MAT247H1S ALGEBRA II Term Test Solutions for 1 - 3

1.(a) Take a basis of V including $(1,0,2)$, then apply Gram-Schmidt. For example, take $\{(1,0,2),(1,0,0),(0,1,0)\}$, then after Gram-Schmidt we get $\{\frac{1}{\sqrt{5}}(1,0,2), \frac{1}{\sqrt{5}}(2,0,-1),(0,1,0)\}$. So $\{\frac{1}{\sqrt{5}}(2,0,-1),(0,1,0)\}$ is an orthonormal basis for $W^{\perp}$.

(b) $U(a,b,c) = \left\langle (a,b,c), \frac{1}{\sqrt{5}}(2,0,-1) \right\rangle \frac{1}{\sqrt{5}}(2,0,-1) + \langle (a,b,c),(0,1,0)\rangle (0,1,0)$ $= (\frac{2}{5}(2a-c), b, \frac{-1}{5}(2a-c))$.

(c) We proved in class that a 3×3 matrix is orthogonal iff its rows (or columns) form an orthogonal basis for $\mathbb{R}^3$ relative to the standard inner product. One approach is to take T such that $T(e_1) = x_1$ and $T(e_2) = x_2$, where $\{x_1, x_2\}$ is an orthonormal basis for $W^{\perp}$

Hence we take $T = \begin{pmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \end{pmatrix}$. Note the columns are the basis we got in (a). We have $T \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}$.

2.(a) $[T]_\beta = \begin{pmatrix} & 1 & \\ & & 2 \\ -1 & & \end{pmatrix}$, so $[T^*]_\beta = \begin{pmatrix} & & -1 \\ 1 & & \\ & 2 & \end{pmatrix}$.

(b) $T^*(ax^2+bx+c) = (c+a)(x^2-1) + ax + 2b$.

(c) $[TT^* - T^*T]_\beta = \begin{pmatrix} 0 & & \\ & 3 & \\ & & -3 \end{pmatrix} \neq 0$, so T is not normal.

3. Simplest approach: Since T is normal, theorems proved in class tell us that T is unitary if and only if $|\lambda| = 1$ for every eigenvalue λ of T, and T is self adjoint if and only if $\lambda \in \mathbb{R}$ for every eigenvalue λ of T.

(a) Characteristic polynomial of T is $\lambda^4 - 1$, whose roots are $1, i, -1, -i$. They are all possible eigenvalues for T and have modulus 1. So T is unitary.

(b) Since $T^2 \neq I_V$, eigenvalues of T can only be i adn $-i$, which are not real. So T is not self-adjoint.

(c) T is normal, so $T^2 + 3T^*$ is normal, and is invertible if and only if all eigenvalues of $T^2 + 3T^*$ are non zero. If λ is an eigenvalue of T, i.e. $T(x) = \lambda x$ for some $x \neq 0$, then T is normal $\Rightarrow T^*(x) = \overline{\lambda} x$. The eigenvalues of $T^2 + 3T^*$ have the form $\lambda^2 + 3\overline{\lambda}$ for λ an eigenvalue of T.

One compute $\lambda^2 + 3\overline{\lambda} \neq 0$ for $\lambda = i$ or $-i$, so $T^2 + 3T^*$ is invertible.