A p-adic Jacquet-Langlands Correspondence

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1 Introduction

1.1 Main Results

Let $D/\mathbb{Q}_p$ be the unique quaternion algebra, and let $E/\mathbb{Q}_p$ be a finite extension that is "large enough". Classical work of Jacquet and Langlands allows one to classify smooth representations of $D^\times$ in terms of smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. Recently, there has been a lot of interest in a different class of representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, namely continuous unitary representations valued in Banach spaces over E. The goal in this paper is to construct an analogue of the Jacquet-Langlands correspondence for continuous unitary representations valued in E-Banach spaces. The following theorem shows that there is an analogue of the classical theory, a "p-adic Jacquet-Langlands correspondence."

Theorem 1.1. *Let π be a continuous unitary irreducible admissible representation of $GL_2(\mathbb{Q}_p)$ in an E-Banach space. Then there is a continuous unitary representation $J(\pi)$ of $D^\times$ valued in an E-Banach space.*

The representation $J(\pi)$ is constructed purely locally. The above theorem is vacuous without any properties of $J(\pi)$, and the following theorems are meant to give some characterizing properties of $J(\pi)$.

Let $\Delta/\mathbb{Q}$ be a division algebra such that $\Delta_{\mathbb{Q}_p} = D$ and $\Delta_{\mathbb{R}} = M_2(\mathbb{R})$. Let $G = \Delta^\times$ as an algebraic group over $\mathbb{Q}$ and S_G be the set of all primes $\ell \neq p$ such that $G(\mathbb{Q}_\ell) \neq \mathrm{GL}_2(\mathbb{Q}_\ell)$. I will assume from here on out that $\ell \not\equiv \pm 1 \pmod p$ for all $\ell \in S_G$. Now, let S_0 be a finite set of primes disjoint from

S_G, and let $S = S_0 \cup \{p\}$. Finally, let $G_{S_0} = \prod_{\ell \in S_0} G(\mathbb{Q}_\ell)$. Choose a maximal compact $K_0^S \subset G(\mathbb{A}_f^S)$, and to any compact open subgroup $K_p \times K_{S_0} \subset D^\times \times G_{S_0}$, we can associate a Shimura curve $Sh_{K_p K_{S_0} K_0^S}/\mathbb{Q}$. Following Emerton, define the completed cohomology as follows:

Definition 1 (Completed Cohomology). *The completed cohomology for G, denoted $\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0})$ is defined as*

$$\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0}) = \left(\varprojlim_s \varinjlim_{K_p} H^1_{ét}(Sh_{K_p K_{S_0} K_0^S, \overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s) \right).$$

Also, let $\hat{H}^1_{E,G}(K_{S_0}) = \hat{H}^1_{\mathcal{O}_E,G}(K_{S_0}) \otimes_{\mathcal{O}_E} E$. This is a Banach space over E with unit ball given by $\hat{H}^1_{\mathcal{O}_E,G}(K_{S_0})$. Finally, let

$$\hat{H}^1_{*,G,S} = \varinjlim \hat{H}^1_{*,G}(K_{S_0})$$

where $*$ is either $\mathcal{O}_E$ or E, and the limit is taken over all compact open subgroups $K_{S_0} \subset G_{S_0}$.

There are commuting unitary actions of $G_\mathbb{Q}$, $D^\times$, and a Hecke algebra $\mathbb{T}$ on $\hat{H}^1_{*,G}(K_{S_0})$ and $\hat{H}^1_{*,G,S}$. Additionally, there is an action of G_{S_0} on $\hat{H}^1_{*,G,S}$ that commutes with all of the aforementioned actions. Let ρ be a 2-dimensional continuous representation of $G_\mathbb{Q}$ that is unramified at all places $\ell \notin S \cup S_G$. Assume further that ρ is promodular (conjecturally, this is equivalent to being odd), that $\overline{\rho}|_{G_{\mathbb{Q}_\ell}} = \begin{pmatrix} \chi\overline{\epsilon} & * \\ & \chi \end{pmatrix}$ for all $\ell \in S_G$ ($\overline{\epsilon}$ is the mod p cyclotomic character, and $*$ is non-zero), and that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is neither $\begin{pmatrix} \chi & * \\ & \chi\overline{\epsilon} \end{pmatrix}$, $\begin{pmatrix} \chi & * \\ & \chi \end{pmatrix}$, nor $\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}$ (where χ is any mod p character of $G_{\mathbb{Q}_p}$, $\overline{\epsilon}$ is the mod p cyclotomic character, and $*$ may be zero or nonzero in this case). Let $\pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})$ be the representation of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ associated to $\rho|_{G_{\mathbb{Q}_\ell}}$ under the local Langlands correspondence, as modified by Emerton and Helm (see [EH14]). The assumption on ρ at primes in S_G implies that $\rho|_{G_{\mathbb{Q}_\ell}}$ is an extension of characters $\begin{pmatrix} \chi_\ell\overline{\epsilon} & * \\ & \chi_\ell \end{pmatrix}$ for some character χ_ℓ of $G_{\mathbb{Q}_\ell}$, and we will view this as a character of $G(\mathbb{Q}_\ell)$ by factoring through the reduced norm and then using local class field theory.

Theorem 1.2. *With the above assumptions on ρ, there is a representation $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ such that*

$$\mathrm{Hom}_{G_\mathbb{Q}}(\rho, \hat{H}^1_{E,G,S}) \cong J(B(\rho|_{G_{\mathbb{Q}_p}})) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho|_{\mathbb{Q}_\ell}) \otimes \bigotimes_{\ell \in S_G} \chi_\ell.$$

The key point of this theorem is that the representation $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ depends only on the local Galois representation $\rho|_{G_{\mathbb{Q}_p}}$. The fact that it doesn't depend on all of ρ, or even any of the choices made in constructing G is the main thrust of the theorem. Additionally, in the above isomorphism, there is an action of $\mathbb{T}$ on $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho, \hat{H}^1_{E,G,S})$. Importantly in the above theorem, one has that T_ℓ acts via $\mathrm{tr}(\mathrm{Frob}_\ell|_\rho)$; this is a compatability for the Hecke algebra.

A natural question to ask is "What are the locally algebraic vectors in $J(B(\rho|_{G_{\mathbb{Q}_p}}))$?" More precisely, assume that $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$. Then I can construct a Weil-Deligne representation $WD_{\rho|_{G_{\mathbb{Q}_p}}}$. If $WD_{\rho|_{G_{\mathbb{Q}_p}}}$ is indecomposable, then there is an associated smooth admissible $D^\times$ representation Sm given by using the local Langlands and Jacquet-Langlands correspondences on $WD_{\rho|_{G_{\mathbb{Q}_p}}}$. Additionally, the weights $w_1 + 1$ and w_2 are weights for an algebraic representation Alg of $D^\times$. This construction is similar to the one performed in [BS07].

Theorem 1.3. *Let ρ be as in Theorem 1.2. If the assumptions on $\rho|_{G_{\mathbb{Q}_p}}$ in the above construction hold, then the space of locally algebraic vectors $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \cong Sm \otimes Alg$. If one of the assumptions does not hold, then $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \cong 0$.*

Some more interesting notes about the locally algebraic vectors: following through the construction of the locally algebraic vectors, one sees that there are no locally algebraic vectors for crystalline representations. Additionally, the locally algebraic vectors are still finite dimensional for a fixed representation $J(\rho|_{G_{\mathbb{Q}_p}})$, and so they are closed in $J(B(\rho|_{G_{\mathbb{Q}_p}}))$. However, $J(B(\rho|_{G_{\mathbb{Q}_p}}))$ is believed to be infinite dimensional for all ρ. Thus, even when the associated Galois representation $\rho|_{G_{\mathbb{Q}_p}}$ is irreducible, one should have a proper closed subrepresentation of $J(B(\rho|_{G_{\mathbb{Q}_p}}))$. Thus, one would not expect a statement like $J(\pi)$ is "as irreducible as" π to be true. This also suggests that $J(\rho|_{G_{\mathbb{Q}_p}})$ is not directly characterized by the space of locally algebraic vectors together with a continuous unitary admissible norm on said space.

There is an auxillary group $\overline{G}/\mathbb{Q}$ that arises in the proof of these theorems. This group is an inner form of G with invariants at p and ∞ swapped. Since $\overline{G}(\mathbb{A}_f^p) = G(\mathbb{A}_f^p)$, I can identify subgroups of $\overline{G}(\mathbb{A}_f^p)$ with subgroups of $G(\mathbb{A}_f^p)$. Continuous functions on the double coset space $X_{\overline{K}_{S_0}} := \overline{G}(\mathbb{Q})/\backslash\overline{G}(\mathbb{A}_f)/\overline{K}_{S_0}\overline{K}_0^S$ valued in E gives an analogue of the completed cohomology of G for $\overline{G}$.

The idea behind the proof of Theorem 1.2 is as follows: there is a p-adic analytic uniformization (called the Cerednik-Drinfel'd uniformization) $\Sigma^n \times X_{\overline{K}_{S_0}}/\mathrm{GL}_2(\mathbb{Q}_p) \cong Sh_{K_p^n K_{S_0} K_0^S, \mathbb{C}_p}$, where Σ^n is a cover of the Drinfel'd upper half plane. The plan is thus to first analyze the space of continuous functions $\mathcal{C}^0(X_{\overline{K}_{S_0}}, E)$, and then apply that knowledge to understanding what the uniformization says about completed cohomology. It turns out that the Hecke algebra $\mathbb{T}$ will act on the space of automorphic forms for $\overline{G}$, and so the first step is to describe this space as a $\mathbb{T}[\mathrm{GL}_2(\mathbb{Q}_p)]$-module. This can be done by an adaptation of the main argument in [Eme11]. There is a spectral sequence relating the cohomology of $Sh_{K_p^n K_{S_0} K_0^S}$ to that of Σ^n and $X_{\overline{K}_{S_0}}$, and analysis of this spectral sequence will give Theorem 1.2.

As for Theorem 1.3, the argument is an application of the results of [Eme06]. That paper gives a spectral sequence relating the locally algebraic vectors to the cohomology of various local systems $\mathcal{V}_W$ on $Sh_{K_p^n K_{S_0} K_0^S}$. Since the cohomology of these local systems is well understood, the only thing that is needed to compute the locally algebraic vectors is to perform an analysis of the spectral sequence.

There is a more general version of the above theorems. Let $F/\mathbb{Q}$ be a totally real field with one place v/p. If π is a representation of $\mathrm{GL}_2(F_v)$, then one can construct a representation $J'(\pi)$ of $D_{F_v}^\times \times G_{F_v}$. Again, this construction is purely local and depends only on F_v and not on any other choices. Now, one considers a unitary group G/F that is $D_{F_v}^\times$ at F_v, $U(1,1)$ at exactly one infinite place, and $U(2)$ at every other infinite place. There is a Shimura curve $Sh_{K_v K^v}/F$, and one may talk about $\hat{H}^1_{E,G}(K^v)$, as before. Additionally, there is an auxiliary group $\overline{G}$ with $\overline{G}(F) = \mathrm{GL}_2(F)$ that arises in the uniformization of $Sh_{K_v K^v}$.

In this situation, there is the following weaker version of Theorem 1.2:

Theorem 1.4. *If π arises as a representation of $\overline{G}$, then $J'(\pi)$ arises in $\hat{H}^1_{E,G}(K^v)$.*

The proof of Theorem 1.4 follows the same lines as the proof of Theorem 1.2, with the weakening coming from the fact that there is no p-adic Langlands correspondence to understand the space $\mathcal{C}^0(X_{\overline{K_{S_0}}}, E)$ as there was in the $F_v = \mathbb{Q}_p$ case.

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2 The Drinfel'd Upper Half Plane

This section will introduce the local object that will be important throughout the rest of the paper. We will keep the same notation as in the introduction; in particular, $F/\mathbb{Q}$ will be a totally real field with exactly one place v over p, and $\varpi_{D_{F_v}} \in \mathcal{O}_{D_{F_v}} \subset D_{F_v}$ will be the quaternion algebra over F_v with its ring of integers and its uniformizer. Some more notation that wasn't mentioned in the introduction but will be used here: $v_{F_v} : F_v \to \mathbb{Z} \cup \{\infty\}$ will be the $\mathbb{Z}$-valued valuation associated to F_v, and $\nu : D_{F_v} \to F_v$ will be the reduced norm.

As a rigid space, the Drinfel'd upper half plane $\Omega^2_{F_v}/\widehat{F_v^{ur}}$ (which I will use Ω for for the rest of this section) is just $\mathbb{P}^1 \backslash \mathbb{P}^1(F_v)$. This is a rigid analytic variety and has the structure of a formal scheme $\breve{\Omega}/\mathrm{spf}(\mathcal{O}_{\widehat{F^{ur}}})$. The relationship between $\breve{\Omega}$ and Ω is that Ω is the rigid generic fiber of $\breve{\Omega}$. For the rest of this article, Ω will be viewed as an adic space, primarialy due to the theory of étale cohomology for adic spaces.

This exposition will be based on the exposition in [Car90]. The results that are discussed were originally proved in [Dri74], [Dri77], and [Dri76], and there is a vast generalization that can be found in [RZ96].

2.1 The Moduli Interpretation and the Level Covers

The formal scheme $\breve{\Omega}$ represents the following moduli problem. Recall that D_{F_v} is the unique quaternion algebra over F_v, and let $\nu : D_{F_v} \to F_v$ be the reduced norm. Let $\mathcal{M}$ be the functor associating to an $\mathcal{O}_{\widehat{F_v^{ur}}}$-scheme S with $\varpi_{\widehat{F_v^{ur}}}\mathcal{O}_S$ locally nilpotent (such as scheme S is said to be in the category $\mathrm{Nilp}_{\mathcal{O}_{\widehat{F_v^{ur}}}}$) the set of triples $(G, \iota_{D_{F_v}}, \varrho)$. Here G is a two-dimensional formal $\mathcal{O}_{F_v}$-module over S with F_v-height four, $\iota_{D_{F_v}} : \mathcal{O}_{D_{F_v}} \hookrightarrow \mathrm{End}(X)$ gives an action of $\mathcal{O}_{D_{F_v}}$ on G. $\iota_{D_{F_v}}$ is assumed to satisfy the following condition: let F'_v be the unramified quadratic extension of F_v. Then $\mathcal{O}_{F'_v} \hookrightarrow \mathcal{O}_{D_{F_v}}$, so, via $\iota_{D_{F_v}}$, one gets an action of $\mathcal{O}_S \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{F'_v}$ on $\mathrm{Lie}(G)$. The assumption is that this makes $\mathrm{Lie}(G)$ a locally free sheaf of rank one over $\mathcal{O}_S \otimes_{\mathcal{O}_{F_v}} \mathcal{O}_{F'_v}$. Finally, over $\overline{k}_{F_v}$, there is only one two-dimensional formal $\mathcal{O}_{F_v}$-module $\mathbb{G}$ of height 4 with an action of $\mathcal{O}_{D_v}$. Then

$\varrho : G \times_S \overline{S} \to \mathbb{G} \times_{\overline{k_{F_v}}} \overline{S}$ is chosen to be a quasi-isogeny of height zero. While this description has no reference to polarizations, there is a canonical polarization on any such triple $(G, \iota_{D_{F_v}}, \varrho)$, satisfying a compatability condition for the action of $\mathcal{O}_{D_{F_v}}$ on G. The details may be found in [BC91].

There is a universal formal $\mathcal{O}_{F_v}$-module $\mathscr{G} \to \Omega$. Inside $\mathscr{G}$ is the universal $\varpi^n_{D_{F_v}}$-torsion, written $\mathscr{G}[\varpi^n_{D_{F_v}}]$. $\mathscr{G}[\varpi^n_{D_{F_v}}]$ is a free $\mathcal{O}_{D_{F_v}}/\varpi^n_{D_{F_v}}$-module over Ω of rank one. There are maps $\mathscr{G}[\varpi^n_{D_{F_v}}] \hookrightarrow \mathscr{G}[\varpi^{n+1}_{D_{F_v}}]$ and $\mathscr{G}[\varpi^{n+1}_{D_{F_v}}] \twoheadrightarrow \mathscr{G}[\varpi^n_{D_{F_v}}]$. The first is the natural inclusion, and the second is multiplication by $\varpi_{D_{F_v}}$. Let $\Sigma^n = \mathscr{G}[\varpi^n_{D_{F_v}}] \backslash \mathscr{G}[\varpi^{n-1}_{D_{F_v}}]$ for $n > 0$. As a note, Σ^n is the first object in this paragraph that is not naturally a formal scheme, as the removal of $\mathscr{G}[\varpi^{n-1}_{D_{F_v}}]$ cannot be done integrally. Multiplication by $\varpi_{D_{F_v}}$ gives rise to maps $\Sigma^{n+1} \to \Sigma^n$. I will let $\Sigma = \varprojlim \Sigma^n$. Recently this was shown to be a well defined object in [SW13], but for the reader unfamiliar with the theory of perfectoid spaces, the only operation that will be done to Σ is taking cohomology. Since any reasonable notion of inverse limits would have $H^i(\Sigma, *) = \varinjlim H^i(\Sigma^n, *)$, this may be viewed as the definition of $H^i(\Sigma, *)$.

2.2 Group Actions

There are three different groups that act on Σ. The first group that will be discussed is $\mathrm{GL}_2(F_v)$. PGL_2 is the group of automorphisms of $\mathbb{P}^1$, and so there is a natural action of GL_2 on $\mathbb{P}^1$. Since $\mathbb{P}^1(F_v)$ is a closed F_v-orbit of the action of $\mathrm{GL}_2(F_v)$, the action of $\mathrm{GL}_2(F_v)$ on $\mathbb{P}^1$ preserves Ω. This action, however, doesn't have a moduli theoretic interpretation, and so will not naturally extend to $\mathscr{G}$ (and thus to Σ^n). If one twists the action by $g \to \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g))}$, where $\mathrm{Frob}_{F_v} : \widehat{F_v^{ur}} \to \widehat{F_v^{ur}}$ is geometric Frobenius, one gets another action of $\mathrm{GL}_2(F_v)$ on Ω. This action doesn't preserve the structure morphism $\Omega \to \widehat{F_v^{ur}}$. This is well-defined, as the original action did preserve the structure morphism, and so one has that, letting $\cdot$ be used for the original $\mathrm{GL}_2(F_v)$-action on Ω, $\mathrm{Frob}_{F_v}(g \cdot x) = g \cdot \mathrm{Frob}_{F_v}(x)$ and thus $g_1 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_1))}(g_2 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_2))}(x)) = g_1 \cdot g_2 \cdot \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g_1 g_2))}(x)$. This new action has a moduli interpretation ([Dri74] and [Dri77] have the details), and thus extends to $\mathscr{G}$ (and thus to Σ^n and thus to Σ).

Since $\Sigma^n \to \Omega$ is an $(\mathcal{O}_{D_{F_v}}/\varpi^n_{D_{F_v}})^\times$-torsor, one gets an action of $\mathcal{O}^\times_{D_{F_v}}$ on Σ^n. It is straightforward to see that the maps $\Sigma^n \to \Sigma^m$ are $\mathcal{O}^\times_{D_{F_v}}$-equivariant, and thus one has an action of $\mathcal{O}^\times_{D_{F_v}}$ on Σ. If (x_n) is a sequence of points $x_n \in \Sigma^n$ with $\varpi_{D_{F_v}} x_n = x_{n-1}$ (i.e. (x_n) is a point of Σ), then it is natural to define $\varpi_{D_{F_v}} \cdot (x_n) = (x_n)$. This may look like the choice of $\varpi_{D_{F_v}}$ would be important, but the choice of $\varpi_{D_{F_v}}$ is also baked in to the map from $\Sigma^n \to \Sigma^m$. But, as before, the "correct" action is not this one, but rather this one twisted by $d \mapsto \mathrm{Frob}_F^{v_F(\nu(d))}$. This action of $D^\times_{F_v}$ commutes with the aforementioned action of $\mathrm{GL}_2(F_v)$, giving rise to an action of $\mathrm{GL}_2(F_v) \times D^\times$ on Σ.

While the action of $\mathrm{GL}_2(F) \times D^\times$ doesn't respect the structure morphism $\Sigma \to \widehat{F_v^{ur}}$, it does respect the the morphism $\Sigma \to \widehat{F_v^{ur}} \to F_v$. Thus, one gets an action of G_{F_v} on $H^i(\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}(\Sigma) \times_{F_v} \mathbb{C}_p, *)$ that automatically commutes with the action of $\mathrm{GL}_2(F_v) \times D^\times_{F_v}$. Going a little deeper into this action, one has that $\mathrm{Res}_{F_v}^{\widehat{F_v^{ur}}}(\Sigma) \times_{F_v} \mathbb{C}_p$ is the union of $\hat{\mathbb{Z}}$ copies of $\Sigma \times_{\widehat{F_v^{ur}}} \mathbb{C}_p$, with an element $g \in G_{F_v}$ shifting the index based on the image of g in $G_{k_{F_v}} = \mathrm{Frob}_{F_v}^{\hat{\mathbb{Z}}}$.

For $(g,d) \in \mathrm{GL}_2(F_v) \times D^\times$, choose a $w \in G_{F_v}$ such that $w|_{G_{k_{F_v}}} = \mathrm{Frob}_{F_v}^{v_{F_v}(\det(g)\nu(d))}$. From the definition of the above action, one has that $(g,d) : \Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p \to \left(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p\right)^w$ is an isomorphism.

2.3 Connected Components

This version of the space Σ^n is not connected. There are two ways known to the author of how to determine the connected components. The first is global in nature: one applies the results of the next section and uses the reciprocity law for Shimura varieties to compute the connected components. The details of this may be found in [BZ] for example. Alternatively, one may use local calculations on the connected components for the Lubin-Tate tower (these can be found in e.g. [Wei14]) and then appeal to the isomorphism in [SW13]. In either case, the result is the following:

Claim 2.1. *There is an identification*

$$\varprojlim_n \pi_0(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p) \cong \mathcal{O}_{F_v}^\times.$$

If one lets $P = \{(m,d,g) \in GL_2(F_v) \times D_{F_v}^\times \times G_{F_v} | v_{F_v}(\det(m)\nu(d)) = v_{F_v}(g)\}$, then $(m,d,g) \in P$ acts on $\varprojlim \pi_0(\Sigma^n)$ by multiplication by $\det(m)\nu(d)\mathrm{cl}(g)^{-1}$ (which is guarenteed to be a unit due to the definition of P).

Here, cl is the Artin map of local class field theory. Just to note: P is the largest subgroup of $\mathrm{GL}_2(F_v) \times D_{F_v}^\times \times G_{F_v}$ that could reasonably be expected to act on the connected components of Σ^n, as this is the subgroup that preserves the morphism $\Sigma^n \to \widehat{F_v^{ur}}$.

3 The Global Picture

Choose a number field $F/\mathbb{Q}$ such that F is totally real, with real places $v_{\infty,1}, \ldots, v_{\infty,d}$, and there is exactly one plave v over p. If $F = \mathbb{Q}$, let Δ and G be as in the introduction. If $F \neq \mathbb{Q}$, we need to properly define the unitary group mentioned in the introduction. Choose a CM extension L/F such that v splits as $v_1 v_2$ in L. Then choose a quaternion algebra Δ/L such that $\Delta(L_{v_1}) = \Delta(L_{v_2}) = D_{F_v}$, such that there is an involution i of the second kind with signature $(1,1)$ at $v_{\infty,1}$ and $(2,0)$ at $v_{\infty,j}$ for $j > 1$, and such that $\Delta(L_w) = M_2(L_w)$ for all w that are nonsplit finite places of L/F (the final assumption is not necessary for the results discussed in this section but is convienent to make in the next one). Then $G = \{d \in \Delta | d \cdot i(d) = 1\}$, an algebraic group over F. The assumptions listed imply that $G(F_v) = D_{F_v}^\times$, $G(F_{v_{\infty,1}}) = U(1,1)$, and $G(F_{v_{\infty,j}}) = U(2)$ for $j > 1$. If $K \subset G(\mathbb{A}_{F,f})$ is a compact open subgroup, then there is a unitary Shimura curve Sh_K/L. It will be convienent to define $K_v^n = \{d \in \mathcal{O}_{D_{F_v}}^\times | d \equiv 1 \pmod{\varpi_{D_{F_v}}^n}\}$ and to choose $K = K_v^n K^v$ with K^v a compact open subgroup of $G(\mathbb{A}_{F,f}^v)$. The goal of this section is to describe a p-adic analytic uniformization of $Sh_{K_v^n K^v}$.

3.1 The Group $\overline{G}$

To that end, we will introduce another group $\overline{G}$ over F. If $F = \mathbb{Q}$, let $\overline{\Delta}/\mathbb{Q}$ be the division algebra that has the same invariants as Δ away from p and ∞, and is now $M_2(\mathbb{Q}_p)$ at p and H (here, H is the real quaternion algebra) at ∞. Define $\overline{G} = \overline{\Delta}^{\times}$ as an algebraic group over $\mathbb{Q}$. If $F \neq \mathbb{Q}$, this group is an inner form of G that is isomorphic to G away from v and $v_{\infty,1}$, is GL_2 at v, and $U(2)$ at $v_{\infty,1}$. Visibly, one may identify $G(\mathbb{A}^v_{F,f}) \cong \overline{G}(\mathbb{A}^v_{F,f})$ in both cases. Choose an isomorphism that comes from an algebra anti-isomorphism of the underlyling algebras. Then one may identify K^v with a subgroup $\overline{K}^v$ in $\overline{G}(\mathbb{A}^v_{F,f})$.

The theory of automorphic forms for $\overline{G}$ is related to the double coset spaces $\overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}_f$ where $\overline{K}_f$ is a compact open subgroup of $\overline{G}(\mathbb{A}_{F,f})$. These spaces are finite sets of points which are sometimes called "Hida varieties." In this paper, the spaces $X_{\overline{K}^v} := \overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}^v$ will also be relevant. This space is a compact $\mathrm{GL}_2(F_v)$-set and functions on this space will come from automorphic forms with any level at p. Additionally, $X_{\overline{K}^v}$ breaks up into finitely many orbits under the $\mathrm{GL}_2(F_v)$ action. The orbits are parameterized by the finite set $\overline{G}(F)\backslash\overline{G}(\mathbb{A}^v_{F,f})/\overline{K}^v$, and if one chooses a set of double coset representatives $\overline{g}_i$, these orbits are of the form $\Gamma_i\backslash\mathrm{GL}_2(F_v)$, where $\Gamma_i = \{\gamma \in \overline{G}(F) | \overline{g}_i\gamma\overline{g}_i^{-1} \in \overline{K}^v\}$ (inclusion is under the natural map from $\overline{G}(F) \to \overline{G}(\mathbb{A}^v_{F,f})$). The Γ_is are discrete cocompact subgroups of $\mathrm{GL}_2(F_v)$.

3.2 Čerednik-Drinfel'd Uniformization

The main result is the following:

Theorem 3.1 (Čerednik-Drinfeld Uniformization)**.** *With notation as above, there is an isomorphism*

$$(Sh_{K^vK^n_v} \times_F \mathbb{C}_p)^{an} \cong \left(\left(\mathrm{Res}^{\widehat{F_v^{ur}}}_{F_v}\Sigma^n \times_{F_v} \mathbb{C}_p\right) \times X_{\overline{K}^v}\right)/GL_2(F_v).$$

This theorem was originally proved by Čerednik in the case that $F = \mathbb{Q}$ and $n = 0$ and was proved in general for $F = \mathbb{Q}$ by Drinfel'd (just as in the previous section, see [Dri74], [Dri77], and [Dri76]). Later, a vast generalization which includes all of the cases above was proved by Rapoport and Zink (again, see [RZ96]). The interested reader can read **??** to get an exposition on Drinfel'd's proof.

A few remarks are in order. Since there is a decomposition $X_{\overline{K}^v} = \coprod \Gamma_i\backslash\mathrm{GL}_2(F_v)$, the right hand side of the isomorphism may be written as $\coprod \Gamma_i\backslash\left(\mathrm{Res}^{\widehat{F_v^{ur}}}_{F_v}\Sigma^n \times_{F_v} \mathbb{C}_p\right)$. Additionally, if one lets $\Gamma'_i = \Gamma_i \cap \{g \in \mathrm{GL}_2(F_v) | \det(g) \in \mathcal{O}^{\times}_{F_v}\}$ and n_i to be the smallest positive integer in $v_F(det(\Gamma_i))$, then there is an isomorphism $\Gamma_i\backslash\left(\mathrm{Res}^{\widehat{F_v^{ur}}}_{F_v}\Sigma^n \times_{F_v} \mathbb{C}_p\right) \cong \coprod_{j=0}^{n_i-1}\left(\Gamma'_i\backslash(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p)\right)^{\mathrm{Frob}^j_{F_v}}$. In order to give a G_{F_v} action on $\coprod_{j=0}^{n_i-1}\left(\Gamma'_i\backslash(\Sigma^n \times_{\widehat{F_v^{ur}}} \mathbb{C}_p)\right)^{\mathrm{Frob}^j_{F_v}}$, one can give a model of $\coprod_{j=0}^{n_i-1}(\Gamma'_i\backslash\Sigma^n)^{\mathrm{Frob}^j_{F_v}}$ over F_v. One thus needs to give a Frob_{F_v}-semilinear map $\varphi : \coprod_{j=0}^{n_i-1}(\Gamma'_i\backslash\Sigma^n)^{\mathrm{Frob}^j_{F_v}} \to \coprod_{j=0}^{n_i-1}(\Gamma'_i\backslash\Sigma^n)^{\mathrm{Frob}^j_{F_v}}$. If $j < n_i-1$, then define $\varphi : (\Gamma'_i\backslash\Sigma^n)^{\mathrm{Frob}^j_{F_v}} \to (\Gamma'_i\backslash\Sigma^n)^{\mathrm{Frob}^{j+1}_{F_v}}$ to just be Frobenius. Otherwise, choose

$g \in \Gamma_i$ such that $v_{F_v}(det(g)) = n_i$, and define $\varphi : (\Gamma_i'\backslash\Sigma^n)^{\mathrm{Frob}_{F_v}^{n-1}} \to (\Gamma_i'\backslash\Sigma^n)$ to be $g^{-1} \cdot \mathrm{Frob}_{F_v}$. It is easy to see that this is isomorphic to $\Gamma_i\backslash Res_{F_v}^{\widehat{F_v}^{ur}} \Sigma^n$ over F_v.

Additionally, there are Hecke operators acting on both the Shimura curve, and the set $X_{\overline{K}^v}$. The above isomorphism is an isomorphism of analytic varieties, together with an action of G_{F_v} and of the Hecke operators. The action of all of $G_{\mathbb{Q}}$ or G_L in the $F \neq \mathbb{Q}$ case is not completely lost, as one must have that there is compatibility between the Hecke operators and Galois representation.

4 Local-Global Compatibility for $\overline{G}$ over $\mathbb{Q}$

Let S_0 be a finite set of places of F not containing v nor any infinite place, nor any place where $\overline{G}$ is nonsplit. It is also useful to let $S = S_0 \cup \{v\}$ and S_G to be the set of places away from p where G is nonsplit if $F = \mathbb{Q}$ and the set of all places away from v that split in L where G is nonsplit if $F \neq \mathbb{Q}$. We will assume that our tame level $\overline{K}^v$ is of the form $\overline{K}_{S_0}\overline{K}_S^0$, where K_{S_0} is a compact open in $\prod_{w \in S_0} \overline{G}(F_w)$ and $\overline{K}_S^0$ is a maximal compact open subgroup of ${\prod_{\substack{w \notin S \\ w \nmid \infty}}}' \overline{G}(F_w)$. The goal of this section is to understand the space $X_{\overline{K}_{S_0}} := \overline{G}(F)\backslash\overline{G}(\mathbb{A}_{F,f})/\overline{K}_{S_0}\overline{K}_0^S$. It is a well known fact that $X_{\overline{K}_{S_0}}$ is a compact $\mathrm{GL}_2(F_v)$-set, which is equivalent to the pair of facts that $X_{\overline{K}_{S_0}}/\overline{K}_v$ is finite for all compact open subgroups $\overline{K}_v \subset \mathrm{GL}_2(F_v)$ and that $X_{\overline{K}_{S_0}} = \varprojlim_{\overline{K}_v} X_{\overline{K}_{S_0}}/\overline{K}_v$. The ideas in this section are inspired heavily by chapters 5 and 6 of [Eme11]. Additionally, [CS13] and [CS14] are invaluable as references, especially to highlight some of the simplifications that arise in this section.

4.1 The Completed H^0 for $X_{\overline{K}_{S_0}}$

The main object of study in this section is the Banach space

$$\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}) := \left(\varprojlim_s \varinjlim_{\overline{K}_v} H^0(X_{\overline{K}_{S_0}}/\overline{K}_v, \mathcal{O}_E/\varpi_E^s) \right) \otimes_{\mathcal{O}_E} E,$$

and its cousin $\hat{H}^0_{E,\overline{G},S} = \varinjlim_{\overline{K}_{S_0}} \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$. There are also the analogues of these spaces $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})$ and $\hat{H}^0_{\mathcal{O}_E,\overline{G},S}$ defined similarly but without tensoring with E.

Proposition 4.1. *$\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}^0_E(X_{\overline{K}_{S_0}})$, where $\mathcal{C}^0_E(X_{\overline{K}_{S_0}})$ is the space of continuous E-valued functions on $X_{\overline{K}_{S_0}}$ endowed with the sup-norm.*

Proof. Let $f : X_{\overline{K}_{S_0}} \to E$ be a continuous function. Since $X_{\overline{K}_{S_0}}$ is compact, there is an integer i such that f factors as $X_{\overline{K}_{S_0}} \xrightarrow{f'} \varpi_E^i\mathcal{O}_E \hookrightarrow E$. Then f is continuous if and only if f' is continuous, and the latter is equivalent to the functions $f_j' : X_{\overline{K}_{S_0}} \to \varpi_E^i\mathcal{O}_E \to \varpi_E^i\mathcal{O}_E/\varpi_E^{i+j}\mathcal{O}_E$ being continuous for all j.

The set $\varpi_E^i \mathcal{O}_E / \varpi_E^{i+j} \mathcal{O}_E$ is discrete, so f_j' is continuous if and only if it is locally constant. However, f_j' is locally constant if and only if for all $x \in X_{\overline{K}_{S_0}}$ there is a compact open subgroup $\overline{K}_p$ depending on x such that f_j' is constant on $x \cdot \overline{K}_p$. However, by compactness, I may choose the subgroup $\overline{K}_p$ so that it doesn't depend on x. Thus, f is continuous if and only if f_j' is smooth for all j.

The above process starts from an element of $\mathcal{C}_E^0(X_{\overline{K}_{S_0}})$ and produces an element of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$. Moreover the process is clearly reversible due to all the equivalences in the proof, which shows the desired equality. $\square$

4.2 Completed Hecke Algebras

For a fixed $\overline{K}_v \subset \overline{G}(F_v)$, there is a Hecke algebra $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. This is just the $\mathcal{O}_E$ subalgebra of $\mathrm{End}(H^0(X_{\overline{K}_{S_0}}/\overline{K}_v, E))$ generated by the Hecke operators T_w and S_w for $w \notin S \cup S_G$ such that w splits in L if $F \neq \mathbb{Q}$. If $\overline{K}_v' \subset \overline{K}_v$, then there is a natural surjection $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v') \to \mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. Define the completed Hecke algebra $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}) = \varprojlim_{\overline{K}_v} \mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$. Giving the algebras $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}\overline{K}_v)$ the ϖ_E-adic topology, $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ inherits the inverse limit topology from those topolgies.

It is a theorem that $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ is a semilocal ring; i.e. $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0}) = \prod_{\mathfrak{m}} \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\mathfrak{m}}$ where the product runs over all of the finitely many maximal ideals $\mathfrak{m} \subset \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$. Well known results about $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ imply that the maximal ideals of $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$ are in correspondence with the finite set of $\overline{G}$-modular k_E-valued representations $\overline{\rho}$ of $G_{\mathbb{Q}}$ if $F = \mathbb{Q}$ and G_L if $F \neq \mathbb{Q}$, with a $\overline{G}$-modular lift ρ that is unramified outside of $S \cup S_G$, with ramification at primes in S_0 specified by the level $\overline{K}_{S_0}$ and with $\overline{\rho}|_{G_{F_w}}$ being an extension of an unramified character χ by $\chi\epsilon$ for all $w \in S_G$. However, it is possible that such a representation is decomposable when restricted to G_{F_w} and reduced mod ϖ_E even though it is indecomposable in $\mathcal{O}_E$. On the other hand, if $\overline{\rho}|_{G_{F_w}}$ is a nontrivial extension of $\overline{\chi}$ by $\overline{\chi\epsilon}$, then the following lemma tells you that any (GL_2-)modular lift of $\overline{\rho}$ will be $\overline{G}$-modular.

Lemma 4.2. *Assume that $\#(k_w) \not\equiv \pm 1 \pmod p$, and let $\overline{\chi}$ be a mod ϖ_E character of G_{F_w}. If $\overline{\rho}_w : G_{F_w} \to GL_2(k_E)$ is a nonzero extension of $\overline{\chi}$ by $\overline{\chi\epsilon}$, then any $\rho_w : G_{F_w} \to GL_2(\mathcal{O}_E)$ with $\rho_w \otimes_{\mathcal{O}_E} k_E = \overline{\rho}_w$ must be an extension of χ by $\chi\epsilon$ for some unramified character χ.*

Proof. Let ρ_w and $\overline{\rho}_w$ be as above. Notice that I_{F_w} must act on ρ_w through $\{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right) \pmod{\varpi_E}\}$, which is a pro-p group. Thus, I_{F_w} acts through the $\mathbb{Z}_p(1)$ factor in tame inertia. Let γ be a topological generator of $\mathbb{Z}_p(1)$. First, we need to show that γ does not act semisimply on ρ_w.

If γ acted semisimply on ρ_ℓ, then one must have two eigenspaces generated by vectors e_1 and e_2, and with eigenvalues ζ_1 and ζ_2. If Frob_w does not switch the eigenspaces, then the action of G_{F_w} on ρ_w and thus $\overline{\rho}_w$ must be through the abelian quotient of G_{F_w}. But this would imply that $\overline{\epsilon}$ is the trivial character and thus $\#(k_w) \equiv 1 \pmod p$. Thus, Frob_w must switch the eigenspaces.

Now, using the fact that $\mathrm{Frob}_\ell \gamma \mathrm{Frob}_\ell^{-1} = \gamma^\ell$, one gets that $\zeta_1^{\#(k_w)} e_1 = \gamma^{\#(k_w)} e_1 = \mathrm{Frob}_w \gamma \mathrm{Frob}_w^{-1} e_1 = \zeta_2 e_1$, and similarly $\zeta_2^{\#(k_w)} e_2 = \zeta_1 e_2$. Thus, $\zeta_1^{\#(k_w)^2-1} = 1$. Since ζ_1 is a p-power root of unity, one has $\#(k_w)^2 - 1 \equiv 1 \pmod p$, a contradiction.

Since γ does not act semisimply, ρ_ℓ must be an extension of χ by $\chi\epsilon$ for some character χ. Since $\overline{\chi}$ is unramified and $\#(k_w) \not\equiv 1 \pmod p$, $\mathcal{O}_{F_w}^\times$ has no quotients onto a p-group. Thus, χ must also be unramified, proving the lemma. $\square$

The above lemma is an incredibly simple case of "Jacquet-Langlands in Families," an analogue of Emerton and Helm's work on local Langlands in families for representations of quaternion algebras. This lemma forces that one has that the set of places S_G contains no places v with $\#(k_v) \equiv \pm 1 \pmod p$. From now on, we will assume that S_G contains no places v with $\#(k_v) \equiv \pm 1 \pmod p$. The nonexistence of this work in full generality is the obstruction to letting there be level at the places in S_G.

With the above discussion in mind, assume that $\overline{\rho}$ is modular, absolutely irreducible, unramified outside of $S \cup S_G$, and has $\overline{\rho}|_{G_{F_w}} = \begin{pmatrix} \overline{\chi\epsilon} & * \\ & \overline{\chi} \end{pmatrix}$ with $*$ nonzero for all $w \in S_G$. If $\overline{\rho}$ corresponds to $\mathfrak{m}$, write $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0}) = \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\mathfrak{m}}$. There is a deformation $\rho(\overline{K}_{S_0})$ over $\mathbb{T}_{\overline{G},\overline{\rho}}$ such that $tr(\mathrm{Frob}_w|_{\rho(\overline{K}_{S_0})}) = T_w$ and $\det(\mathrm{Frob}_w|_{\rho(\overline{K}_{S_0})}) = \#(k_w)S_w$. for all $w \notin S \cup S_G$. Since $\overline{\rho}$ is absolutely irreducible, there is a ring $R_{\overline{\rho},S\cup S_G}$ parameterizing lifts of $\overline{\rho}$ unramified outside of S, together with a universal deformation $\rho^u/R_{\overline{\rho},S\cup S_G}$. One has a map from $R_{\overline{\rho},S\cup S_G} \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. This map is surjective, as $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$ is topologically generated by the T_ℓ and S_ℓs, which must be the image of $tr(\mathrm{Frob}_\ell|_{\rho^u})$ and $\ell^{-1}\det(\mathrm{Frob}_\ell|_{\rho^u})$ respectively.

If $\overline{K}'_{S_0} \subset \overline{K}_{S_0}$, then one has a natural map from $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K'_{S_0}}) \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. Moreover, the map is surjective, as the following diagram commutes and all other maps are surjective:

$$\begin{tikzcd} R_{\overline{\rho},S\cup S_G} \arrow[d] \arrow[dr] & \\ \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}'_{S_0}) \arrow[r] & \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0}). \end{tikzcd}$$

Any modular lift of $\overline{\rho}$ unramified outside of $S \cup S_G$ will have conductor away from $\{p\} \cup S_G$ dividing some integer $N_{\overline{\rho},S_0}$ by results of Carayol and Livné in [Car89] and [Liv89]. This implies that if $\overline{K}_{S_0}$ is sufficiently small (for example, having $\overline{K}_{S_0}$ contained in the principal congruence subgroup mod $N_{\overline{\rho},S_0}$ is sufficient), then the map $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K'_{S_0}}) \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$ is an isomorphism. Define $\mathbb{T}_{\overline{G},\overline{\rho},S} = \varprojlim_{\overline{K}_{S_0}} \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$, and notice that the transition maps are eventually isomorphisms. Thus, one has that $\mathbb{T}_{\overline{G},\overline{\rho},S}$ is a complete noetherian $\mathcal{O}_E$ algebra. Moreover, there is a surjection from $R_{\overline{\rho},S\cup S_G} \to \mathbb{T}_{\overline{G},\overline{\rho},S}$, which gives rise to a universal modular representation $\rho_S^m : G_F \to \mathrm{GL}_2(\mathbb{T}_{\overline{G},\overline{\rho},S})$. The characterizing property of ρ_S^m is that $T_w = tr(\mathrm{Frob}_w|_{\rho_S^m})$ and $\#(k_w)S_w = \det(\mathrm{Frob}_w|_{\rho_S^m})$. We will say $\overline{K}_{S_0}$ is allowable if $\mathbb{T}_{\overline{G},\overline{\rho},S} \to \mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$ is an isomorphism.

Since $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})$ is a $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$-module, one may localize at $\mathfrak{m}$ and obtain a $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$-module, denoted $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho}}(\overline{K}_{S_0})$. Passing to the inverse limit over $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})$, one gets a $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module denoted $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$. This should be thought of as the $\overline{\rho}$-part of $\hat{H}^0_{\mathcal{O}_E,\overline{G},S}$.

4.3 Local-Global Compatibility

At this point, the discussion will focus on the $F = \mathbb{Q}$ case, as everything that will be said relies on the existence of a p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Let $\pi_S^m = B(\rho_S^m|_{G_{\mathbb{Q}_p}})$, the admissible unitary representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ over $\mathbb{T}_{\overline{G},\overline{\rho},S}$ given by the p-adic Langlands correspondence. We will let $\overline{\pi_S^m}$ be $\pi_S^m \otimes_{\mathbb{T}_{\overline{G},\overline{\rho},S}} (\mathbb{T}_{\overline{G},\overline{\rho},S}/\mathfrak{m})$, the mod p representation that π_S^m is a deformation of. Additionally, there is a representation $\pi_{S_0}(\rho_S^m)$ of $\overline{G}_{S_0}$, which is a smooth coadmissible representation of $\overline{G}_{S_0}$ over $\mathbb{T}_{\overline{G},\overline{\rho},S}$. Recall that $\pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m) := \varinjlim_{\overline{K}_{S_0}} \pi_S^m \hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m)^{\overline{K}_{S_0}}$.

Theorem 4.3. *There is an isomorphism of $\mathbb{T}_{\overline{G},\overline{\rho},S}$-modules with an action of $\mathrm{GL}_2(\mathbb{Q}_p) \times \overline{G}_{S_0}$:*

$$\pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m) \tilde{\to} \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}.$$

4.4 Proof of Theorem 4.3

Let $X = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p),\mathbb{T}_{\overline{G},\overline{\rho},S}}(\pi_S^m, \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S})$. There is an action of $\overline{G}_{S_0}$ on X. There is a natural evaluation map $\mathrm{ev}_X : \pi_S^m \overset{\curlywedge}{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} X \to \hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$, and if Y is a submodule of X then one may consider ev_Y as well. These maps will be the object of study for the rest of the section.

Say that a maximal prime $\mathfrak{p} \subset \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]$ is weakly allowable if $\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}$ is crystalline with distinct Hodge-Tate weights, and allowable if it is absolutely irreducible and non-exceptional as well. While a direct proof that the allowable points are Zariski-dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$ would be desirable, it is easier to show that the closure of the allowable points contains the weakly allowable points and the weakly allowable points are dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

We will first determine the structure of the locally algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$. Let $\overline{W}$ be an irreducible algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ and $\overline{K}_p$ be a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$. Then the $\overline{W},\overline{K}_p$-algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$ are the image of the evaluation map

$$\overline{W} \otimes_E \mathrm{Hom}_{\overline{K}_p}(\overline{W}, \hat{H}^0_{E,\overline{G},\overline{\rho},S}) \to \hat{H}^0_{E,\overline{G},\overline{\rho},S}.$$

The $\overline{W}$-algebraic vectors are the union over all $\overline{K}_p$ of the $\overline{W},\overline{K}_p$-algebraic vectors and the $\overline{K}_p$-algebraic vectors are the direct sum over all $\overline{W}$ of the $\overline{W},\overline{K}_p$-algebraic vectors.

Lemma 4.4. *For sufficiently small $\overline{K}_{S_0}$, one has the following:*

1. *$\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}^{GL_2(\mathbb{Z}_p)-alg} = \bigoplus_{\mathfrak{p}} \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}[\mathfrak{p}]^{GL_2(\mathbb{Z}_p)-alg}$ with the sum taken over all weakly allowable primes $\mathfrak{p}$.*
2. *The $GL_2(\mathbb{Z}_p)$-algebraic vectors are dense in $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$.*

Proof. Let $\overline{W}$ be an algebraic representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. There is a local system $\mathcal{V}_{\overline{W}}$ over $X_{\overline{K}_{S_0}}/\overline{K}_p$ whose sections are $H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{V}_{\overline{W}}) = \{f : X_{\overline{K}_{S_0}} \to \overline{W} | f(gk) = k^{-1} \cdot f(g) \, \forall k \in \overline{K}_p\}$. Let $H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\overline{W}}) = \varinjlim_{\overline{K}_p} H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{V}_{\overline{W}})$. The space has a decomposition $H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\overline{W}}) = \bigoplus_\pi \pi_p \otimes (\pi_f^p)^{\overline{K}_{S_0}\overline{K}_0^S}$, where the sum is taken over all automorphic representationsπ of $\overline{G}(\mathbb{A})$ with $\pi_\infty = \check{\overline{W}}$ (this makes sense after choosing an embedding $E \hookrightarrow \mathbb{C}$). This is where the significance of 4.2 comes up: it is a priori possible that these representations arise with higher or lower multiplicity due to the primes where $\overline{G}$ is split, but because of lemma 4.2, we know that any lift of $\overline{\rho}$ must be an extension of characters over all those primes and thus the associated smooth representation will be a character.

Applying Corolloray 2.2.25 of [Eme06], one sees that $\mathrm{Hom}_{\mathfrak{gl}_2}(\overline{W}, \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0}))^{la} = H^0(X_{\overline{K}_{S_0}}, \mathcal{V}_{\check{\overline{W}}})$. Thus, one has $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^{alg} = \oplus_{\overline{W},\pi} \overline{W} \otimes \pi_p \otimes (\pi_f^p)^{\overline{K}_{S_0}\overline{K}_0^S}$, the sum being taken over all algebraic representations $\overline{W}$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ and automorphic representations π of $\overline{G}(\mathbb{A})$ with $\pi_\infty = \overline{W}$. Comparing Hecke actions, one sees that if π corresponds to a representation $\rho(\mathfrak{p})$, then the image of the π part of the above decomposition must be $\mathfrak{p}$-torsion. The $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors arise when π_p has a $\mathrm{GL}_2(\mathbb{Z}_p)$ fixed vector. That happens only when $\mathfrak{p}$ is crystalline with distinct Hodge-Tate weights, namely when $\mathfrak{p}$ is weakly allowable. This shows part 1 of the lemma.

For part 2, we claim that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is an injective $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module. To that end, let M be a smooth finitely generated $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$ module. Define $M^\vee = \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, \mathcal{O}_E/\varpi_E^s)$, the Pontrjagin dual of M. If $\overline{K}_{S_0}$ is sufficiently small, there is a local system $\mathcal{M}^\vee$ over $X_{\overline{K}_{S_0}}/\overline{K}_p$ given by $\mathcal{M}^\vee := (M \times X_{\overline{K}_{S_0}})/\overline{K}_p \to X_{\overline{K}_{S_0}}/\overline{K}_p$. Then one has that $\mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[\overline{K}_p]}(M, \hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})) = H^0(X_{\overline{K}_{S_0}}/\overline{K}_p, \mathcal{M}^\vee)$. Thus, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of smooth finitely generated $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-modules, there is a short exact sequence of local systems $0 \to \mathcal{M}_3^\vee \to \mathcal{M}_2^\vee \to \mathcal{M}_1^\vee \to 0$. This gives rise to a long exact sequence on cohomology, but there are no higher H^is. Thus, the functor $M \to \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[\overline{K}_p]}(M, \hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0}))$ is exact, so $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is an injective $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module.

The injectivity of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ as an $\mathcal{O}_E/\varpi_E^s[\overline{K}_p]$-module is equivalent to the projectivity of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee$ as an $\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]]$-module. It is well known that $\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]]$ is a (noncommutative) noetherian local ring if $\overline{K}_p$ is a pro-p group. Thus, one has that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})^\vee = (\mathcal{O}_E/\varpi_E^s[[\overline{K}_p]])^{r_s}$ for an r_s that depends on s. Consequently, $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0}) \cong \mathcal{C}^0(\overline{K}_p, \mathcal{O}_E/\varpi_E^s)^{r_s}$. Tensoring both sides with k_E, one gets $H^0_{k_E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}^0(\overline{K}_p, k_E)^{r_s}$. The first side is visibly independent of s and so the second side must be too. Letting $r = r_s$, one gets that $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0}) = \mathcal{C}(\overline{K}_p, \mathcal{O}_E)^r$, i.e. $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})^\vee = \mathcal{O}_E[[\overline{K}_p]]^r$, a free $\mathcal{O}_E[[\overline{K}_p]]$-module.

Now, choose $\overline{K}_p$ to be a pro-p normal subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$. The natural map between the functors $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]])}(\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^\vee, *)$ and $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\overline{K}_p]])}(\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^\vee, *)^{\mathrm{GL}_2(\mathbb{Z}_p)/\overline{K}_p}$ is an isomorphism (here, dual is now Schikhof dual). However, the second functor is exact, as $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^\vee$ is free as an $E \otimes (\mathcal{O}_E[[\overline{K}_p]])$-module and taking invariants of a finite group is an exact functor in characteristic 0. Thus, $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})^\vee$ is projective as an $E \otimes (\mathcal{O}_E[[\mathrm{GL}_2(\mathbb{Z}_p)]])$-module. Since projectivity is equivalent to being a summand of a free module, and taking Schikhof duals, one gets that $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$ is a summand of $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)^t$ for some t. Since $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is a summand of $\hat{H}^0_{\mathcal{O}_E/\varpi_E^s,\overline{G}}(\overline{K}_{S_0})$, to prove part 2, it is sufficient to show that the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors are dense in $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)$. However, the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors in $\mathcal{C}(\mathrm{GL}_2(\mathbb{Z}_p), E)$ are the polynomial functions, and the theory of Mahler expansions shows that polynomials are dense in the space of continuous functions. Thus, part 2 holds. □

Corollary 4.5. *The allowable points are Zariski-dense in* $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

Proof. Let $t \in \cap_{\mathfrak{p}}\mathfrak{p}$, the intersection being taken over all weakly allowable points. Then one has that $t\cdot v = 0$ for all $v \in (\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}})^{\mathrm{GL}_2(\mathbb{Z}_p)-alg}$. Thus, by the above lemma, one has $t\cdot\hat{H}^0_{E,G}(\overline{K}_{S_0})_{\overline{\rho}} = 0$. If $\overline{K}_{S_0}$ is sufficiently small, one has that $\mathbb{T}_{\overline{G},\overline{\rho},S} = \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ and since the action of $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ on $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is faithful, $t = 0$. This implies that the weakly allowable points are Zariski-dense in $\mathrm{Spec}(\mathbb{T}_{\overline{G},\overline{\rho},S})$.

Finally, one wants to use Proposition 5.4.9 in [Eme11]. In our case, the eigenvariety that is needed is a union of components of the standard eigencurve for $\mathrm{GL}_2(\mathbb{Q})$: the Hecke eigensystems for $\overline{G}$ with tame level $\overline{K}_{S_0}\overline{K}_0^S$ are exactly given by the eigensystems for $\mathrm{GL}_2(\mathbb{Q})$ with level $\overline{K}_{S_0}$ at all the primes in S_0, maximal level at all primes not in S_0 where $\overline{G}$ is split, and new of level $\Gamma_0(\ell)$ for all ℓ where $\overline{G}$ is nonsplit. □

For a maximal prime $\mathfrak{p} \subset \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]$, define $M(\mathfrak{p})$ to be the colsure of $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$ in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]$. Additionally, define $\rho^m_S(\mathfrak{p}) = \rho^m_S \otimes_{\mathcal{T}_{G,\overline{\rho},S}} \mathcal{T}_{G,\overline{\rho},S}/\mathfrak{p}$ and similarly for $\pi^m_S(\mathfrak{p})$. If $\mathfrak{p}$ is an allowable prime, then by the description of $H^1(X_{\overline{K}_{S_0}}, \mathcal{V}_{\overline{W}})$ given in the proof of lemma 4.4, one has that

$$\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg} = BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}}) \otimes \bigotimes_{\ell\in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}),$$

where $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$ is the locally algebriac representation defined in [BS07]. The seminal result of Berger and Breuil in [BB10] shows that for allowable points, $\pi^m_S(\mathfrak{p})$ is the universal unitary completion of $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$. Thus, taking the $M(\mathfrak{p})^{\overline{K}_{S_0}}$, one gets a complete Banach space that contains $BS(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_p}}) \otimes \left(\bigotimes_{\ell\in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})\right)^{\overline{K}_{S_0}}$ as a dense subspace. After taking the limit over K_{S_0}, one gets

$$M(\mathfrak{p}) = \pi^m_S(\mathfrak{p}) \otimes \bigotimes_{\ell\in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}).$$

At this point, we are in a position to compute $(X \otimes E)[\mathfrak{p}]$. Using the fact that $\pi^m_S(\mathfrak{p})$ is the universal unitary completion of $BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}})$, one gets

$$\begin{aligned}(X \otimes E)[\mathfrak{p}] &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_S^m(\mathfrak{p}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(BS(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_p}}), \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}) \\ &= \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_S^m(\mathfrak{p}), M(\mathfrak{p})) \\ &= \bigotimes_{\ell \in S_0} \pi_{LL}(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})\end{aligned}$$

From the above chain, it is visible that $\mathrm{ev}_X : \pi_S^m(\mathfrak{p}) \otimes \bigotimes_{\ell \in S_0} \pi_{LL}(\rho_S^m(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}}) \to M(\mathfrak{p})$ is an isomoprphism.

Recall the following proposition from [Eme11] (proposition 6.4.2 there):

Proposition 4.6. *Let Y be a saturated coadmissible $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X. Then the following are equivlant:*

1. *Y is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.*
2. *For all allowable $\overline{K}_{S_0}$, $Y^{\overline{K}_{S_0}}$ is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.*
3. *For all allowable primes $\mathfrak{p}$, the inclusion $Y[\mathfrak{p}] \hookrightarrow X[\mathfrak{p}]$ is an isomorphism.*
4. *For all allowable primes $\mathfrak{p}$ and allowable levels $\overline{K}_{S_0}$, the inclusion $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \hookrightarrow X^{\overline{K}_{S_0}}[\mathfrak{p}]$ is an isomorphism.*
5. *$Y \supset X_{ctf}$, where X_{ctf} is the maximal cotorsion free submodule of X as in definition C.39 in [Eme11].*

Proof. The proof will be recalled here as well. It is immediate that 2) $\Rightarrow$ 1) and that 3) $\Leftrightarrow$ 4). Thus, all that is needed for the equivalence of 1) through 4) is 1) $\Rightarrow$ 3) and 4) $\Rightarrow$ 2).

The above description of $X[\mathfrak{p}]$ shows that $X[\mathfrak{p}]$ is an irreducible $\overline{G}_{S_0}$-representation for $\mathfrak{p}$ an allowable prime. Thus, $Y[\mathfrak{p}] \neq 0$ if and only if $Y[\mathfrak{p}] \hookrightarrow X[\mathfrak{p}]$ is an isomorphism. But Proposition C.36 of [Eme11] shows that, if Y is faithful, then $Y[\mathfrak{p}] \neq 0$. Thus, 1) $\Rightarrow$ 3).

Similarly, if $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \hookrightarrow X^{\overline{K}_{S_0}}[\mathfrak{p}]$ is an isomorphimsm for allowable levels $\overline{K}_{S_0}$ and allowable primes $\mathfrak{p}$, then one has that $Y^{\overline{K}_{S_0}}[\mathfrak{p}] \neq 0$ for all allowable primes. Proposition C.22 of [Eme11] implies that the allowable primes are in the cosupport of $Y^{\overline{K}_{S_0}}$ and thus by Zariski density of allowable primes, one has that the cosupport of $Y^{\overline{K}_{S_0}}$ is (0), i.e. Y is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module.

The final step is to show that 5) is equivalent to any of the other parts. To that end, it is useful to show that X_{ctf} is the unique saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X. Proposition C.40 of [Eme11] shows that, since X is a saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-module, so is X_{ctf}. Letting Y be a saturated cotorsion-free faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X, one has that the image of Y in X lies in X_{ctf}, and by 3), $Y[\mathfrak{p}] \hookrightarrow X_{ctf}[\mathfrak{p}]$ is an isomorphism. Proposition C.41 in [Eme11] shows that $Y \hookrightarrow X_{ctf}$ is an isomorphism.

Thus, if Y is any saturated faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}[\overline{G}_{S_0}]$-submodule of X, then Y_{ctf} is faithful as well. Since Y_{ctf} is cotorsion-free, one must have that $Y_{ctf} = X_{ctf}$ and thus $Y \supset X_{ctf}$. This shows that 1) $\Rightarrow$ 5). Conversely, if $Y \supset X_{ctf}$, then Y contains a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module and thus must be faithful itself, showing 5) $\to$ 1). This completes the proof of the proposition. $\square$

Proposition 4.7. *Let Y be a saturated submodule of X that satisfies the equivalent conditions of the above proposition. Then one has that* $\mathrm{ev}_{Y,E}$ *is surjective.*

Proof. We will show that $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E} : \pi_S^m \otimes Y^{K_{S_0}} \to \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is surjective for any allowable $\overline{K}_{S_0}$. This implies the result after taking the limit over all $\overline{K}_{S_0}$.

If $\mathfrak{p}$ is a weakly allowable prime, then one has that $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$ is irreducible as a $\mathrm{GL}_2(\mathbb{Q}_p) \times \overline{G}_{S_0}$-representation. Thus, in order to show that $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]^{alg}$, it is necessary and sufficent to show that it contains one locally algebraic vector. However, one has that Y is faithful as a $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module, and so $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains a Jordan-Holder factor of $\pi_S^m(\mathfrak{p})$. If $\rho_S^m(\mathfrak{p})$ is irreducible, then so is $\pi_S^m(\mathfrak{p})$ and remarks made above show that $\mathrm{im}(\mathrm{ev}_{Y,E})$ contains locally algebraic vectors in the $\mathfrak{p}$-torsion. Otherwise, the other possible Jordan-H "older factors are either a principal series associated to locally algebraic characters, a twist of the Steinberg by a locally algebraic character, or a locally algebraic character. "By hand" calculations on all three of the possibilities show that they all have locally algebraic vectors, and so the image of $\mathrm{ev}_{Y,E}$ contains one (hence all) of the locally algebraic vectors in $\hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{p}]$. Taking $\overline{K}_{S_0}$-invariants, one gets that the image of $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E}$ contains all of the $\mathrm{GL}_2(\mathbb{Z}_p)$-algebraic vectors.

Thus, one has that the image of $\mathrm{ev}_{Y^{\overline{K}_{S_0}},E}$ contains a dense subspace of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$. Lemma 3.1.16 in [Eme11] implies that $\pi_S^m \hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} Y^{\overline{K}_{S_0}}$ is an admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. Moreover, the finiteness of the class group of $\overline{G}$ implies that $H^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})_{\overline{\rho}}$ is admissible. Proposition 3.1.3 of Emerton shows that the image of $\mathrm{ev}_{Y,E}$ is thus closed, and so must be everything. $\square$

Lemma 4.8. *The following are equivalent:*

1. ev_Y *is an isomorphism.*
2. *Y is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module, and* $\mathrm{ev}_Y(\mathfrak{m}) : (Y/\varpi_E)[\mathfrak{m}] \otimes_{k_E} \overline{\pi_S^m} \to \hat{H}^0_{E,\overline{G},\overline{\rho},S}[\mathfrak{m}]$ *is injective.*

If either of these conditions holds, then $Y = X_{ctf}$.

Proof. Assume first that ev_Y is an isomorphism. Because $\hat{H}^0_{E,\overline{G},\overline{\rho},S}$ is a faithful $\mathbb{T}_{\overline{G},\overline{\rho},S}$-module, Y must be as well. Moreover, since ev_Y is an isomorphism, the associated map mod ϖ_E is injective and this remains true when passing to $\mathfrak{m}$ torsion. Thus, 1) $\Rightarrow$ 2).

If $\mathrm{ev}_Y(\mathfrak{m})$ is injective, the lemma C.46 of [Eme11] implies that ev_Y is injective as well, with saturated image. Lemma C.52 of [Eme11] implies that Y must necessarily be saturated in X. Since Y is faithful, Y satisfies the conditions of Proposition 4.6, and thus one has that $\mathrm{ev}_{Y,E}$ is surjective. But, as before, one has that the image of ev_Y is saturated, so $\mathrm{ev}_{Y,E}$ is surjective if and only if ev_Y is. Thus, ev_Y is an isomorphism, showing 2) $\Rightarrow$ 1).

Now assume that ev_Y is an isomorphism. Since Y is faithful, Proposition 4.6 shows that $Y \supset X_{ctf}$. Additionally, one has that $X_{ctf}/\varpi_E X_{ctf} \hookrightarrow Y/\varpi_E Y$ as X_{ctf} is saturated in X and thus Y. This remains injective when passing to $\mathfrak{m}$ torsion, and so one has that X_{ctf} satisfies the conditions of the lemma. Consequently one has that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism and so since $\mathrm{ev}_{X_{ctf}} = \mathrm{ev}_Y \circ \iota_{X_{ctf},Y}$, one has that $\iota_{X_{ctf},Y}$ is an isomorphism as well. That is, $Y = X_{ctf}$. $\square$

A key point to make in the above lemma is that it does not prove that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism unconditionally. It only proves that if there is a submodule $Y \subset X$ such that ev_Y is an isomorphism, then $Y = X_{ctf}$. Indeed, the proof that X_{ctf} satisfies condition 2) in Lemma 4.8 needs that there is a Y such that Y does.

Now, we will recall another key result in [Eme11]. Recall that a Serre weight is a representation of $\mathrm{GL}_2(\mathbb{F}_p)$ over k_E. Such a weight is of the form $Sym^a(\mathrm{Std})\otimes\det^b$, with $0 \le a \le p-1$ and $0 \le b \le p-2$. If V is a Serre weight, then V is a global Serre weight for ρ if $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \hat{H}^1_{E,\overline{G},\overline{\rho},S}) \neq 0$. The set of such weights is denoted $W^{gl}(\overline{\rho})$. A weight V of the form $Sym^a(\mathrm{Std}) \otimes \det^b$ in $W^{gl}(\overline{\rho})$ is called good if either $a < p-1$ or $a = p-1$ and det^b is not in $W^{gl}(\overline{\rho})$. Notice that if V is not good, then necessarily det^b is a global Serre weight for $\overline{\rho}$ and that is a good Serre weight, so $W^{gl}(\overline{\rho})$ contains a good Serre weight.

Theorem 4.9. *1. $W^{gl}(\overline{\rho}) \subset W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$.*

2. If $V \in W^{gl}(\overline{\rho})$ is a good weight, there is an isomorphism of $\mathcal{H}(V)$-modules

$$F_{S_0}\left(soc_{\mathcal{H}(V)}\left(\mathrm{Hom}_{k_E[GL_2(\mathbb{Z}_p)]}(V, H^0_{k_E,\overline{G},\overline{\rho},S})[\mathfrak{m}]\right)\right) \cong soc_{\mathcal{H}(V)} m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}}).$$

3. For any weight $V \in W^{gl}(\overline{\rho})$, the $\overline{G}_{S_0}$-representation $\mathrm{Hom}_{k[GL_2(\mathbb{Z}_p)]}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic.

Part one of this theorem is one direction in the weight part of Serre's conjecture, part two is a mod p multiplicity one result, and part three is Ihara's lemma.

Corollary 4.10. *$\mathrm{Hom}_{GL_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is a generic $\overline{G}_{S_0}$ representation.*

Proof. The proof breaks up into three cases: $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension of χ_1 by χ_2 with $\chi_1\chi_2^{-1} \neq \overline{\epsilon}$, or $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is an extension of χ by $\chi\overline{\epsilon}^{-1}$.

In the first case, one then has that $\overline{\pi^m_S}$ is irreducible. Let V be a weight in $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\overline{\pi^m_S})$, chosen to be one-dimensional if possible. Since $\overline{\pi^m_S}$ is irreducible, V generates $\overline{\pi^m_S}$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$ representation, and so the natural restriction map $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is injective. If the target is nonzero, then V is in $W^{gl}(\overline{\rho})$ and is good, so part three of the above theorem shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic and thus so is $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$. If the target is zero, then so is the source, and thus $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is trivially generic.

In the second case, one has that $\overline{\pi^m_S}$ is an extension of irreducible representations $0 \to \overline{\pi}_1 \to \overline{\pi^m_S} \to \overline{\pi}_2 \to 0$. Letting V_i be a weight for $\overline{\pi}_i$, the exact same argument as above shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_i, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic. Thus, since there is an exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_2, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \\ &\to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_1, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])\end{aligned}$$

which shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ has a quotient by a generic representation that lies inside a generic representation, and thus is generic.

In the final case, one has that there is a short exact sequence as above $0 \to \overline{\pi}_1 \to \overline{\pi^m_S} \to \overline{\pi}_2 \to 0$, where $\overline{\pi}_2 = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi\overline{\epsilon}^{-1} \boxtimes \chi\overline{\epsilon})$ and $\overline{\pi}_1$ now lies in a short exact sequence $0 \to (\chi \circ \det) \otimes St \to \overline{\pi}_1 \to \chi \circ \det \to 0$. A simple calculation shows that $\mathrm{soc}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\overline{\pi}_2)$ has one weight that is not in $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$, and so one has that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_2, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$. Thus, the map $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) \to \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi}_1, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is an injection.

If $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is trés ramifieé, then $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$ contains no one dimensional weights. Thus, $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(\chi \circ \det, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$ and so $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi \circ \det, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]) = 0$. Conseqently, we get that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ embeds into $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}((\chi \circ \det) \otimes St, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$. Moreover, any weight for $(\chi \circ \det) \otimes St$ is good because there are no one dimensional weights in $W^{gl}(\overline{\rho})$, and the argument used above shows that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic.

If, on the other hand, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is peu ramefieé, then the natural surjection $\overline{\pi}_1 \to \chi \circ \det$ admits a $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant section whose image generates $\overline{\pi}_1$ as a $\mathrm{GL}_2(\mathbb{Q}_p)$-representation. Thus, $\chi \circ \det$ is a weight for $\overline{\rho}|_{G_{\mathbb{Q}_p}}$, which must be good. Thus, arguing as above, one sees that $\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}])$ is generic. □

Lemma 4.11. *If $\mathrm{ev}_{X_{ctf}}$ is an isomorphism, then one has that $X_{ctf} = \pi_{S_0}(\rho^m_S)$*

Proof. We will let $\mathcal{C}$ be the set of weakly allowable primes in $\mathbb{T}_{\overline{G},\overline{\rho},S}$. Then, with this choice of $\mathcal{C}$, one needs to show that both X_{ctf} and $\pi_{S_0}(\rho^m_S)$ satisfy the conditions of Theorem 4.4.1 in [Eme11]. This will show that there is an isomorphism $X_{ctf} \cong \pi_{S_0}(\rho^m_S)$.

Because X_{ctf} is saturated in X, there is a chain of embeddings

$$(X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] \hookrightarrow (X/\varpi_E X)[\mathfrak{m}] \hookrightarrow \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi^m_S}, H^0_{k_E,\overline{G},\overline{\rho},S}[\mathfrak{m}]).$$

Since the last term in the chain is generic as a $\overline{G}_{S_0}$ representation by Corollary 4.10, so is the first.

By the assumption on $\mathrm{ev}_{X_{ctf}}$, one has that $\mathrm{ev}_{X_{ctf}}(\mathfrak{m})$ is an embedding of $(X_{ctf}/\varpi_E X_{ctf}) \otimes_{k_E} \overline{\pi^m_S} \hookrightarrow H^0_{k_E,\overline{G},\overline{\rho},S}$. Letting $V \in W^{gl}(\overline{\rho})$ be a good weight, one has an isomorphism $m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}}) \cong$

$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m})$. Then there is the following chain of maps which are all embeddings (indeed, all but one are isomorphisms):

$$\begin{aligned}
&\mathrm{soc}_{\mathcal{H}(V)}(m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}})) \otimes_{k_E} F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]) \\
&= \mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m})) \otimes_{k_E} F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]) \\
&= F_{S_0}(\mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, \overline{\pi_S^m} \otimes_{k_E} (X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}]))) \\
&\hookrightarrow F_{S_0}(\mathrm{soc}_{\mathcal{H}(V)}(\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Z}_p)}(V, H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]))) \\
&\cong \mathrm{soc}_{\mathcal{H}(V)}(m(V, \overline{\rho}|_{G_{\mathbb{Q}_p}})).
\end{aligned}$$

The last isomorphism is from part two of Theorem 4.9. This implies that $F_{S_0}((X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}])$ is at most one-dimensional. Thus, condition 1 of Theorem 4.4.1 in [Eme11] is satisfied.

By the discussion about allowable primes $\mathfrak{p}$, one has that $X_{ctf}[\mathfrak{p}] \otimes_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}]/\mathfrak{p} = X_{ctf}[\mathfrak{p}] \otimes_{\mathcal{O}_E} E = \otimes_{\ell \in S_0} \pi_{LL}(\rho(\mathfrak{p})|_{G_{\mathbb{Q}_\ell}})$, and then [JL70] shows that that representation is generic. This shows parts a and b of 2 of Theorem 4.4.1 in [Eme11]. All that is left is to show that the closure of the saturation of $\Sigma_{\mathfrak{p} \in \mathcal{C}} X_{ctf}[\mathfrak{p}]$ is all of X_{ctf}.

If one lets Y be the closure of the saturation of $\Sigma_{\mathfrak{p} \in \mathcal{C}} X_{ctf}[\mathfrak{p}]$, then one has that $E \otimes_{\mathcal{O}_E} Y[\mathfrak{p}] = E \otimes_{\mathcal{O}_E} X[\mathfrak{p}]$ for all allowable primes $\mathfrak{p}$, and thus $Y \supset X_{ctf}$. But one also has that Y is the closure of a subspace of X_{ctf}, and thus $Y = X_{ctf}$, showing part c of 2. Thus, one has that $X_{ctf} \cong \pi_{S_0}(\rho_S^m)$. □

Lemma 4.12. $\mathrm{ev}_{X_{ctf}}(\mathfrak{m})$ *is injective.*

Proof. Recall that $\mathrm{ev}_{X_{ctf}} : \overline{\pi_S^m} \otimes_{k_E} (X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$. Since $(X_{ctf}/\varpi_E X_{ctf})[\mathfrak{m}] = \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\overline{\pi_S^m}, H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}])$, Lemma 6.4.15 of [Eme11] shows that it is sufficient to show that any non-zero map $\overline{\pi_S^m} \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$ is injective. Again, the proof breaks up into three cases: $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is irreducible, $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is the extension of χ_1 by χ_2 with $\chi_1 \chi_2^{-1} \neq \overline{\epsilon}$, or $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is the extension of χ by $\chi \overline{\epsilon}^{-1}$.

In the first case, one has that $\overline{\pi_S^m}$ is irreducible, and so any non-zero $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant map must be injective.

In the second case, one has that $\overline{\pi_S^m}$ is an extension of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2 \overline{\epsilon})$ by $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1 \overline{\epsilon})$. $W(\overline{\rho}|_{G_{\mathbb{Q}_p}})$ consists of a single Serre weight, and that weight corresponds to the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2 \overline{\epsilon})$. Since the $\mathrm{GL}_2(\mathbb{Z}_p)$-socle of $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1 \overline{\epsilon})$ is not the same, one has that there is no non-zero map from $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_2 \boxtimes \chi_1 \overline{\epsilon}) \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$. Thus, any non-zero map from $\overline{\pi_S^m} \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$ must be non-zero on $\mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi_1 \boxtimes \chi_2 \overline{\epsilon})$ and thus is injective.

In the final case, there is a filtration $0 \subset \overline{\pi}_1 \subset \overline{\pi}_2 \subset \overline{\pi_S^m}$ with $\overline{\pi}_1 = (\chi \circ \det) \otimes_{k_E} St$, $\overline{\pi}_2/\overline{\pi}_1 = \chi \circ det$, and $\overline{\pi_S^m}/\overline{\pi_2} = \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{Q}_p)}(\chi \overline{\epsilon}^{-1} \boxtimes \chi \overline{\epsilon})$. The same Serre weight considerations as above show that there is no non-zero map from $\overline{\pi_S^m}/\overline{\pi}_2 \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$. Additionally, Ihara's lemma guarentees that there is no non-zero map from $\overline{\pi}_2/\overline{\pi}_1 \to H^0_{k_E, \overline{G}, \overline{\rho}, S}[\mathfrak{m}]$, as the image would be one dimensional. Thus, any non-zero map must have trivial kernel, and thus is injective. □

Proof of Theorem 4.3. Lemmas 4.8 and 4.12 show that $\mathrm{ev}_{X_{ctf}}$ is an isomorphism. Lemma 4.11 shows that, if $\mathrm{ev}_{X_{ctf}}$ is an isomorphism, then $X_{ctf} = \pi_{S_0}(\rho_S^m)$. Thus, $\mathrm{ev}_{X_{ctf}}$ provides an isomorphism between $\pi_S^m \hat{\otimes}_{\mathbb{T}_{\overline{G},\overline{\rho},S}} \pi_{S_0}(\rho_S^m)$ and $\hat{H}^0_{\mathcal{O}_E,\overline{G},\overline{\rho},S}$, showing the theorem. □

5 Analysis of the Completed H^1 for G

The aim of this section is to understand the completed cohomology for G. The first main theorem of the paper will be shown here. We have moved back to allowing $F \neq \mathbb{Q}$.

5.1 Completed Cohomology and Completed Hecke Algebras

As before, let S_0 be a finite set of places of F. Assume that $v \notin S_0$ and that any place w where $G(F_w) \neq \mathrm{GL}_2(F_w)$ is not in S_0. Then let $S = S \cup \{v\}$ and $S_G = \{w | G(F_w) \neq \mathrm{GL}_2(F_w)\}$, and will choose a maximal compact subgroup $K_0^S \subset G(\mathbb{A}^S_{F,f})$. Let $G_{S_0} = \prod_{w \in S_0} G(F_w)$. Then recall the following set of definitions:

Definition 2. *If $K_{S_0} \subset G_{S_0}$ is a compact open subgroup, then define*

$$\hat{H}^i_{\mathcal{O}_E,G}(K_{S_0}) = \varprojlim_{s} \varinjlim_{K_v} H^i_{\acute{e}t}(Sh_{K_v K_{S_0} K_0^S,\overline{L}}, \mathcal{O}_E/\varpi_E^s).$$

Additionally, let $\hat{H}^i_{E,G}(K_{S_0}) = \hat{H}^i_{\mathcal{O}_E,G}(K_{S_0}) \otimes_{\mathcal{O}_E} E$ and $\hat{H}^i_{*,G,S} = \varinjlim_{K_{S_0}} \hat{H}^i_{*,G}(K_{S_0})$.

$\hat{H}^1_{E,G}(K_{S_0})$ is an admissible unitary representation of $D^\times_{F_v}$. Additionally, there is an action of G_L (or in the case of $F = \mathbb{Q}$ one gets an action of $G_\mathbb{Q}$) on $\hat{H}^1_{*,G}(K_{S_0})$ that commutes with the action of $D^\times_{F_v}$. Moreover, G_{S_0} acts smoothly on $H^1_{E,G,S}$. The last action that needs to be discussed is that of a Hecke algebra. Let $\mathbb{T}_G(K_v K_{S_0})$ be the $\mathcal{O}_E$ subalgebra of $\mathrm{End}(H^1_{\acute{e}t}(Sh_{K_v K_{S_0} K_0^S,\overline{F}}, E))$ generated by the operators T_w and S_w for $w \notin S \cup S_G$. As before, there is a surjection $\mathbb{T}_G(K'_v K_{S_0}) \to \mathbb{T}_G(K_v K_{S_0})$ if $K'_v \subset K_v$. As before, let $\mathbb{T}_G(K_{S_0}) = \varprojlim_{K_v} \mathbb{T}_G(K_v K_{S_0})$, the completed Hecke algebra for K_{S_0}. If $\overline{\rho}$ is a G-modular, absolutely irreducible 2-dimensional k_E-valued representation of G_L that is unramified outside of $S \cup S_G$, then there is a maximal ideal $\mathfrak{m} \subset \mathbb{T}_G(K_{S_0})$ such that $T_w \equiv tr(\mathrm{Frob}_w|_{\overline{\rho}}) \pmod{\mathfrak{m}}$ and $S_w \equiv \#(k_w)^{-1} \det(\mathrm{Frob}_w|_{\overline{\rho}})$. We will let $\mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ be $\mathbb{T}_G(K_{S_0})_\mathfrak{m}$. The same argument from section 4.2 shows that there is a compact open subgroup $K_{S_0} \subset G_{S_0}$ such that if $K'_{S_0} \subset K_{S_0}$, then the map from $\mathbb{T}_{G,\overline{\rho}}(K'_{S_0}) \to \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ is an isomorphism. Let $\mathbb{T}_{G,\overline{\rho},S}$ be $\varprojlim_{K_{S_0}} \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$ and say that K_{S_0} is an allowable level if $\mathbb{T}_{G,\overline{\rho},S} = \mathbb{T}_{G,\overline{\rho}}(K_{S_0})$.

Recall from section 4.2 that there is also a Hecke algebra $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$. Since the two Hecke algebras are determined by the corresponding Hecke eigensystems (that is, maps from $\mathbb{T}$ to finite extensions of E) and any Hecke eigensystem for G necessarily gives one for $\overline{G}$ by the global Jacquet-Langlands correspondence, there is a surjection from $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\frac{1}{p}] \to \mathbb{T}_G(K_{S_0})[\frac{1}{p}]$. Additionally, this map commutes with the localization mentioned above, so there is a surjection $\mathbb{T}_{\overline{G},\overline{\rho}}(\overline{K}_{S_0})[\frac{1}{p}] \to \mathbb{T}_{G,\overline{\rho}}(K_{S_0})[\frac{1}{p}]$, and this extends to a surjection $\mathbb{T}_{\overline{G},\overline{\rho},S}[\frac{1}{p}] \to \mathbb{T}_{G,\overline{\rho},S}[\frac{1}{p}]$.

5.2 Statement of Results

Let $\mathfrak{p}$ be a maximal ideal in $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\frac{1}{p}]$. Then, by looking at $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}]$, one gets a representation of $\mathrm{GL}_2(F_v)$ that depends only on $\mathfrak{p}$ and K_{S_0} which will be denoted $\pi_{\mathfrak{p},K_{S_0}}$. Conjecturally, this representation should be of the form $\pi_{\mathfrak{p},K_{S_0}} = \pi(\rho^m_S(\mathfrak{p})|_{G_{F_v}}) \otimes (\bigotimes_{w\in S_0} \pi_{LL}(\rho^m_S(\mathfrak{p})|_{G_{F_w}}))^{\overline{K}_{S_0}}$ with $\pi(\rho^m_S(\mathfrak{p})|_{G_{F_v}})$ being a unitary representation of $\mathrm{GL}_2(F_v)$ that depends only on $\rho^m_S(\mathfrak{p})|_{G_{L_{v_1}}}$, but the incomplete knowledge of the p-adic Langlands program prevents such a decomposition from being known. Since $\mathbb{T}_G(K_{S_0})[\frac{1}{p}]$ is a $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\frac{1}{p}]$-algebra, one may also talk about $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}]$, which is a representation of $D^\times_{F_v} \times G_L$. The first main result is the following:

Theorem 5.1. *The space $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}]$ depends only on $\pi_{\mathfrak{p},K_{S_0}}$ as a $D^\times_{F_v} \times G_{F_v}$-representation.*

Since the main difficulty in getting the optimal result in Theorem 5.1 is the lack of a p-adic Langlands correspondence in generality, one would hope that there is a stronger theorem for the case when $F = \mathbb{Q}$ and $F_v = \mathbb{Q}_p$. Indeed, there is, which is given by the following:

Theorem 5.2. *If ρ is a two-dimensional promodular representation of $G_{\mathbb{Q}}$, unramified away from S, and such that $\overline{\rho}$ such that $\overline{\rho}|_{G_{\mathbb{Q}_p}}$ is indecomposable and not of the form $\begin{pmatrix} \chi & * \\ & \chi \end{pmatrix}$ nor $\begin{pmatrix} \chi & * \\ & \chi\overline{\epsilon} \end{pmatrix}$, $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ is of the form $\begin{pmatrix} \chi\overline{\epsilon} & * \\ & \chi \end{pmatrix}$ for all $\ell \in S_G$, and ρ is unramified away from $S \cup S_G$, then*

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G,\overline{\rho},S}) = J(\rho|_{G_{\mathbb{Q}_p}}) \otimes (\bigotimes_{\ell\in S_0} \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}})),$$

where $J(\rho|_{G_{\mathbb{Q}_p}})$ is a unitary representation of $D^\times$ that depends only on $\rho|_{G_{\mathbb{Q}_p}}$ and the isomorphism is as a $D^\times \times G_{S_0}$-representation

5.3 Proofs of Main Theorems

Recall that the Čerednik-Drinfel'd uniformization breaks up as $Sh_{K^n_v K_{S_0} K^S_0,\mathbb{C}_p} = \coprod_i \Gamma'_i\backslash\Sigma^n_{\mathbb{C}_p}$, with the Γ'_i being discrete cocompact subgroups of $\{g \in \mathrm{GL}_2(F_v) | \det(g) \in \mathcal{O}^\times_F\}$. We will drop the $\mathbb{C}_p$-subscript and will write Σ^n for $\Sigma^n_{\mathbb{C}_p}$ until otherwise noted. Letting $\pi : \Sigma^n \to \Gamma'_i\backslash\Sigma^n$ be the natural projection. Letting $\acute{e}tSh(\Gamma'_i\backslash\Sigma^n)$ be the category of étale sheaves on $\Gamma'_i\backslash\Sigma^n$, $\Gamma'_i - \acute{e}tSh(\Sigma^n)$ be the category of Γ'_i-equivariant sheaves on Σ^n, $\Gamma'_i - Mod$ the category of Γ'_i-modules, and Ab the category of abelian groups, there is a commutative diagram of functors:

$$\begin{tikzcd}
\Gamma'_i - \acute{e}tSh(\Sigma^n) \arrow[r, "\Gamma_{\Sigma^n}"] & \Gamma'_i - Mod \arrow[d, "(\cdot)^{\Gamma'_i}"] \\
\acute{e}tSh(\Gamma'_i\backslash\Sigma^n) \arrow[u, "\pi^*"] \arrow[r, "\Gamma_{\Gamma'_i\backslash\Sigma^n}"] & Ab.
\end{tikzcd}$$

If K_{S_0} is sufficiently small, then the action of Γ'_i is free on Σ^n and thus π^* is an equivalence of categories. Additionally, $\pi^*\left(\underline{\mathcal{O}_E/\varpi_E^s}\right) = \underline{\mathcal{O}_E/\varpi_E^s}$, so applying the Grothendieck-Leray spectral sequence, one gets $R^a(\cdot)^{\Gamma'_i}\left(R^b\Gamma_{\Sigma^n}\left(\underline{\mathcal{O}_E/\varpi_E^s}\right)\right) \Rightarrow R^{a+b}\Gamma_{\Gamma'_i\backslash\Sigma^n}\left(\underline{\mathcal{O}_E/\varpi_E^s}\right)$. Giving the functors their more common names, one gets $H^a(\Gamma'_i, H^b_{ét}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \Rightarrow H^{a+b}_{ét}(\Gamma'_i\backslash\Sigma^n, \mathcal{O}_E/\varpi_E^s)$. In low degree terms, there is an exact sequence coming from the spectral sequence

$$\begin{aligned} 0 &\to H^1(\Gamma'_i, H^0_{ét}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \to H^1_{ét}(\Gamma'_i\backslash\Sigma^n, \mathcal{O}_E/\varpi_E^s) \to H^1_{ét}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)^{\Gamma'_i} \\ &\to H^2(\Gamma'_i, H^0_{ét}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)). \end{aligned}$$

The condition that Γ'_i is a discrete cocompact subgroup of $\mathrm{SL}_2(F_v)$ means that Γ'_i is an essentially free group. If one choose K_{S_0} to be sufficiently small, then one has that Γ'_i is a free group. Thus, for sufficiently small K_{S_0}, there is no $H^2(\Gamma'_i, *)$ for any choice of $*$.

The other term that admits easy analysis is the $H^1(\Gamma'_i, *)$ term. Since (again, if K_{S_0} is sufficiently small) Γ'_i acts freely on the Bruhat-Tits tree $\mathcal{T}$ for $\mathrm{PGL}_2(F_v)$, one has that $H^1(\Gamma'_i, *) = H^1(\Gamma'_i\backslash\mathcal{T}, *)$ for any choice of $*$. As a point of clarity, the cohomology on the right in that equality is simply Betti cohomology of a graph. At the end of the day, we will be taking $\varprojlim_s\left(\varinjlim_n H^1(\Gamma'_i\backslash\mathcal{T}, H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s))\right)$. Choosing n' such that $\pi_0(\Sigma^n) = (\mathcal{O}_{F_v}/\varpi_{F_v}^{n'})^\times$, one has that $H^1(\Gamma'_i\backslash\mathcal{T}, H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) = H^1(\Gamma'_i\backslash(\mathcal{T}\times(\mathcal{O}_{F_v}/\varpi_{F_v}^{n'})^\times), \mathcal{O}_E/\varpi_E^s)$. Now, $\Gamma'_i\backslash(\mathcal{T}\times(\mathcal{O}_{F_v}/\varpi_{F_v}^{n'})^\times)$ is a finite graph, and so we can replace that term with $H^1(\Gamma'_i\backslash(\mathcal{T}\times(\mathcal{O}_{F_v}/\varpi_{F_v}^{n'})^\times), \mathcal{O}_E)\otimes\mathcal{O}_E/\varpi_E^s$. Finally, tensor product commutes with colimits, so the term that we will ultimately be considering is $\varprojlim_s\left(\left(\varinjlim_{n'} H^1((\Gamma'_i\backslash\mathcal{T}\times(\mathcal{O}_{F_v}/\varpi_{F_v}^{n'})^\times), \mathcal{O}_E)\right)\otimes\mathcal{O}_E/\varpi_E^s\right)$, which visibly satisifies the Mittag-Leffler condition.

We will consider the space of harmonic cochains on the graphs $\Gamma'_i\backslash\mathcal{T}$. Since the top dimensional simplex is 1-dimensional, there is only one condition for a cochain f being harmonic, namely that $\sum_{v'} f(\overrightarrow{vv'}) = 0$ for all vertivies v, where the sum is taken over all verticies v' adjacent to v. If H is a graph and M is an abelian group, then let $\mathrm{Harm}^1(H, M)$ be the space of M-valued harmonic 1-cochains on H.

There always is a map from $\mathrm{Harm}^1(H, M) \to H^1(H, M)$ sending a cocycle to its image in H^1.

Claim 5.3. *There exists an integer N depending only on H such that the map from* $\mathrm{Harm}^1(H, M) \to H^1(H, M)$ *has kernel and cokernel killed by N.*

The proof of this claim will take us too far afield and will be deferred to the end of the section.

If there is a nonzero kernel after taking the inverse limit over s, that would produce arbitrarly large p-power torsion in the maps from the space of harmonic cochains to H^1, since the space of harmonic cochains is torsion-free. That lies in contradiction with the bound on torsion produced by claim 5.3. Additionally, tensoring with E will kill the cokernel, so one has an isomorphism between $\left(\varprojlim_s\varinjlim_n H^1(\Gamma'_i, H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s))\right)\otimes_{\mathcal{O}_E} E$ and $\left(\varprojlim_s\varinjlim_n \mathrm{Harm}^1(\Gamma'_i\backslash\mathcal{T}, H^0(\Sigma^n, \mathcal{O}_E/\varpi_E^s))\right)\otimes_{\mathcal{O}_E} E$.

One then has that $\mathrm{Harm}^1(\Gamma_i'\backslash\mathcal{T}, H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) = \mathrm{Harm}^1(\mathcal{T}, H^0_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s))^{\Gamma_i'}$ because the condition of being harmonic is a local condition on the graph. Additionally, we may also replace that term with $(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}(\mathcal{O}_F^\times, \mathcal{O}_E/\varpi_E^s))^{\Gamma_i'}$ by the remarks on the cohomology of a graph made above. When summing over all the connected components, its possible to replace

$$\bigoplus_{\Gamma_i'} \left(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}(\mathcal{O}_F^\times, \mathcal{O}_E/\varpi_E^s)\right)^{\Gamma_i'}$$

with

$$\bigoplus_{\Gamma_i} \left(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}(F^\times, \mathcal{O}_E/\varpi_E^s)\right)^{\Gamma_i}$$

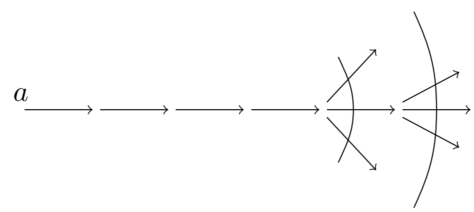
just from the definition of the Γ_is and Γ_i's.

Now, since $X_{\overline{K}_{S_0}} = \oplus_{\Gamma_i}\Gamma_i\backslash\mathrm{GL}_2(F_v)$, we have that $\oplus_{\Gamma_i}\Gamma_i\backslash\mathcal{T} = (\mathcal{T} \times X_{\overline{K}_{S_0}})/\mathrm{GL}_2(F_v)$, and so we may replace the space above with $\left(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}(F^\times \times X_{\overline{K}_{S_0}}, \mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}$.

The following lemma is useful for the analysis of this space.

Lemma 5.4. *There are no smooth harmonic $\mathcal{O}_E/\varpi_E^s$-valued 1-cochains on $\mathcal{T}$.*

Proof. It is useful to recall what all the adjectives in the above lemma mean. A 1-cochain on $\mathcal{T}$ is a function f on directed edges such $f\left(\overrightarrow{ab}\right) = -f(\overrightarrow{ba})$. Being harmonic means $\sum_b f\left(\overrightarrow{ab}\right) = 0$ for all verticies a, where the sum is taken over all vertices b that are adjacent to a. Finally, smooth means that, for one (equivalently any) choice of vertex a to be the center of the tree, there is an integer n such that, if the distance from e to a is greater than n, then $f(e)$ depends only on the distance from e to a and the edge of distance n from a along the shortest path from a to e. The picture below shows the edges that must have the same value for $p = 3$ and $n = 4$.



Now, let e be an edge that is distance $> n$ from a and oriented away from a (using the notation of the previous paragraph). Then one has that $f(e) = \sum_{e'} f(e')$, where the sum is over all edges adjacent to e and one unit farther from a. But smoothness implies that the values of $f(e')$ are constant. Thus, one has that $p|f(e)$. The same argument applies to $f(e')$ for e' one unit further from a, and thus one gets $p^2|f(e)$. One can repeat this process until one has that $\varpi_E^s|f(e)$. But at that point, one must have $f(e) = 0$. Thus, $f(e) \equiv 0$ for all edges e distance greater than n from a. If e is distance exactly n from a, then for all e' adjacent to e and distance $n+1$ from a, one has that $f(e') = 0$. Thus, $f(e) = 0$ by harmonicity. This argument again repeats until one sees that $f \equiv 0$. $\square$

Since, in the term $\left(\mathrm{Harm}^1(\mathcal{T}, \mathcal{O}_E/\varpi_E^s) \otimes_{\mathcal{O}_E/\varpi_E^s} \mathcal{C}(F^\times \times X_{\overline{K}_{S_0}}, \mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}$, the left hand side is a smooth representation of $\mathrm{GL}_2(F_v)$, we have that $\mathcal{C}(F^\times \times X_{\overline{K}_{S_0}}, \mathcal{O}_E/\varpi_E^s) = \varinjlim_{K_p} \mathcal{C}(F^\times \times$

$X_{\overline{K}_{S_0}}, \mathcal{O}_E/\varpi_E^s)^{K_p}$. Direct limits commute with tensor product, so we get that space is equal to

$$\left(\varinjlim_{K_p}\left(\mathrm{Harm}^1(\mathcal{T},\mathcal{O}_E/\varpi_E^s)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}(F^\times\times X_{\overline{K}_{S_0}},\mathcal{O}_E/\varpi_E^s)\right)^{K_p}\right)^{\mathrm{GL}_2(F_v)}$$

. But then there are no elements in the inner part, as anything on the left hand side is invariant under a fixed compact open subgroup and nothing on the right hand side is. Thus, the space $\left(\mathrm{Harm}^1(\mathcal{T},\mathcal{O}_E/\varpi_E^s)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}(F^\times\times X_{\overline{K}_{S_0}},\mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}$ is trivial.

This means that, while there might be no control on $H^1(\Gamma_i', H^0_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s))$, when we take

$$\left(\varprojlim_s\bigoplus_{\Gamma_i'}H^1(\Gamma_i',H^0_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s))\right)\otimes_{\mathcal{O}_E}E$$

, we may replace that with

$$\left(\varprojlim_s\left(\mathrm{Harm}^1(\mathcal{T},\mathcal{O}_E/\varpi_E^s)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}(F^\times\times X_{\overline{K}_{S_0}},\mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}\right)\otimes_{\mathcal{O}_E}E$$

which is 0 by the remarks above.

To finish the initial analysis of the exact sequence, one needs to simplify the following two objects:

$$\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s\varinjlim_n H^1_{\acute{e}t}(\Gamma_i'\backslash\Sigma^n,\mathcal{O}_E/\varpi_E^s)\right)\otimes_{\mathcal{O}_E}E\right)\ \text{and} \tag{1}$$

$$\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s\varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)^{\Gamma_i'}\right)\otimes_{\mathcal{O}_E}E\right). \tag{2}$$

Notice that, by definition of Γ_i', there is an isomorphism $Sh_{K_v^nK_{S_0}K_0^S,\mathbb{C}_p}\equiv\coprod_{\Gamma_i'}\Gamma_i'\backslash\Sigma^n$. Moreover, this decomposition respects connected components. Thus, one has that $\bigoplus_{\Gamma_i'}H^1_{\acute{e}t}(\Gamma_i'\backslash\Sigma^n,\mathcal{O}_E/\varpi_E^s)=H^1_{\acute{e}t}(Sh_{K_p^nK_{S_0}K_0^S,\mathbb{C}_p},\mathcal{O}_E/\varpi_E^s)$, and consequently, the term in (1) is $\hat{H}^1_{E,G}(K_{S_0})$.

Term (2) can be simplified as follows:

$$\begin{aligned}&\bigoplus_{\Gamma_i'}\left(\left(\varprojlim_s\varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)^{\Gamma_i'}\right)\otimes_{\mathcal{O}_E}E\right)\\ =&\bigoplus_{\Gamma_i}\left(\left(\varprojlim_s\left(\left(\varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)\right)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}(\Gamma_i\backslash\mathrm{GL}_2(F_v),\mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}\right)\otimes_{\mathcal{O}_E}E\right)\\ =&\left(\left(\varprojlim_s\left(\left(\varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)\right)\otimes_{\mathcal{O}_E/\varpi_E^s}\mathcal{C}(X_{\overline{K}_{S_0}},\mathcal{O}_E/\varpi_E^s)\right)^{\mathrm{GL}_2(F_v)}\right)\otimes_{\mathcal{O}_E}E\right)\end{aligned}$$

Define $\hat{H}^1_{\mathcal{O}_E}(\Sigma):=\varprojlim_s\left(\varinjlim_n H^1_{\acute{e}t}(\Sigma^n,\mathcal{O}_E/\varpi_E^s)\right)$, $\hat{H}^1_E(\Sigma):=\hat{H}^1_{\mathcal{O}_E}(\Sigma)\otimes_{\mathcal{O}_E}E$, and $\hat{H}^1_E(\Sigma)^\circ=\mathrm{im}(\hat{H}^1_{\mathcal{O}_E}(\Sigma)\to\hat{H}^1_E(\Sigma))$.

Lemma 5.5. • $\hat{H}^1_{\mathcal{O}_E}(\Sigma)$ *is ϖ_E-adically complete.*

- $\varprojlim_s \left(\left(\varinjlim_n H^1_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)\right) \otimes M\right)$ *is isomorphic to $\hat{H}^1_{\mathcal{O}_E}(\Sigma)\hat{\otimes} M$ for any ϖ_E-adically complete $\mathcal{O}_E$-module M.*

Proof. For the first part, consider the short exact sequence $0 \to \underline{\mathcal{O}_E} \to \underline{\mathcal{O}_E} \to \underline{\mathcal{O}_E/\varpi_E^s} \to 0$ of sheaves on Σ^n. This gives rise to the short exact sequence

$$0 \to H^1(\Sigma^n, \mathcal{O}_E)/\varpi_E^s \to H^1(\Sigma^n, \varpi_E^s) \to H^2(\Sigma^n, \mathcal{O}_E)[\varpi_E^s] \to 0.$$

Taking the direct limit over n preserves exactness, as this is a filtered colimit. Additionally, one has that the sequence $\varinjlim_n H^1(\Sigma^n, \mathcal{O}_E)/\varpi_E^s$ has all transition maps being surjective, so the sequence remains exact upon taking the inverse limit over s. Thus, one has that there is a short exact sequence

$$0 \to \left(\varinjlim_n H^1(\Sigma, \mathcal{O}_E)\right)^{\wedge} \to \hat{H}^1_{\mathcal{O}_E}(\Sigma) \to T_{\varpi_E}\left(\varinjlim_n H^2(\Sigma, \mathcal{O}_E)\right) \to 0.$$

The left and right terms of this exact sequence are ϖ_E-adically complete and therefore the middle term is too.

Now, to prove the second part, let X be any ϖ_E-adically complete $\mathcal{O}_E$ module, and let $X_s = X/\varpi_E^s$, and similarly for M. Then one has that

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{O}_E}(X, \hat{H}^1_{\mathcal{O}_E}(\Sigma)\hat{\otimes} M) \\
=&\varprojlim_s \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(X_s, \left(\hat{H}^1_{\mathcal{O}_E}(\Sigma)/\varpi_E^s\right) \otimes_{\mathcal{O}_E/\varpi_E^s} M_s) \\
=&\varprojlim_s \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(X_s \otimes_{\mathcal{O}_E/\varpi_E^s} M_s^\vee, \hat{H}^1_{\mathcal{O}_E}(\Sigma)/\varpi_E^s) \\
=&\varprojlim_s \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(X_s \otimes_{\mathcal{O}_E/\varpi_E^s} M_s^\vee, \varinjlim_n H^1(\Sigma^n, \mathcal{O}_E/\varpi_E^s)) \\
=&\varprojlim_s \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}\left(X_s, \left(\varinjlim_n H^1(\Sigma^n, \mathcal{O}_E/\varpi_E^s)\right) \otimes_{\mathcal{O}_E/\varpi_E^s} M_s\right) \\
=&\mathrm{Hom}_{\mathcal{O}_E}\left(X, \varprojlim_s \left(\left(\varinjlim_n H^1_{\acute{e}t}(\Sigma^n, \mathcal{O}_E/\varpi_E^s)\right) \otimes_{\mathcal{O}_E/\varpi_E^s} M\right)\right)
\end{aligned}$$

Here, $M_s^\vee$ is Pontryagin dual. There are maps from $M_s^\vee \to M_{s-1}^\vee$ by viewing these as reductions of the Schikhof dual mod ϖ_E^s. The middle isomorphism arises from the fact that the maps from $\hat{H}^1_{\mathcal{O}_E}(\Sigma)/\varpi_E^s \to \varinjlim_n H^1(\Sigma^n, \mathcal{O}_E/\varpi_E^s)$ give rise to an isomorphism in the inverse limit. This identification of homs shows the desired isomorphism. □

Putting this all together, one gets

$$\hat{H}^1_{E,G}(K_{S_0}) \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})\right)^{\mathrm{GL}_2(F_v)}. \tag{$*$}$$

As remarked, this is an isomorphism as $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[D_{F_v}^\times \times G_{L_{v_1}}]$-modules. We now prove the main theorems of the section.

Theorem 5.6. *Let $\mathfrak{p} \in \operatorname{maxSpec}(\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\frac{1}{p}])$. Letting $\pi_{\mathfrak{p},K_{S_0}} = \hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}]$, a $GL_2(F_v)$-representation, there is an isomorphism $\hat{H}^1_{E,G}(K_{S_0})[\mathfrak{p}] \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}\pi_{\mathfrak{p},K_{S_0}}\right)^{GL_2(F_v)}$.*

Proof. The main issue is showing $\left(\hat{H}^1_E(\Sigma)\hat{\otimes}\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})\right)^{\mathrm{GL}_2(F_v)}[\mathfrak{p}]$ is the same as $\left(\hat{H}^1_E(\Sigma)\hat{\otimes}\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}]\right)^{\mathrm{GL}_2(F_v)}$. Expanding out what the second term is and letting $\mathfrak{p}' = \mathfrak{p} \cap \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})$, one gets

$$\left(\left(\varprojlim_s \left(\left(\hat{H}^1_E(\Sigma)^\circ/\varpi_E^s\right) \otimes_{\mathcal{O}_E/\varpi_E^s} \left(\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}']/\varpi_E^s\right)\right)\right) \otimes_{\mathcal{O}_E} E\right)^{\mathrm{GL}_2(F_v)}.$$

We can pull $\mathfrak{p}'$-torsion past quotienting out by ϖ_E^s; this may change things at finite level but will produce an isomorphism after taking inverse limits. We can rewrite the internal tensor product as $(\hat{H}^1_E(\Sigma)^\circ \otimes_{\mathcal{O}_E} \mathbb{T}_{\overline{G}}(\overline{K}_{S_0})/\varpi_E^s) \otimes_{\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})/\varpi_E^s} \hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})/\varpi_E^s[\mathfrak{p}']$. The left hand side is flat as a $\mathbb{T}_{\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}']/\varpi_E^s$-module because $\hat{H}^1_E(\Sigma)^\circ$ is flat as an $\mathcal{O}_E$-module. Now, if A is a flat R-module, B is any R-module, and I is an ideal in R generated by $\{r_i\}$, then there is a short exact sequence $0 \to B[I] \to B \to \oplus_i B$ where the final map sends b to $\oplus_i r_i b$. Since A is flat, the sequence remains exact when tensoring with A, and so one gets that $(A \otimes_R B)[I] = A \otimes_R (B[I])$. Thus, we may also pull $\mathfrak{p}'$-torsion past the first tensor product.

Pulling $\mathfrak{p}'$-torsion past the inverse limit is also fine: using the same notation and generality as above, $B[I] = \operatorname{Hom}_R(R/I, B)$ and Hom commutes with limits. Pulling $\mathfrak{p}'$-torsion past tensoring with E is exactly the same as above, as E is a flat $\mathcal{O}_E$-module. Additionally, this turns the $\mathfrak{p}'$-torsion into $\mathfrak{p}$-torsion. Finally, since the action of $\mathrm{GL}_2(F_v)$ and the Hecke algebra commute, you can pull the $\mathfrak{p}$-torsion past the $\mathrm{GL}_2(F_v)$-invariants. Thus, one gets that $\hat{H}^0_{\mathcal{O}_E,\overline{G}}(\overline{K}_{S_0})[\mathfrak{p}']/\varpi_E^s$ is isomorphic to $\left(\hat{H}^1_E(\Sigma)\hat{\otimes}\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})\right)^{\mathrm{GL}_2(F_v)}[\mathfrak{p}]$. Appealing to $*$ and the notation defined in the statement of the theorem, the theorem itself follows. □

Theorem 5.7. *If $F = \mathbb{Q}$ and thus $F_v = \mathbb{Q}_p$, there is an isomorphism*

$$\hat{H}^1_{E,G,\overline{\rho},S}[\mathfrak{p}] \cong \left(\hat{H}^1_E(\Sigma)\hat{\otimes}B(\rho_S^m(\mathfrak{p}))\right)^{GL_2(\mathbb{Q}_p)} \overset{\curlywedge}{\otimes} \pi_{S_0}(\rho_S^m(\mathfrak{p})).$$

Finally, if one defines $J(\pi) := \operatorname{Hom}_{G_{\mathbb{Q}_p}}\left(\mathbb{V}(\pi), \left(\hat{H}^1_E(\Sigma)\hat{\otimes}\pi\right)^{\mathrm{GL}_2(\mathbb{Q}_p)}\right)$ for π a unitary continuous Banach representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then the following corollary is immediate from Theorem 5.7:

Corollary 5.8. *Let ρ be a 2-dimensional promodular E-representation of $G_{\mathbb{Q}}$. Assume further that $\overline{\rho}$ satisfies all of the running assumptions. Then there is an isomorphism*

$$\operatorname{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G,\overline{\rho},S}) \cong J(B(\rho|_{G_{\mathbb{Q}_p}})) \overset{\curlywedge}{\otimes} \pi_{S_0}(\rho).$$

Proof. The main issue here is explaining how to extract the action of $G_{\mathbb{Q}}$ here. However, one has that the ρ part of the completed cohomology lies in the $t_\ell = tr(Frob_\ell|_\rho)$-eigenspace, letting us use the results of section 4. □

Finally, we will close this section off with the proof of claim 5.3.

Proof. Recall that we are interested in proving the following result: let H be a finite graph, and M a $\mathbb{Z}$-module. Then there exists an integer N depending only on H such that the kernel and cokernel of the natural map from $\mathrm{Harm}^1(H,M) \to H^1(H,M)$ has kernel and cokernel killed by N. While this result will only be applied for the case of $p+1$ regular graphs and cohomology with coefficients in a $\mathbb{Z}_p$-module, the proposition is true under the much weaker assumptions listed here. The definition of harmonic is that $\sum_b f\left(\overrightarrow{ab}\right) = 0$ for all verticies a, where the sum is taken over all verticies b adjacent to a. In particular, there is no normalization going on.

The first step is to prove the proposition in the case of $\mathbb{R}$-valued cochains. Let f be an $\mathbb{R}$-valued cochain, and define $\Delta f = \sum_a \left(\sum_b f\left(\overrightarrow{ab}\right)\right)^2$ with the outer sum taken over all verticies and the inner sum taken over all vertecies adjacent to a. Then f is harmonic if and only if $\Delta f = 0$. The goal now is to show that if you have a cochain f that is not harmonic, it is possible to vary f along the space of cochains cohomologous to it such that you can reduce the value of $\Delta(f)$. Then, if we view Δ as a function on the space of cochains cohomologous to f, then this is the restriction of a positive definite quadratic form on a vector space to an affine subspace and so it has a nonnegative minimum. Assuming that you can do the variation mentioned, the minimum value can't be positive, and hence it must be 0. Thus, every cohomology class has a representative that is harmonic and so the map from harmonic cochains to cohomology is surjective.

To that end, let f be a cochain that is not harmonic. Let a be a vertex in H, and t to be a small real number. Defining $g_{a,t}$ to be the 0-cochain that is t at a and 0 everywhere else, we get that $\frac{d}{dt}\left(\Delta(f+g_{a,t})\right) = 2deg(a)\sum_b f\left(\overrightarrow{ab}\right) - 2\sum_{a'}\sum_b f\left(\overrightarrow{a'b}\right)$ (here, the first and second sum are over verticies adjacent to a and the third is over verticies adjacent to a'). We want to choose a so that this is forced to be nonzero. If a is chosen so that the value $\left|\left(\sum_b f\left(\overrightarrow{ab}\right)\right)\right|$ is maximized, then this works as long as $\left|\left(\sum_b f\left(\overrightarrow{ab}\right)\right)\right|$ is strictly larger than one of the values of $\left|\left(\sum_b f\left(\overrightarrow{a'b}\right)\right)\right|$ for some a' adjacent to a. If $\left|\left(\sum_b f\left(\overrightarrow{ab}\right)\right)\right| = \left|\left(\sum_b f\left(\overrightarrow{a'b}\right)\right)\right|$ for all a' adjacent to a, then we are still fine unless $\left(\sum_b f\left(\overrightarrow{ab}\right)\right) = \left(\sum_b f\left(\overrightarrow{ab}\right)\right)$ for all a' adjacent to a. However, that situation cannot persist across the whole graph: if it did then $\sum_a\sum_b f\left(\overrightarrow{ab}\right)$ would be nonzero, which cannot be the case because each edge appears twice with opposite directions in that sum. Thus, its possible to choose a vertex a such that $\frac{d}{dt}\left(\Delta(f+g_{a,t})\right) \neq 0$, and so it is possible to reduce the value of $\Delta(f)$ within the same cohomology class, completing the proof of surjectivity.

To show injectivity, assume that f is harmonic and also that $f = dg$ for some 0-cochain g. Harmonicity for f implies harmonicity for g: $deg(a)g(a) = \sum_b g(b)$ for all a, where the sum is taken over all b adjacent to a. Choosing a with the largest value, one sees that $g(b) = g(a)$ for all b adjacent to a. Thus, g is locally constant, and so $(f =)dg = 0$. Thus, the map is an isomorphism over $\mathbb{R}$.

Now, let M be any $\mathbb{Z}$-module. Let X be the space of $\mathbb{Z}$-valued 1-cochains of H, and Y be the space of $\mathbb{Z}$-valued 0-cochains of H. Consider the chain complex $0 \to X \to Y \to 0$ with the middle map sending f to the function $g_f(a) = \sum_b f\left(\overrightarrow{ab}\right)$. The first homology of this complex is just $\mathrm{Harm}_1(H, \mathbb{Z})$, and the zeroth homology is just Y. Then theorem 3.6.1 in [Wei94] shows that there is a short exact sequence $0 \to \mathrm{Harm}_1(H, \mathbb{Z}) \otimes M \to \mathrm{Harm}_1(H, M) \to \mathrm{Tor}_1(Y, M) \to 0$. However, Y is finitely-generated and so $\mathrm{Tor}_1(Y, M)$ is a torsion $\mathbb{Z}$-module and thus killed by some integer N'.

Finally, we to show that there is an integer N'' depending only on H such that the map from $\mathrm{Harm}_1(H, \mathbb{Z}) \otimes M \to H^1(H, M)$ has kernel and cokernel killed by N''. By the fact that the theorem is true for $\mathbb{R}$, we see that there is a short exact sequence $0 \to \mathrm{Harm}_1(H, \mathbb{Z}) \to H^1(H, \mathbb{Z}) \to B \to 0$ where B is a finite torsion $\mathbb{Z}$-module. Thus, tensoring with M, we get $0 \to \mathrm{Tor}_1(B, M) \to \mathrm{Harm}_1(H, \mathbb{Z}) \otimes M \to H^1(H, \mathbb{Z}) \otimes M \to B \otimes M \to 0$. But $H^1(H, \mathbb{Z}) \otimes M = H^1(H, M)$. Thus, if N'' is the exponent of B, we get that the map from $\mathrm{Harm}_1(H, \mathbb{Z}) \otimes M \to H^1(H, M)$ has kernel and cokernel killed by N''. Putting this together with the fact that the map from $\mathrm{Harm}_1(H, \mathbb{Z}) \otimes M \to \mathrm{Harm}_1(H, M)$ is injective with cokernel killed by an integer N', we get the claim. □

6 Locally Algebraic Vectors

Throughout this section, the assumption that E is "large enough" will include the assumption that all algebraic representations of $D_{F_v}^\times$ as an algebraic group over $\mathbb{Q}_p$ that are defined over $\overline{\mathbb{Q}_p}$ are defined over E. This is equivalent to the assumption that there is a fixed embedding $E \to \overline{\mathbb{Q}_p}$ such that every embedding $\iota : F_v \to \overline{\mathbb{Q}_p}$ factors through $E \to \overline{\mathbb{Q}_p}$ and for every ι there is a field H_ι such that $\iota(F) \subset H_\iota \subset E$ and $[H_\iota : \iota(F)] = 2$. In particular, this can always be realized by, e.g., taking E to be the unramified quadratic extension of the Galois closure of $F/\mathbb{Q}_p$. If π is a Banach space representation of a p-adic group, then π^{alg} will be the locally algebraic vectors in π, and π^{la} will be the locally analytic vectors in π.

This aim of this section is to describe the locally algebraic vectors in the representations $J(\pi)$ in the $F_v = \mathbb{Q}_p$ case and $J'(\pi)$ in the general case. In order to describe the main result, we will introduce a Jacquet-Langlands map from finite length locally algebraic representations of $\mathrm{GL}_2(F_v)$ to finite length locally algebraic representations of $D_{F_v}^\times$. Since $D_{F_v}^\times \times_{\mathbb{Q}_p} E = \prod_{\iota: F_v \to E} \mathrm{GL}_2/E$, the algebriac representations of $D_{F_v}^\times$ and $\mathrm{GL}_2(F_v)$ over E are naturally identified. Every finite length locally algebraic representation of $\mathrm{GL}_2(F_v)$ is of the form $\bigoplus_i \pi_i \otimes W_i$, where π_i is an indecomposable smooth representation of $\mathrm{GL}_2(F_v)$ and W_i is an algebraic representation of $\mathrm{GL}_2(F_v)$. There are 5 possibilities for π_i: π_i may be a character, an irreducible principle series, a twist of the Steinberg representation, an extension of a character by the Steinberg, or a supercuspidal representation.

Definition 3. *Let $\pi = \bigoplus_i \pi_i \otimes W_i$ as above. Define $JL(\pi_i \otimes W_i)$ to be 0 unless π_i is either a twist of the Steinberg representation or a spercuspidal representation. If $\pi_i = (\chi \circ det) \otimes St$, then define $JL(\pi_i \otimes W_i) = (\chi \circ \nu) \otimes W_i$, where W_i is viewed as a representation of $D_{F_v}^{\times}$ as above. If π_i is a supercuspidal representation, then define $JL(\pi_i \otimes W_i) = JL^{cl}(\pi_i) \otimes W_i$, where JL^{cl} is the classical Jacquet-Langlands correspondence for smooth supercuspidal representations of $GL_2(F_v)$. Finally, define $JL\left(\bigoplus_i \pi_i \otimes W_i\right) = \bigoplus_i JL(\pi_i \otimes W_i)$.*

The following is the main result of this section:

Theorem 6.1.

- *If $F = \mathbb{Q}$, assume that $\rho : G_{\mathbb{Q}} \to GL_2(E)$ is as in the introduction. Then $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$.*
- *For a general L/F as in section 3, one has that, if $\rho : G_L \to GL_2(E)$ arises in the cohomology of $Sh_{K_v K_{S_0} K_0^S}$, and π is the ρ part of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$, then $J'(\pi)^{alg} = JL(\pi^{alg})$.*

This section is similar to section 4.2 in [Eme06], but there are three main remarks to be made. First of all, several of the results will be slightly less explicit. This is primarily an issue of focus: the aim here is to determine what the locally algebraic vectors are, whereas the aim in [Eme06] is to elucidate the structure of the eigencurve. Secondly, there is no difference between compactly supported cohomology and regular cohomology in this case. This means that the arguments that were specific to H^i_c or H^i in [Eme06] will now be needed to be made at the same time. Finally, while one might imagine that differences between the groups GL_2 and $D_{F_v}^{\times}$ would come up, this is not the case. The group theoretic properties of GL_2 that are used in [Eme06] and thus the properties of $D_{F_v}^{\times}$ that are used in this paper are all about the algebraic representation theory. Since the algebraic representation theory of a group over a sufficiently large base field depends only on the geometric isomorphsim type of the group, none of those arguments will change.

6.1 Notation and Elementary Calculations

For this section, $\mathfrak{d}_{F_v}$ will be the Lie algebra of $D_{F_v}^{\times}$. As remarked before, this will be viewed as a group over $\mathbb{Q}_p$. Let $\mathfrak{sd}_{F_v}$ be the Lie algebra of the group $SD_{F_v}^{\times} := \{d \in D_{F_v} | \nu(d) = 1\}$ and $\mathfrak{z}_{F_v}$ be the Lie algebra of $Z(D_{F_v}^{\times}) = F_v^{\times}$. As remarked at the start of this section, one has that $\mathfrak{d}_{F_v}$ is a form of $\mathfrak{gl}_{2,F_v}$ and $\mathfrak{sd}_{F_v}$ is a form of $\mathfrak{sl}_{2,F_v}$. There is a natural isomorphism $\mathfrak{d}_{F_v} = \mathfrak{sd}_{F_v} \oplus \mathfrak{z}_{F_v}$. This isomorphism arises from the group homomorphism $SD_{F_v}^{\times} \times Z(D_{F_v}^{\times}) \to D_{F_v}^{\times}$ sending $(d, x) \to dx$, which has finite kernel and cokernel. Notice that $\mathfrak{z}_{F_v}$ is abelian, as it is the Lie algebra of an abelian group, and $\mathfrak{sd}_{F_v}$ is semisimple, as it is a form of a semisimple Lie group. Let $H^i(\mathfrak{g}; W) = Ext^i_{\mathfrak{g}}(\check{W}, E)$; this is the Lie algebra cohomology of $\mathfrak{g}$. An important result is the following:

Proposition 6.2. *Let W be an irreducible algebraic representation of $SD_{F_v}^{\times}$. Then one has that $H^i(\mathfrak{sd}_{F_v}; W) = 0$ unless W is the trivial representation, $3|i$, and $i \leq 3\deg(F/\mathbb{Q}_p)$.*

Proof. For irreducible representations W of $SD_{F_v}^{\times}$, there is a tensor product decomposition $W = \bigotimes_{\iota : F_v \hookrightarrow E} W_\iota$, where the terms W_ι are the base change of irreducible representations of $SD_{F_v}^{\times}$ as a group over F_v to E along ι. On the level of Lie algebras, this decomposition arises becuase one has that $\mathfrak{so}_{F_v} \otimes_{\mathbb{Q}_p} E = \bigoplus_\iota \mathfrak{so}_{F_v} \otimes_{\iota(F_v)} E$. As a notational convience, let $\mathfrak{so}_\iota$ be the summands of the direct sum decomposition of $\mathfrak{so}_{F_v} \otimes_{\mathbb{Q}_p} E$. There is a Künneth formula, showing that $H^*(\mathfrak{so}_{F_v}; W) = \bigotimes_\iota H^*(\mathfrak{so}_\iota; W_\iota)$. Thus, the statement in the theorem is reduced to the statement that, for any irreducible representation W_ι of $\mathfrak{so}_\iota$, then $H^i(\mathfrak{so}_\iota, W_\iota) = 0$ unless $i = 0$ or 3 and W_ι is trivial.

$H^0(\mathfrak{so}_\iota; W_\iota) = W_\iota^{\mathfrak{so}_\iota}$. This is visibly 0 unless W_ι is the trivial representation. Since $\mathfrak{so}_\iota$ is semisimple, there are no non-trivial extensions between finite dimensional representations of $\mathfrak{so}_\iota$, so $H^1(\mathfrak{so}_\iota; W_\iota) = 0$ for all W_ι. Poincaré duality says that $H^3(\mathfrak{so}_\iota; E) = E$, that the cup product pairing from $H^i(\mathfrak{so}_\iota; W_\iota) \times H^{3-i}(\mathfrak{so}_\iota; \check{W}_\iota) \to H^3(\mathfrak{so}_\iota; E)$ is a perfect pairing for $0 \leq i \leq 3$, and that $H^i(\mathfrak{so}_\iota; W_\iota) = 0$ for all $i > 3$. Since $H^1(\mathfrak{so}_\iota, \check{W}_\iota) = 0$ for all W_ι, one has that $H^2(\mathfrak{so}_\iota; W_\iota) = 0$. If W_ι is not the trivial representation, then $H^0(\mathfrak{so}_\iota; \check{W}_\iota) = 0$ so $H^3(\mathfrak{so}_\iota; W_\iota) = 0$ as well. Finally, if $i > 3$, then $H^i(\mathfrak{so}_\iota; W_\iota) = 0$ directly because of Poincaré duality. $\square$

There are also a couple important results that are needed in order to apply the results in [Eme06].

Proposition 6.3. *$\hat{H}^i_{\mathcal{O}_E,G}(K_{S_0})$ is the ϖ_E-adic completion of $\varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{L}}, \mathcal{O}_E)$.*

Proof. $Sh_{K_{S_0}K_p}$ is a complete curve, and thus one has that $H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E^s = H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E/\varpi_E^s)$ as any obstruction to this being true must come from a torsion class in $H^{i+1}_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E)$, of which there are none. This remains true when passing to the limit along the K_ps, and since the ϖ_E-adic completion of $\varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E)$ is definitionally

$\varprojlim_s \varinjlim_{K_p} H^i_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{O}_E) \otimes_{\mathcal{O}_E} \mathcal{O}_E/\varpi_E^s$, the result follows. $\square$

The importance of this proposition is that it identifies what we denote $\hat{H}^i$ with what is denoted $\hat{H}^i$ in [Eme06]. The other result is the following calculation:

Proposition 6.4. *The space $\hat{H}^2_{E,G}(K_{S_0})$ vanishes.*

Proof. We will show that, for all K_p and s, there is a compact open subgroup $K'_p \subset K_p$ such that the map from $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p}, \mathcal{O}_E/\varpi_E^s) \to H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s)$ is 0. This implies that $\varinjlim_{K_p} H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s) = 0$ and thus that $\hat{H}^2_{E,G}(K_{S_0}) = 0$.

Fixing K_p and s, choose $K'_p \subset K_p$ such that $p^s | [(K'_p \cap SD^\times_{F_v}) : (K_p \cap SD^\times_{F_v})]$. $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p}, \mathcal{O}_E/\varpi_E^s)$ is freely generated by classes $[C']$ for each connected curve C' of $Sh_{K_{S_0}K_p}$, and $H^2_{\acute{e}t}(Sh_{K_{S_0}K_p}, \mathcal{O}_E/\varpi_E^s)$ is freely generated by by classes $[C]$ for each connected curve C of $Sh_{K_{S_0}K_p}$. If the preimage of C is $\coprod_i C'_i$ with the degree of the map from $C'_i \to C$ being d_i, then the image of $[C]$ in $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p.\overline{L}}, \mathcal{O}_E/\varpi_E^s)$ is $\sum_i d_i[C'_i]$. But, over a fixed connected curve C of $Sh_{K_{S_0}K_p}$, the map from $Sh_{K_{S_0}K'_p} \to Sh_{K_{S_0}K_p}$ sends $[K'_p : K_p]/[(K'_p \cap SD^\times_{F_v}) : (K_p \cap SD^\times_{F_v})]$ connected curves C' onto C with degree $[(K'_p \cap SD^\times_{F_v}) : (K_p \cap SD^\times_{F_v})]$. Thus, the image of $H^2_{\acute{e}t}(Sh_{K_{S_0}K_p,\overline{L}}, \mathcal{O}_E/\varpi_E^s)$ in $H^2_{\acute{e}t}(Sh_{K_{S_0}K'_p,\overline{L}}, \mathcal{O}_E/\varpi_E^s)$ is divisible by p^s, and thus ϖ_E^s, and thus is 0. $\square$

Proposition 6.5. *There is an integer r and a compact open subgroup $H \subset F_v^\times$ such that $\hat{H}^0_{E,G}(K_{S_0}) \cong \mathcal{C}^0(H, \mathcal{O}_E)^r$.*

Proof. In the $F = \mathbb{Q}$ case, the connected components of $Sh_{K_pK_{S_0}K_0^S}$ are parameterized by the double coset space $\mathbb{R}_{>0} \times \mathbb{Q}^\times \backslash \mathbb{A}^\times / \nu(K_pK_{S_0}K_0^S)$. It is well known that $\mathbb{A}^\times = \mathbb{R}_+ \times \mathbb{Q}^\times \times \prod_{\ell \neq p} \mathbb{Z}_\ell^\times \times \mathbb{Z}_p^\times$. If I let $S^p = \prod_{\ell \neq p} \mathbb{Z}_\ell^\times / \nu(K_{S_0})$ and $r = \#(S^p)$, then one gets that the connected components are parameterized by $S^p \times (\mathbb{Z}_p^\times / \nu(K_p))$. Additionally, as K_p shrinks to $\{1\}$, $\nu(K_p)$ shrinks to $\{1\}$ as well. An argument entirely analogous to Claim 4.1 shows that $\hat{H}^0_{\mathcal{O}_E,G}(K_{S_0}) = \mathcal{C}^0(\mathbb{Z}_p^\times \times S^p, \mathcal{O}_E) = \mathcal{C}^0(\mathbb{Z}_p^\times, \mathcal{O}_E)^r$.

In the other case, let $T = \{x \in L^\times | x\overline{x} = 1\}$, a torus over F that is the determinant group of G. Then the connected components of $Sh_{K_vK_{S_0}}$ are parameterized by the double coset space $T(F)\backslash T(\mathbb{A}_{F,f})/\det(K_vK_{S_0})$. Choose $H \subset F_v^\times = T(F_v)$ sufficiently small that $T(\mathcal{O}_F) \cap H \det(K_{S_0}) = \{1\}$. If $S^v = T(F)\backslash T(\mathbb{A}_{F,f})/H\det(K_{S_0})$ and $r = \#(S^v)$, then the description of the connected components shows that, for K_v sufficiently small, the connected components are parameterized by $S^v \times H/\det(K_v)$ as an H-set. Now, a similar argument as before shows that $\hat{H}^0_{\mathcal{O}_E,G}(K_{S_0}) = \mathcal{C}^0(H, \mathcal{O}_E)^r$. ∎

Corollary 6.6. *Viewing $\hat{H}^0_{E,G}(K_{S_0})$ as a representation of $SD_{F_v}^\times \times Z(D_{F_v}^\times)$, one has that $\hat{H}^0_{E,G}(K_{S_0}) = 1_{SD_{F_v}^\times} \boxtimes \hat{H}^0_{E,G}(K_{S_0})|_{Z(D_{F_v}^\times)}$.*

Proof. This claim is equivalent to the triviality of the action of $SD_{F_v}^\times$ on $\hat{H}^0_{E,G}(K_{S_0})$. However, this is exactly the content of 6.5. ∎

Proposition 6.7. *Let W be an irreducible algebraic representation of $D_{F_v}^\times$. Then*

$$\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) = 0$$

unless W is one dimensional, $3|i$, and $i \leq 3\deg(F/\mathbb{Q})$

Proof. Let χ be the central character of W. One may then write $\check{W} = E(\chi^{-1}) \boxtimes \check{W}|_{\mathfrak{sd}_{F_v}}$ where the first factor is a representation of $\mathfrak{z}_{F_v}$. Since the action of $\mathfrak{sd}_{F_v}$ on $\hat{H}^0_{E,G}(K_{S_0})^{la}$ is trivial, one may write $\hat{H}^0_{E,G}(K_{S_0})^{la} = \hat{H}^0_{E,G}(K_{S_0})^{la}|_{\mathfrak{z}_{F_v}} \boxtimes E$ where the second factor is the trivial representation of $\mathfrak{sd}_{F_v}$. Now, one can appeal to the Künneth formula and get

$$\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) = \bigoplus_{a+b=i} \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E(\chi^{-1}), \hat{H}^0_{E,G}(K_{S_0})^{la}) \otimes \mathrm{Ext}^b_{\mathfrak{sd}_{F_v}}(\check{W}, E).$$

The second term is equal to $H^b(\mathfrak{sd}_{F_v}; W)$, which is zero unless $3|b$ and $\dim(W) = 1$. Thus, we may assume that $\dim(W) = 1$. To compute the first term, choose a compact $H \subset Z(D_{F_v}^\times)$ sufficiently small so that $\nu|_H : H \to F_v^\times$ is an isomorphism onto its image (this is possible because ν restricts to an isomorphism of Lie algebras $\mathfrak{z}_{F_v} \to \mathrm{Lie}(F_v^\times)$ and such that $\hat{H}^0_{E,G}(K_{S_0}) = \mathcal{C}^0(H,E)^r$ as H-representations for some integer r. Then there is a sequence of isomorphisms

$$\begin{aligned}\mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E(\chi^{-1}), \hat{H}^0_{E,G}(K_{S_0})^{la}) &\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, \hat{H}^0_{E,G}(K_{S_0})^{la} \otimes E(\chi)) \\ &\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, (\mathcal{C}^0(H,E)^{la})^r \otimes E(\chi)) \\ &\cong \mathrm{Ext}^a_{\mathfrak{z}_{F_v}}(E, (\mathcal{C}^0(H,E)^{la})^r) \\ &\cong H^a(\mathfrak{z}_{F_v}; (\mathcal{C}^0(H,E)^{la})^r).\end{aligned}$$

The third isomorphism arises because $f(h) \mapsto \chi(h)f(h)$ is an automorphism on $\mathcal{C}^0(H,E)$, and the fourth isomorphism is an alternative definition of Lie algebra cohomology. Then by theorems 1.1.12 (v) and 1.1.13 in [Eme06], one sees that $H^a(\mathfrak{z}_{F_v}; (\mathcal{C}^0(H^E)^{la})^r) = 0$ unless $a = 0$. Thus, the only nonzero terms in the Kunneth description are when $a = 0$ and $b = 0$ or $b = 3$. That is, $\mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la})$ is only nonzero when $3|i$ and $i \leq 3\deg(F/\mathbb{Q})$. $\square$

6.2 Conculsions About Locally Algebraic Vectors

The following notation will be useful in this section. Recall that there is a local system $\mathcal{V}_W$, defined in [Del71] for the $F = \mathbb{Q}$ case and in (much) more generality in e.g. [Eme06], on $Sh_{K_{S_0}K_p}$. We will let $H^i(\mathcal{V}_W, K_{S_0}) = \varinjlim_{K_p} H^i_{ét}(Sh_{K_{S_0}K_p,\overline{\mathbb{Q}}}, \mathcal{V}_W)$. This is a smooth E representation of $D^\times_{F_v}$ that can be understood in terms of automorphic forms. In particular, one has that $H^i(\mathcal{V}_W, K_{S_0})$ vanishes if $i > 2$. Additionally, Corollary 2.2.18 in [Eme06] constructs a spectral sequence with E_2 page given by $E_2^{ij} = \mathrm{Ext}^i_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^j_{E,G}(K_{S_0})^{la})$ which converges to $H^{i+j}(\mathcal{V}_W, K_{S_0})$. Since the $E_2^{i,j}$ terms are zero for $j > 2$, the spectral sequence collapses on the E_3 page. Proposition 6.7 says that the $E_2^{i,0}$ terms are 0 unless $3|i$ and $i \leq 3\deg(F/\mathbb{Q})$. Since $H^n(\mathcal{V}_W, K_{S_0}) = 0$ for $n \geq 3$, one has that $E_2^{i,1} = 0$ for $i \geq 2$. Additionally, one has that $d : E^{1,1} \to E^{3,0}$ is surjective with kernel H^2, that $E_2^{0,1} = H^1$, and that $E_2^{0,0} = H^0$. Unwinding the terms in the spectral sequence, one has the following equalities:

$$H^0(\mathcal{V}_W, K_{S_0}) = \mathrm{Hom}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}), \tag{3}$$

$$H^1(\mathcal{V}_W, K_{S_0}) = \mathrm{Hom}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}), \text{ and} \tag{4}$$

$$\mathrm{Ext}^{3j-2}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) = \mathrm{Ext}^{3j}_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}). \qquad (2 \leq j \leq \deg(F/\mathbb{Q})) \tag{5}$$

and the following short exact sequence:

$$0 \to H^2(\mathcal{V}_W, K_{S_0}) \to \mathrm{Ext}^1_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) \to \mathrm{Ext}^3_{\mathfrak{d}_{F_v}}(\check{W}, \hat{H}^0_{E,G}(K_{S_0})^{la}) \to 0.$$

Finally, notice that, unless W is one-dimensional, the left-hand term and hence the right-hand term in the first equality is 0.

For the reader that is unfamiliar with spectral sequences, here is a picture of the E_2-page with (possibly) non-zero terms written as $E_2^{i,j}$ to keep in mind when reading the above:

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \dots \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & \dots \\
E_2^{0,0} & 0 & 0 & E_2^{3,0} & 0 & \dots
\end{array}$$

Recall the following fact proved using local-global compatibility at p and multiplicity one results:

Fact 1. *Let W have weights $\{(w_{\iota,1}, w_{\iota,2})\}$ where $w_{\iota,1} < w_{\iota,2}$. Then one has*

$$H^1(\mathcal{V}_W, K_{S_0}) = \bigoplus_{\rho} \rho \otimes JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes \left({\bigotimes_{\ell \in S_0}}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}},$$

with the direct sum running over all G-modular representations ρ of $G_{\mathbb{Q}}$ in the $F = \mathbb{Q}$ case and G_L case for general F with Hodge-Tate weights $-w_{\iota,2} - 1$ and $-w_{\iota,1}$.

This follows from resluts of Carayol in [Car89], Saito in [Sai97], and the global Jacquet-Langlands correspondence. Explicitly, Carayol and Saito explain how to get the corresponding result for $\mathrm{GL}_2(\mathbb{Q})$, and the global Jacquet-Langlands correspondence explains how to transfer this result to G. Here, ρ being G-modular means that there is an automorphic form on G such that $tr(Frob_w|_\rho)$ is equal to the T_w Hecke eigenvalue for the automorphic form for almost all w.

The following precise formulations of Theorem 6.1 can now be proved:

Theorem 6.8. *Let $\rho : G_{\mathbb{Q}} \to GL_2(E)$ be a promodular representation of $G_{\mathbb{Q}}$ unramified outside of S. Assume further that $\overline{\rho}|_{G_{\mathbb{Q}_\ell}}$ is an extension of a character χ by $\chi\epsilon$ for all $\ell \neq p$ such that G is ramified at ℓ. Then one has that $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$.*

Proof. Consider $\mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg})$. On one hand, this is equal to

$$\left(J(B(\rho|_{G_{\mathbb{Q}_p}})) \otimes \left({\bigotimes_{\ell \in S_0}}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}} \right)^{alg} = \left(J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \otimes \left({\bigotimes_{\ell \in S_0}}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}} \right)$$

by theorem 5.7. On the other hand, this space is the union of the spaces $\check{W} \otimes \mathrm{Hom}_{G_{\mathbb{Q}}, \mathfrak{d}_{F_v}}(\rho \otimes \check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la})$ over all algebraic representations W. As noted above, one has $\mathrm{Hom}_{G_{\mathbb{Q}}, \mathfrak{d}_{F_v}}(\rho \otimes \check{W}, \hat{H}^1_{E,G}(K_{S_0})^{la}) = \mathrm{Hom}_{G_{\mathbb{Q}}}(\rho, H^1(\mathcal{V}_W))$. At this point, it is important to make sure that K_{S_0} is chosen small enough that $\left(\bigotimes'_{\ell \in S_0} \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}}$ is non-zero.

If ρ is G-modular, then $\rho|_{G_{\mathbb{Q}_p}}$ is potentially semistable with distinct Hodge-Tate weights $w_1 < w_2$ and moreover one has that $WD(\rho|_{G_{\mathbb{Q}_p}})$ is indecomposable. In this case, one has that ρ arises in $H^1(\mathcal{V}_W)$ where W has weights $-w_2 \leq -w_1 - 1$ and only for this W. For such a W, $\check{W}$ has weights w_2 and $w_1 + 1$, which means that $\check{W} = Sym^{w_2 - w_1 - 1}(Std) \otimes det^{w_1+1}$. Importantly, $B(\rho|_{G_{\mathbb{Q}_p}})^{alg} = \pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes \check{W}$. Thus, one has that $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg}) = JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes \check{W}$. Plugging this into $\mathrm{Hom}_{G_\mathbb{Q}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg})$, the following chain of equalities holds:

$$
\begin{aligned}
\left(J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} \otimes \left(\bigotimes_{\ell \in S_0} {}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}} \right) &= \mathrm{Hom}_{G_\mathbb{Q}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg}) \\
&= \mathrm{Hom}_{G_\mathbb{Q}}(\rho, H^1(\mathcal{V}_W, K_{S_0})) \otimes \check{W} \\
&= JL^{cl}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes \left(\bigotimes_{\ell \in S_0} {}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}} \otimes \check{W} \\
&= JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg}) \otimes \left(\bigotimes_{\ell \in S_0} {}' \pi_{LL}(\rho|_{G_{\mathbb{Q}_\ell}}) \right)^{K_{S_0}}.
\end{aligned}
$$

Finally, it will suffice to show that if ρ is not G-modular, then $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg}$ and $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg})$ are both 0. Notice that, in the above argument, if ρ is not G-modular, then $\mathrm{Hom}_{G_\mathbb{Q}}(\rho, H^1(\mathcal{V}_W)) = 0$ for all W, and so $\mathrm{Hom}_{G_\mathbb{Q}}(\rho, \hat{H}^1_{E,G}(K_{S_0})^{alg}) = 0$ which implies that $J(B(\rho|_{G_{\mathbb{Q}_p}}))^{alg} = 0$.

One has that $B(\rho|_{G_{\mathbb{Q}_p}})^{alg}$ is non-zero if and only if $B(\rho|_{G_{\mathbb{Q}_p}})$ is potentially semistable with disticnt Hodge-Tate weights. Theorem 4.3 implies that ρ is $\overline{G}$-modular. Moreover, if ρ is not G-modular, one gets that $WD(\rho|_{G_{\mathbb{Q}_p}})$ is the sum of two characters, and thus $\pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}}))$ is either a principal series or the extension of the Steinberg by a character. But then one has that $JL(\pi_{LL}(WD(\rho|_{G_{\mathbb{Q}_p}})) \otimes W) = 0$ for all algebraic representations W of $\mathrm{GL}_2(\mathbb{Q}_p)$, which implies that $JL(B(\rho|_{G_{\mathbb{Q}_p}})^{alg}) = 0$, showing the theorem. □

Theorem 6.9. *Let $\rho : G_L \to GL_2(E)$ be a G-promodular representation. If π is the ρ-part of $\hat{H}^0_{E,\overline{G}}(\overline{K}_{S_0})$, then $J'(\pi)^{alg} = \rho \otimes JL(\pi^{alg})$.*

The proof is almost identical to the above proof, with the weaker results coming from the weaker Theorem 5.6.

6.3 Conclusions about the representations $J(\pi)$

At this point, all results will be specialized to the $F = \mathbb{Q}$ case. The main result to be shown is that $(\hat{H}^1_{E,G,\overline{\rho},S})^{alg}$ is dense in $\hat{H}^1_{E,G,\overline{\rho},S}$. In fact, a stronger version of that will be shown:

Theorem 6.10. *Let K_{S_0} be sufficiently small. Then $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}}^{\mathcal{O}_{D^\times_{F_v}}-alg}$ is dense in $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}}$.*

Proof. Let K_{S_0} be small enough that the action of $G(\mathbb{Q})$ on $G(\mathbb{A}_f) \times \mathbb{H}^{\pm}/K_p K_{S_0} K_0^S$ is fixed point free for all compact open $K_p \subset D_{F_v}^{\times}$. Additionally, choose K_p small enough that K_p is normal in $\mathcal{O}_{D_{F_v}}^{\times}$ and K_p is pro-p. The first goal is to show that $\hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}}$ is injective as a smooth $(\mathcal{O}_E/\varpi_E^s)[K_p]$-module.

To that end, let M be a smooth finitely generated $\mathcal{O}_E/\varpi_E^s[K_p]$-module. Using the smallness assumption on K_{S_0}, there is a local system $\mathcal{M}$ over $Sh_{K_p K_{S_0} K_0^S}$ associated to M. One has that $H^i_{\acute{e}t}(Sh_{K_p K_{S_0} K_0^S, \overline{\mathbb{Q}}}, \mathcal{M})$ is a $\mathbb{T}(K_{S_0})$-module. We claim that $H^0(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M})_{\overline{\rho}} = 0$ and $H^2(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M})_{\overline{\rho}} = 0$. By Poincaré duality, it is sufficient to show that $H^0(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M})_{\overline{\rho}} = 0$ for all M. Moreover, since $\overline{\rho}$ is irreducible, and the action of $\mathbb{T}(K_{S_0})$ on H^0 is through only reducible representations, one must have $H^0(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M})_{\overline{\rho}} = 0$. This implies that $M \mapsto H^1(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M})$ is an exact functor.

Now, I claim that $\mathrm{Hom}_{K_p}(M, \hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}}) = H^1(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})_{\overline{\rho}}$. Choose K'_p small enough that $M^{K'_p} = M$ and such that K'_p is normal in K_p. Then the Hochschild-Serre spectral sequence gives an exact sequence

$$\begin{aligned} 0 &\to H^1(K_p/K'_p, H^0(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})) \\ &\to H^1_{\acute{e}t}(Sh_{K_p K_{S_0} K_0^S \overline{\mathbb{Q}}}, \mathcal{M}^{\vee}) \\ &\to H^1_{\acute{e}t}(Sh_{K_p K_{S_0} K_0^S \overline{\mathbb{Q}}}, \mathcal{M}^{\vee})^{K_p/K'_p} \\ &\to H^2(K_p/K'_p, H^0(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})). \end{aligned}$$

Localizing at $\overline{\rho}$ kills the H^0-terms, so one gets an isomorphism

$$H^1(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})_{\overline{\rho}} \cong (H^1(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})_{\overline{\rho}})^{K_p/K'_p}.$$

The assumption on K'_p implies that there is an equality as K_p/K'_p-representations:

$$H^1(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{M}^{\vee}) = \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, H^1(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{O}_E/\varpi_E^s)).$$

Plugging that into the above isomorphism, one gets an isomorphism

$$H^1(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})_{\overline{\rho}} \cong \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s}(M, H^1(Sh_{K'_p K_{S_0} K_0^S}, \mathcal{O}_E/\varpi_E^s)_{\overline{\rho}})^{K_p/K'_p}.$$

Finally, as K'_p shrinks to $\{1\}$, one gets the isomorphism claimed at the start of the paragraph.

Thus, $M \mapsto \mathrm{Hom}_{\mathcal{O}_E/\varpi_E^s[K_p]}(M, \hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}})$ is exact, as it is the composition of $M \mapsto \mathcal{M}^{\vee} \mapsto H^1(Sh_{K_p K_{S_0} K_0^S}, \mathcal{M}^{\vee})$, both of which are exact. Consequently, $\hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}}$ is an injective $\mathcal{O}_E/\varpi_E^s$-module, so $(\hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}})^{\vee}$ is projective as a $\mathcal{O}_E[[K_p]]$-module. Since K_p is assumed to be pro-p, $\mathcal{O}_E/\varpi_E^s[[K_p]]$ is a local ring, so there is an integer r_s such that $(\hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}})^{\vee} \cong ((\mathcal{O}_E/\varpi_E^s)[[K_p]])^{r_s}$. Dualizing, that says that $\hat{H}^1_{\mathcal{O}_E/\varpi_E^s, G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, \mathcal{O}_E/\varpi_E^s)^{r_s}$.

Tensoring both sides with k_E, one gets that $\hat{H}^1_{k_E, G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, k_E)^{r_s}$. Thus, r_s doesn't depend on s and, after taking the inverse limit over s, there is an isomorphism $\hat{H}^1_{\mathcal{O}_E, G}(K_{S_0})_{\overline{\rho}} \cong$

$\mathcal{C}^0(K_p, \mathcal{O}_E)^r$. Inverting p, one gets $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}} \cong \mathcal{C}^0(K_p, E)^r$. Now, the natural map from $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[\mathcal{O}^\times_{D_{F_v}}]])}((\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee, *)$ to $\mathrm{Hom}_{E\otimes(\mathcal{O}_E[[K_p]])}((\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee, *)^{\mathcal{O}^\times_{D_{F_v}}}$ is an isomorphism. Moreover, the second functor is exact as taking the invariants of a finite group is exact in characteristic 0. Thus, $(\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}})^\vee$ is projective as a $E\otimes(\mathcal{O}_E[[\mathcal{O}^\times_{D_{F_v}}]])$-module. Thus, it is a summand of a free module, and thus, after dualizing, one has that $\hat{H}^1_{E,G}(K_{S_0})_{\overline{\rho}}$ is a summand of $\mathcal{C}^0(\mathcal{O}^\times_{D_{F_v}}, E)^t$ for some t. It is then sufficient to show that the $\mathcal{O}^\times_{D_{F_v}}$-algebraic vectors are dense in the $\mathcal{C}^0(\mathcal{O}^\times_{D_{F_v}}, E)$. But the $\mathcal{O}^\times_{D_{F_v}}$ algebraic vectors in $\mathcal{C}^0(\mathcal{O}^\times_{D_{F_v}}, E)$ are exactly the polynomial functions, and Mahler expansions express polynomials as a dense subspace of continuous functions. $\square$

The above proof, in addition to being similar to the one in [Eme11], models the same philosophy: H^1 becomes exact when localizing at a non-Eisenstein prime. In addition, the above proof shows that $\hat{H}^1_{E,G,\overline{\rho},S}$ is cofinitely generated as a $D^\times_{F_v} \times G_{S_0}$-representation. The theorem is tantalizingly close to showing that $J(\pi)$ is nonzero for a large class of π, but isn't there.

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