MAT 247S - Problem Set 4

Due Thursday February 12th

NOTE: Questions 3b), 3c), 4 and 6 will be marked.

1. Let
$$A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}.$$
   a) Find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$.
   b) Let $V = \mathbb{R}^3$ with the standard inner product. Let T be the linear operator on V such that $[T]_\beta = A$. Compute the spectral decomposition of T. That is, find the formula for each orthogonal projection that occurs in the spectral decomposition of T, and express T as a linear combination of these orthogonal projections.
2. Let $V = \mathbb{C}^4$ with the standard inner product. Let β be the standard ordered basis for V. Let T be the linear operator on V such that
$$[T]_\beta = A = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$
   a) Show that T is a normal operator.
   b) Find a unitary matrix P and a diagonal matrix D such that $PAP^* = D$.
   c) Compute the spectral decomposition of T. That is, find the formula for each orthogonal projection that occurs in the spectral decomposition of T, and express T as a linear combination of these orthogonal projections.
3. Let $V = \mathbb{C}^4$ with the standard inner product. Let β be the standard ordered basis for V. Let T be the linear operator on V such that
$$[T]_\beta = A = \begin{pmatrix} 0 & 0 & 2i & 0 \\ 0 & 0 & 0 & -2i \\ 2i & 0 & 0 & 0 \\ 0 & -2i & 0 & 0 \end{pmatrix}.$$
   a) Show that T is a normal operator.
   b) Find a unitary matrix P and a diagonal matrix D such that $PAP^* = D$.
   c) Compute the spectral decomposition of T. That is, find the formula for each orthogonal projection that occurs in the spectral decomposition of T, and express T as a linear combination of these orthogonal projections.
4. Let $T : V \to V$ be a linear operator on a finite-dimensional complex inner product space. Suppose that $T^2 = -I_V$. Prove that T is unitary if and only if there exists a subspace W of V such that $T(x) = ix$ for all $x \in W$ and $T(x) = -ix$ for all $x \in W^\perp$.
5. Let V be a finite-dimensional real inner product space and let m be a positive integer. Suppose that $T \in \mathcal{L}(V)$, T is normal, and $T^m = T_0$ (that is, $T^m(x) = \mathbf{0}$ for all $x \in V$). Prove that $T = T_0$.
6. Let V be a finite-dimensional real inner product space.
   a) Suppose that $n = \dim V$ is even. Prove that there exists $T \in \mathcal{L}(V)$ such that T is orthogonal and $T^2 = -I_V$. (*Hint*: Can you find an orthogonal matrix $A \in M_{n\times n}(\mathbb{R})$ such that $A^2 = -I_n$?)
   b) Suppose that $T \in \mathcal{L}(V)$ is orthogonal and $T^2 = -I_V$. Prove that $\langle T(x), x\rangle = 0$ for all $x \in V$.