Coloring problems on infinite graphs

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- we work with infinite graphs: countably or uncountably many vertices;
- edge-coloring problems: Ramsey-type results and partitions into monochromatic subgraphs;
- vertex-coloring problems: structural properties of graphs with large chromatic number,
- some problems I would like to solve.
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- **anti-Ramsey theory:**
  - applications in *general topology*: $L$-spaces,
  - applications in *functional analysis*: Banach-spaces and free sequences;

- **chromatic number problems:**
  - theory of expanders,
  - applications in *computer science*;
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Edge-colored complete graphs
The origins

Theorem (R. Rado, 1978)
If the edges of the complete graph on $\mathbb{N}$ are colored with finitely many colors then the vertices can be covered by disjoint monochromatic paths of different color.

P. Erdős on Richard Rado:
"I was good at discovering perhaps difficult and interesting special cases, and Richard was good at generalizing them and putting them in their proper perspective."
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Ideas of the proof

There is a non-trivial \textbf{0-1-valued measure} on $\mathbb{N}$, i.e. $m : \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ such that:

- $m$ is \textbf{finitely additive},
- $m(\mathbb{N}) = 1$ and $m(\{n\}) = 0$ for all $n \in \mathbb{N}$.

\textbf{Fact}

- If $m(U \cup V) = 1$ then either $m(U) = 1$ or $m(V) = 1$.
- If $m(U) = m(V) = 1$ then $m(U \cap V) = 1$. 
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Consider a complete graph on \( \mathbb{N} \) with red and blue edges.

- let \( A_r = \{ u \in \mathbb{N} : m(\{ v \in \mathbb{N} : \{u, v\} \text{ is red} \}) = 1 \} \),
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Any \( u, u' \in A_r \) are connected by infinitely many red paths (of length 2),
\[ \Rightarrow A_r \text{ is covered by a red path}, \]
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Developments on the finite case

**General problem (Gyárfás):** given an $r$-edge coloring of $K_n$ is there a cover by (disjoint) monochromatic paths (of different color)?

Suppose that $r$ is small:

1. ("easy") Every 2-edge colored $K_n$ can be partitioned into 2 monochromatic paths of different color.

2. [K. Heinrich, ??] There are $r$-edge colored copies of $K_n$ for $r \geq 3$ so that there is no partition into $r$ paths of different color.

3. [A. Pokrovskiy, 2013] Every 3-edge colored $K_n$ can be partitioned into 3 monochromatic paths.

**Completely open:** $r = 4$ or larger.
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For arbitrary number of colors:

1. [Gyárfás, 1989] Every $r$-edge colored $K_n$ is covered by $\leq C \cdot r^4$ monochromatic paths (for some small constant $C$).

2. [Gyárfás et al., 1998] Every $r$-edge colored copy of $K_n$ can be partitioned into $\approx 100r \log(r)$ monochromatic cycles.

Significant work done on monochromatic cycle partitions; Lehel’s conjecture and Erdős-Gyárfás-Pyber conjecture.
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Stronger versions of Rado’s theorem
Covers by powers of paths

Definition

Suppose that $G$ is a graph and $k \in \mathbb{N}$. The $k^{th}$ power of $G$ is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a finite path of length $\leq k$ from $v$ to $w$.

What is a power of a path?
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i.e. the graph is **locally complete**.
Motivation

Theorem (Infinite Ramsey)

In every finite edge colored complete graph on \( \mathbb{N} \) there is an infinite monochromatic complete subgraph.

- one cannot always partition into monochromatic complete subgraphs,
- how about partitions into monochromatic locally complete subgraphs?
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Partitions into powers of paths

A $k^{th}$-power of a path is $\{x_i : i < n\}$ so that $x_i, x_j$ is an edge if $|i - j| \leq k$.

Jointly with M. Elekes, L. Soukup and Z. Szentmiklóssy at Rényi Institute:

**Theorem**

Fix natural numbers $k, r$ and an $r$-edge coloring of the complete graph on $\mathbb{N}$. Then the vertices can be covered by $\leq r^{(k-1)r+1}$ disjoint infinite monochromatic $k^{th}$ powers of paths apart from a finite set.

For $k = r = 2$ we actually have a partition into 4 monochromatic second powers of paths and this result is sharp.
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The tools of our proof

- introduce a **game** on edge colored graphs with parameter $W$ (subset of vertices),
  - Adam and Bob chooses disjoint finite sets turn by turn,
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Infinite paths of arbitrary length

**Definition (Rado, 1978)**

For a graph $P = (V, E)$, we say that $P$ is a **path** iff there is a well ordering $\prec$ on $V$ such that any two points $v, w \in V$ are connected by a $\prec$-monotone finite path.

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**Theorem (D.S.)**

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... and what are the **difficulties**?

Our approach:

1. find the **limit points** of the path first,

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Definition

The **chromatic number** of a graph $G$, denoted by $\text{Chr}(G)$, is the least (cardinal) number $\kappa$ such that the vertices of $G$ can be covered by $\kappa$ many independent sets.

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- **Tutte, 1954:** There are $\Delta$-free graphs of arbitrary large finite chromatic number.

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What graphs must occur as subgraphs of uncountably chromatic graphs?

- **Erdős-Rado, 1959**: There are $\Delta$-free graphs with size and chromatic number $\kappa$ for each infinite $\kappa$.

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E.g: $K_{n,\omega_1}$ is $n$-connected. Does having large chromatic number imply the existence of highly connected subgraphs? reflect to some highly connected subgraphs?

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Does every graph $G$ with uncountable chromatic number contain

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- [Komjáth, 1986] If $Chr(G) > \omega$ then there is an $n$-connected uncountably chromatic subgraph of $G$ for each $n \in \omega$.

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The answers - continued

- Question from [Erdős-Hajnal, 1966]: independent. ✓
- Question from [Erdős-Hajnal, 1985]: consistently no.

**Theorem (D.S. 2014)**

There is a graph of **chromatic number** $\omega_1$ and size continuum **without uncountable infinitely connected subgraphs**.

- Question from [Erdős-Hajnal, 1985]: no in ZFC. ✓
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A few words about the proof

- find a rather disconnected graph with large chromatic number:
  - the comparability graph of a non-special tree without uncountable chains;

- thin out the edges to have no uncountable infinitely connected subgraph:
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- [Komjáth, ??] Is there a graph with uncountable chromatic number which contains no infinitely connected subgraphs?
  - Recall: there are $G$ with $Chr(G) > \omega$ where every infinitely connected subgraph is countable.
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- **[Erdős, Hajnal 1975]** Is there a graph with uncountable chromatic number which contains no triangle free subgraphs with uncountable chromatic number?

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- **[Shelah, 1988]** Consistently yes.
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- [Erdős, Hajnal 1975] Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ is increasing. Is there a graph $G$ with uncountable chromatic number such that every $n$-chromatic subgraph of $G$ has at least $f(n)$ vertices (for all $n \geq 3$)?

  - Recall: If $Chr(G)$ is infinite then $\sup \{Chr(H) : H \subseteq G \text{ finite}\}$ is infinite as well.

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Thank you for your attention.

"The infinite we do now, the finite will have to wait a little."

P. Erdős