MAT 247S - Problem Set 7

Due Thursday March 26

Questions 2a), 2b), 4, 5 and 7 will be marked.

1. Let T be the linear operator on $V = \mathbb{R}^8$ whose Jordan canonical form is:

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

a) Find the dot diagram for each eigenvalue of T,

b) Find $\dim N((T - \lambda \cdot 1_V)^j)$, for every positive integer j and every eigenvalue λ of T.

c) Find the minimal polynomial of T.

2. Let $T : \mathbb{R}^8 \to \mathbb{R}^8$ be the linear operator whose matrix $[T]_\beta$ relative to the standard basis β for $\mathbb{R}^8$ is given by

$$[T]_\beta = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 4 \end{pmatrix}.$$

a) Find the dot diagram for each eigenvalue of T.

b) Find the Jordan canonical form of T.

c) Find the minimal polynomial of T.

3. Let $T \in \mathcal{L}(V)$, where V is a complex vector space of dimension 10. Suppose that the eigenvalues of T are i, -1 and $\sqrt{3}$, and

$$\begin{aligned} &\dim K_i = 4, \quad \dim R(T - i \cdot 1_V) = 7 \\ &R(T + 1_V) = R((T + 1_V)^2) \\ &\dim K_{\sqrt{3}} = 3, \quad \dim R(T - \sqrt{3} \cdot 1_V) = 9, \end{aligned}$$

where K_i and $K_{\sqrt{3}}$ are the generalized eigenspaces corresponding to the eigenvalues i and $\sqrt{3}$, respectively.

a) Find the Jordan canonical form of T.

b) Find the minimal polynomial of T.

4. Let V be a real vector space. Suppose that $T \in \mathcal{L}(V)$ has characteristic polynomial $f(t) = -(t-1)^9(t+1)^4$,

$$\text{nullity}(T - 1_V) = 3, \quad N(T - 1_V) \subset R((T - 1_V)^2),$$
$$\text{and } \text{rank}((T + 1_V)^3) = \text{rank}((T + 1_V)^2) = \text{rank}(T + 1_V) - 1.$$

   a) Find the dot diagram associated to each eigenvalue of T.
   b) Find the Jordan canonical form of T.
   c) Find the minimal polynomial of T.

5. Let V be a nonzero n dimensional complex vector space. Assume that n is even. Let $T \in \mathcal{L}(V)$ and let $f(t)$ be the characteristic polynomial of T and let $p(t)$ be the minimal polynomial of T. Assume that $f(t) = (p(t))^2$.
   a) Let λ be an eigenvalue of T and let K_λ be the generalized eigenspace of T corresponding to the eigenvalue λ. Prove that $\dim K_\lambda$ is even.
   b) Prove that T is diagonalizable if and only if T has $n/2$ distinct eigenvalues.

6. Let T_1 and T_2 be linear operators on a nine-dimensional complex vector space V. Suppose that the characteristic polynomials of T_1 and T_2 are both equal to $-(t-i)^6(t+3)^3$ and the minimal polynomials of T_1 and T_2 are both equal to $(t-i)^3(t+3)^2$. Prove that T_1 and T_2 have the same Jordan canonical form if and only if $\dim \text{N}(T_1 - i \cdot 1_V) = \dim \text{N}(T_2 - i \cdot 1_V)$.

7. Let T be a linear operator on an n-dimensional complex vector space. Suppose that the characteristic polynomial of T is equal to $(-1)^n(t-3)^n$. Prove that T and $T^2 - 2T$ have the same Jordan canonical form.

8. §7.2, #13. (*Note*: Results on Jordan canonical form cannot be used to solve this problem because it is not known ahead of time that the characteristic polynomial of T splits over F. For part b), the corollary on page 51 of the text is useful.)

9. §7.2, #14.

10. Let T_1 and T_2 be nilpotent linear operators on a finite-dimensional vector space. Suppose that T_1 and T_2 have the same minimal polynomial and $\dim N(T_1) = \dim N(T_2)$. Let β be an ordered basis of V and set $A = [T_1]_\beta$ and $B = [T_2]_\beta$.
    a) Show that if $\dim V = 6$, then A and B are similar matrices.
    b) Show that if $\dim V = 7$, the matrices A and B might not be similar.

11. Let $T : V \to V$ be a linear operator on a finite-dimensional vector space V. Assume that the characteristic polynomial of T splits over F. Let $\lambda_1, \lambda_2, \ldots \lambda_k$ be the distinct eigenvalues of T. Prove that T is diagonalizable if and only if $N(T - \lambda_j 1_V) = N((T - \lambda_j 1_V)^2)$ for $1 \leq j \leq k$.

12. §7.2, #17.