MAT247 GROUP THEORY SOLUTIONS

Here are brief hints/solutions to some of the problems in group theory on Assignments 8 and 9.

Problem 9: First note that every element in G is it's own inverse, so for any $x, y \in G$, we have $1 = (xy)(xy)$, so $xy = y^{-1}x^{-1} = yx$.

Problem 11: Since unitary operators are diagonalizable, we may consider their matrices in a basis of eigenvectors. Thus a unitary operator has finite order if and only if there is some integer k for which $\lambda^k = 1$ for each eigenvalue λ.

Problem 13: A and B both have finite order, as you can easily verify, but $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has infinite order, since $(AB)^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. In particular, A and B are in H_1, but AB is not.

Problem 15: *a)* Take $x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and check that $\phi(x)\phi(y) \neq \phi(xy)$.

b) You can check directly that in this case, ϕ is a homomorphism. The identity element in G' is $(1, 1)$, so the kernel is given by those elements of G that map to $(1, 1)$:

$$ker\phi = \left\{ \begin{bmatrix} a & b \\ & d \end{bmatrix} \mid |a| = 1 \ , d \in \{\pm 1, \pm i\} \right\}$$

The image of ϕ in G' is

$$im\phi = \left\{ (|a|, c^4) | \ a, \ c \in \mathbb{C}^\times \right\} = \left\{ (a, c) \mid a \in \mathbb{R}, \ a > 0, \ c \in \mathbb{C}^\times \right\}$$

c) If $x = \begin{bmatrix} 2 & \\ & 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix}$, then $\phi(xy) \neq \phi(x)\phi(y)$.

Problem 17: An isomorphism of groups is in particular a bijection of sets - $GL_n(\mathbb{Q})$ is countable, while $GL_n(\mathbb{R})$ is not.

Problem 20: For any $g \in G$, there is an integer k so that $g^k = 1$, since G is finite. So the operator $\phi(g) \in GL_n(\mathbb{C})$ satisfies $\phi(g)^k = \phi(1) = \mathbf{Id}_n$. Therefore, the minimal polynomial of $\phi(g)$ divides the polynomial $t^k - 1 = (t - \zeta)(t - \zeta^2) \dots (t - \zeta^{k-1})$, where $\zeta = e^{\frac{2\pi i}{k}}$. In particular, the minimal polynomial (which is over $\mathbb{C}$) is a product of distinct linear factors.

Problem 22: Note that for any k, we have that $r(r^2ks)r^{-1} = r^{2k+2}s \in < r^2, s >$, for example. You can check similar relations for other typical elements of D_n to see that $< r^2, s >$ is normal. If n is odd, we actually have $< r^2, s > = D_n$, since one can obtain r by taking powers of r^2.

If n is even, then there is an obvious isomorphism between $D_{n/2}$ and $H = < r^2, s >$ in D_n: namely, fix a rotation and a reflection generating $D_{n/2}$, send the rotation to r^2, and send the reflection to s, and note the defining relations for the two groups are the same.

Finally, for any n, let $H = < rs, r^2 s >$. Since $r = (r^2 s)(rs)$ is in H, so is $s = r^{-1} rs$, so $H = D_n$.

Problem 25: If ϕ were a homomorphism, the equation $rs = sr^{-1}$ implies that $\phi(r)\phi(s) = \phi(s)\phi(r)^{-1}$. However, we must also have that $det(\phi(s)) \neq 0$, since $s^2 = 1$, so applying the determinant to the above equation yields a contradiction.

Problem 26: Assume $m, n \geq 3$ to avoid trivial cases.

If m divides n, there is a clear map from D_m to D_n by fixing a reflection and rotation generating each, sending the reflection in D_m to the reflection in D_n, and sending the rotation in D_m to $r^{n/m}$ where r is a reflection of order n in D_n. Convince yourself that this map is an injective homomorphism.

On the other hand, suppose that $\phi : D_m \to D_n$ is an injective homomorphism. If $r \in D_m$ is a rotation of order m, then we must have $\phi(r)^k \neq 1$ for $k < m$, since ϕ is injective. On the other hand, $\phi(r)^n = 1$, since ϕ must send r to a rotation (again, because it is injective). Therefore, m divides n.

Problem 29: If $a, b \in H$, so $a^m = 1 = b^n$ for some m and n, then $(ab^{-1})^{mn} = a^{mn}(b^{-1})^{mn} = (1)^n(1)^m = 1$, so $ab^{-1} \in H$. Note that this is not true when the group is not abelian - see problem 13.