FiveThirtyEight's July 3, 2020 Riddler

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This week's riddler is a number theory exercise:

Question 1. *When N equals 50, N is twice a square and $N+1$ is a centered pentagonal number. After 50, what is the next integer N with these properties?*

There is motivation about putting stars on a flag, but that's tangential to solving the problem. Additionally, the k^{th} centered pentagonal number is $\frac{5k^2+5k+2}{2}$.

Write $2x^2+1 = \frac{5y^2+5y+2}{2}$. After clearing a bunch of denominators and doing some rearranging, one gets $(4x)^2-5(2y+1)^2 = -5$. Letting $X = 4x$ and $Y = 2y+1$, we get a solution to $X^2-5Y^2 = -5$, or $N_{\mathbb{Z}[\sqrt{5}]/\mathbb{Z}}(X+Y\sqrt{5}) = -5$. To find all solutions of this, one takes one solution α and a fundamental totally positive unit u, and then gets that every solution is of the form $\pm u^i\alpha$ for some integer i[1] There is an "obvious" solution: $\alpha = \sqrt{5}$ and that's the one I will use. The fundamental totally positive unit is $9+4\sqrt{5}(= (2+\sqrt{5})^2)$. Thus, every solution is of the form $X+Y\sqrt{5} = \pm(9+4\sqrt{5})^i\sqrt{5}$.

Notice that choosing negative values of i and choosing $+$ or $-$ don't change the underlying value of N; all this does is negate X, Y, or both. To get the small solutions: $i = 0$ gives $0 + 1\sqrt{5}$ for $X = Y = 0$ and $N = 0$ which does indeed check out. $i = 1$ gives $20 + 9\sqrt{5}$ for $X = 5$, $Y = 4$, and $N = 50$, which was the original solution. $i = 2$ will be the solution to the riddler, and it gives $360 + 161\sqrt{5}$ for $x = 90$, $y = 80$, and $N = 16200$. Thus, $N = 16200$ is the solution to the riddler. One can keep on incrementing i to get more solutions but I will not do that here.

Alternatively, the equation can also be reduced to $(2x)^2 = 5y(y+1)$. 5 divides the right hand side and so it must divide the left hand side. The right hand side is a square, so 5^2 divides the right hand side, and so 5 divides either y or $y+1$. If $5|y$, then notice that $y/5$ and $y+1$ are integers who are coprime and whose product is a square, so they are squares themselves. Thus, one has a solution to $a^2-5b^2 = 1$ with $a, b \in \mathbb{Z}$. Similar reasoning shows that if $5|y+1$ then there is a solution to $a^2-5b^2 = -1$ with $a, b \in \mathbb{Z}$. Thus, every solution to this equation is of the form $\pm(2+\sqrt{5})^i$ with $i \in \mathbb{Z}$. As before, choosing $+$ or $-$ and replacing i with $-i$ only negates a and/or

[1] I'm sweeping a couple of details under the rug here. For completeness, here they are: 1) there is only one ideal of norm 5 in $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ so there is only one equivalence class of solutions, and 2) because X is an even integer there are no solutions that come from elements in $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$ that aren't also in $\mathbb{Z}[\sqrt{5}]$.

b and doesn't change the value of y. From a solution of $a^2 - 5b^2 = \pm 1$, one gets y by seeing that y is the smaller of a^2 and $5b^2$.

$i = 0$ again gives $a = 1$, $b = 0$, $y = 0$, $x = 0$, and $N = 0$ (the trivial solution). $i = 1$ gives $a = 2$, $b = 1$, $y = 4$, $x = 5$, and $N = 50$ (the given solution). Next, $i = 2$ gives $a = 9$, $b = 4$, $y = 80$, $x = 90$, and $N = 16200$ (the solution we got above). Again, one can keep on incrementing i and get larger solutions, but that is still something that will not be done here.

But what about the other situation: when is N a centered pentagonal number and $N + 1$ twice a square? The same manipulations as in the first solution give the equation $X^2 - 5Y^2 = 11$. There are now two classes of solution to that: $X + Y\sqrt{5} = \pm(4 + \sqrt{5})(9 + 4\sqrt{5})^i$ and $X + Y\sqrt{5} = \pm(4 - \sqrt{5})(9 + 4\sqrt{5})^j$. After noticing which choices of $X + Y\sqrt{5}$ give the same underlying value of N, one sees that you can always choose $+$ as the sign, and look at when $i \geq 0$ and $j \geq 1$ and get a complete set of solutions in N.

Here are the first few small solutions: $i = 0$ gives $X + Y\sqrt{5} = 4 + \sqrt{5}$ for $x = 1$, $y = 0$, and $N = 1$. $j = 1$ gives $X + Y\sqrt{5} = 16 + 7\sqrt{5}$ for $x = 4$, $y = 3$, and $N = 31$. $i = 1$ gives $X + Y\sqrt{5} = 56 + 25\sqrt{5}$ for $x = 14$, $y = 12$, and $N = 391$. Finally $j = 2$ gives $X + Y\sqrt{5} = 284 + 127\sqrt{5}$ for $x = 71$, $y = 63$, and $N = 10081$.