MAT 247S - Problem Set 3

Due Thursday February 5th

NOTE: Questions 6a), 8c), 9, and 10 will be marked.

1. Let $V = \mathbb{C}^4$ with the standard inner product. Let T be the linear operator on V whose matrix with respect to the standard basis β for V is given by:

$$A = [T]_\beta = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

   a) Show that T is a normal operator.
   b) Find an orthonormal basis β' for V consisting of eigenvectors of T.

2. Let $A = \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}$. In each case below, answer the following three questions: *(i)* Is B self-adjoint? *(ii)* Is B normal? *(iii)* Is B unitary? (Explain your answers fully).

   a) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + iA$
   b) $B = I - A$
   c) $B = (I + A)(I - A)^{-1}$.

4. §6.4, #6
5. §6.5, #9
6. In each case below, answer the following three questions: *(i)* Is T self adjoint? *(ii)* Is T normal? *(iii)* Is T unitary or orthogonal?

   a) Let V be a 4-dimensional complex inner product space. Let β be an orthonormal basis for V. Suppose that $T \in \mathcal{L}(V)$ is such that

$$[T]_\beta = A = \begin{pmatrix} 0 & 0 & i & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ i & -i & 0 & 0 \end{pmatrix}.$$

   b) Let $V = M_{2\times 2}(\mathbb{R})$, with inner product $\langle A, B \rangle = \mathrm{trace}(AB^t)$, for $A, B \in V$. Define $T \in \mathcal{L}(V)$ by $T(A) = -A^t$, $A \in V$.

7. Let $V = \mathbb{C}^2$ with the inner product

$$\langle (a_1, a_2), (b_1, b_2) \rangle = a_1\overline{b_1} + a_1\overline{b_2} + a_2\overline{b_1} + 2a_2\overline{b_2}, \qquad (a_1, a_2), (b_1, b_2) \in V.$$

   Let $T \in \mathcal{L}(V)$ be defined by $T(a_1, a_2) = ((1+i)a_1 + (1-i)a_2, (1-i)a_1 + (1+i)a_2)$, $a_1, a_2 \in \mathbb{C}$.

   a) Show that T is diagonalizable.
   b) Find $T^*(a_1, a_2)$ for all $a_1, a_2 \in \mathbb{C}$.
   c) Show that if β is an orthonormal basis for V, then the matrix $[T]_\beta$ is not a diagonal matrix.

8. Let V be an inner product space. Let W_1 and W_2 be nonzero finite-dimensional subspaces of V. Let T_j be the orthogonal projection of V on W_j, $j = 1, 2$. Let c_1, $c_2 \in F$ and $U = c_1T_1 + c_2T_2$.
   a) Suppose that c_1, $c_2 \in F$ and $c_1\overline{c_2} \in \mathbb{R}$. Prove that U is normal.
   b) Suppose that $T_1T_2 = T_2T_1$. Prove that U is normal.
   c) Let $V = \mathbb{C}^2$ with the standard inner product. Find two one-dimensional subspaces W_1 and W_2 of V and complex numbers c_1, $c_2 \in \mathbb{C}$ such that $U = c_1T_1 + c_2T_2$ is not normal.

9. Let V be a finite-dimensional complex inner product space. Suppose that $T \in \mathcal{L}(V)$ satisfies $T - T^* = 2i\, I_V$, where $I_V : V \to V$ is the identity operator: $I_V(x) = x$ for all $x \in V$.
   a) Prove that T is normal.
   b) Prove that if λ is an eigenvalue of T, then $\lambda - i$ is a real number.
   c) Prove that $T + I_V$ and $T + (-2i + 1)I_V$ are invertible.
   d) Let $U = (T + (-2i + 1)I_V)(T + I_V)^{-1}$. Prove that U is unitary.

10. Let V be a finite-dimensional complex inner product space. Suppose that $T \in \mathcal{L}(V)$ is invertible and normal. Prove that there exist T_1, $T_2 \in \mathcal{L}(V)$ that satisfy:
   (i) T_1 is unitary,
   (ii) T_2 is self adjoint,
   (iii) $T = T_1T_2 = T_2T_1$.
   (*Hint*: If z is a nonzero complex number, then $z = \frac{z}{|z|} \cdot |z|$.)