),
\]

where \((\text{Ind}_{V_w}^{GL_2(\mathcal{O}_{F_w})} 1)_{Z}\) is the maximal quotient of \(\text{Ind}_{V_w}^{GL_2(\mathcal{O}_{F_w})} 1\) on which the center of \(GL_2(\mathcal{O}_{F_w})\) acts by \(w^{-1} I_{F_w}\). Writing each \((\text{Ind}_{V_w}^{GL_2(\mathcal{O}_{F_w})} 1)_{Z}\) as a successive extension of Serre weights for \(GL_2(\mathcal{O}_{F_w})\), an obvious dēvissage shows that it is enough to prove that for all Serre weights \((\sigma_{w})_{w \in S_p\{v\}}\), we have

\[
\dim_{GL_2(F_v)} \left( \lim_{\overset{\longrightarrow}{V_w}} \text{Hom}_{U'/V'} \left( \otimes_{w \in S_p\{v\}} \sigma_{w}, \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})) \right) \right) \leq [F_v : \mathbb{Q}_p].
\]

But this follows from [55] and Theorem 8.4.1. In fact, using

\[
\text{Hom}_{U'/V'} \left( -, \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})) \right) \cong \text{Hom}_{G_F} (\tau, \text{Hom}_{U'/V'} \left( -, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F}) \right))
\]

together with

\[
\text{Hom}_{G_F} (\tau, \text{Hom}_{U'/V'} \left( -, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F}) \right)) \cong \text{Hom}_{G_F} (\tau, \text{Hom}_{U'/V'} \left( -, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})_{m} \right))
\]

(for \(m\) as in Remark 8.1.3(ii)) and the fact that \(H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})_{m}\) is an injective representation of \(U'/V'\) over \(\mathbb{F}\) (since \(m\) is non-Eisenstein), we easily deduce that, in the above dēvissage, \(\pi\) as in \([71]\) contains

\[
\lim_{\overset{\longrightarrow}{V_w}} \text{Hom}_{U'/V'} \left( \otimes_{w \in S_p\{v\}} \sigma_{w}, \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})) \right)
\]

for at least one tuple \((\sigma_{w})_{w \in S_p\{v\}}\) with \(\sigma_{w} \in W(\mathbb{F}^\circ_{w})\) for all \(w \in S_p\{v\}\) (since \(\pi \neq 0\)). (We also use that \(\text{Hom}_{U'/V'} \left( \otimes_{w \in S_p\{v\}} \sigma_{w}, H^1_{et}(X_{V'V_e} \times_F \mathcal{F}, \mathcal{F})_{m} \right) \neq 0\) if and only if \(\sigma_{w} \in W(\mathbb{F}^\circ_{w})\) for all \(w\), by [BDJ0] Lemma 4.10.) This implies \(\dim_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]\) by Theorem 8.4.1 (for \(\pi\) as in \([71]\)).

(ii) We now prove \(\dim_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]\) for \(\pi\) as in the statement. Set \(V'' = \prod_{w \neq v} V'_w\) with \(V'_w = V_w\) if \(w \neq w_1\) and \(V'_w = \text{subgroup of } (\mathcal{O}_D)_{w}^{*}\) of matrices that are upper-triangular unipotent mod \(w_1\). By Ihara's Lemma at the place \(w_1\), which is easy here thanks to all the assumptions on \(w_1\), we have for sufficiently small \(V_w\) that

\[
\text{Hom}_{G_F} (\tau, H^1_{et}(X_{V''V_e} \times_F \mathcal{F}, \mathcal{F})) \cong \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V''V_e} \times_F \mathcal{F}, \mathcal{F}))
\]

and hence a \(GL_2(F_v)\)-equivariant isomorphism

\[
\pi \overset{\oplus 2}{=} \pi' \overset{\text{def}}{=} \lim_{\overset{\longrightarrow}{V_v}} \text{Hom}_{G_F} (\tau, H^1_{et}(X_{V''V_e} \times_F \mathcal{F}, \mathcal{F})).
\]

In particular, \(\dim_{GL_2(F_v)}(\pi) = \dim_{GL_2(F_v)}(\pi')\). Replacing \(V'\) by \(V''\), we can thus assume that \(V_{w_1}\) is the subgroup of \((\mathcal{O}_D)_{w_1}^{*}\) of matrices that are upper-triangular unipotent mod \(w_1\). It is enough to prove \(\dim_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]\) when \(V_w = 1 + p M_2(\mathcal{O}_{F_w})\) for \(w \in S_p\{v\}\) (as \(\dim_{GL_2(F_v)}(\pi)\) for the subgroup \(V_w\) of \(1 + p M_2(\mathcal{O}_{F_w})\) can only grow, but is anyway bounded by \([F_v : \mathbb{Q}_p]\) by (i)). But this follows from the last assertion in part (i) above.

\begin{remark}
If \(V'' = \prod_{w \in S} \mathcal{O}_D_{w}^{*} \prod_{w \in S_p\{v\}} V_w\) for some finite set \(S\) containing \(S_D \cup S_T\) such that \(R_{\tau_{w'}}\) is formally smooth for \(w \in S\{S_p\}\), the same proof gives \(\dim_{GL_2(F_v)}(\pi) = [F_v : \mathbb{Q}_p]\).
\end{remark}
Without assuming $V_w \subseteq 1 + p M_2(\mathcal{O}_{F_w})$ for $w \in S_p\backslash\{v\}$, the above proof still gives the bound $\dim_{GL_2(F_v)}(\pi) \leq [F_v : \mathbb{Q}_p]$. 
References


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