MAT 247S - Problem Set 4

Due Monday February 11th

NOTE: Questions 1 b), 2, 7 b),c) and 8 will be marked.

1. Let $A = \begin{pmatrix} i & 1+i \\ -1+i & -i \end{pmatrix}$. In each case below, answer the following three questions: (i) is B self-adjoint? (ii) Is B normal? (iii) Is B unitary? (Explain your answers fully.)
   a) $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + i\,A$
   b) $B = I - A$
   c) $B = (I + A)(I - A)^{-1}$.
2. Let V be a complex inner product space of dimension n.
   a) Suppose that $n = 2$, β is an orthonormal basis of V, and $T \in \mathcal{L}(V)$ satisfies
   $$[T]_\beta = \begin{pmatrix} -1/2 & i\sqrt{3}/2 \\ i\sqrt{3}/2 & -1/2 \end{pmatrix}.$$
   Show that T and $T + \mathbf{1}_V$ are unitary.
   b) Assume that $n \geq 1$, and $U \in \mathcal{L}(V)$ is unitary. Prove that $U + \mathbf{1}_V$ is unitary if and only if the eigenvalues of U belong to the set $\{\,-(1+\sqrt{3}i)/2, (-1+\sqrt{3}i)/2\,\}$.
3. Let $T : V \to V$ be a linear operator on a finite-dimensional complex inner product space. Suppose that $T^2 = -1_V$. Prove that T is unitary if and only if there exists a subspace W of V such that $T(x) = ix$ for all $x \in W$ and $T(x) = -ix$ for all $x \in W^\perp$.
4. Let T be a self adjoint linear operator on a finite-dimensional complex inner product space.
   a) Let $c \in \mathbb{C}$. Prove that if $c \notin \mathbb{R}$, then $T - c\mathbf{1}_V$ is invertible.
   b) Let $U = (T + i\mathbf{1}_V)(T - i\mathbf{1}_V)^{-1}$. Prove that U is unitary.
5. #9, §6.5.
6. #13, §6.5.
7. Let V be a finite-dimensional inner product space.
   a) Let $T \in \mathcal{L}(V)$. Suppose that T is invertible and T is normal. Show that $T^{-1}T^* = T^*T^{-1}$.
   b) Suppose that T is as in part a). Show that $T^{-1}T^*$ is unitary.
   c) Suppose that $U \in \mathcal{L}(V)$ is unitary. Prove that there exists $T_1 \in \mathcal{L}(V)$ such that T_1 is invertible and normal and $U = T_1^{-1}T_1^*$. (*Hint*: The following fact may be useful (you don't have to prove it). Suppose that $z \in \mathbb{C}$ and $|z| = 1$. Then there exists a nonzero $u \in \mathbb{C}$ such that $z = \bar{u}/u$.)
8. Let W be a subspace of a finite-dimensional inner product spave V. Let $T \in \mathcal{L}(V)$ be the orthogonal projection of V on W. Let $U \in \mathcal{L}(V)$. Assume that U is unitary if $F = \mathbb{C}$ and orthogonal if $F = \mathbb{R}$. Let $W_1 = \{\,U(x) \mid x \in W\,\}$. Set $T_1 = UTU^{-1}$. Prove that T_1 is orthogonal projection of V on W_1.
9. Let V be a finite-dimensional inner product space. Suppose that $T \in \mathcal{L}(V)$, $T^2 = T$ and T is normal. Show that T is orthogonal projection on $W = R(T)$.
10. Let V be a finite-dimensional inner product space. Let $T, U \in \mathcal{L}(V)$. Assume that U is unitary if $F = \mathbb{C}$ and orthogonal if $F = \mathbb{R}$. Prove that T is normal if and only if UTU^{-1} is normal.