FINITE TYPE INVARIANTS BY THE SPECIES

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Abstract. We study the general theory of finite type invariants through the study of examples.

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Part 1. Introduction

In the late 1980s, Vassiliev [Va1, Va2] suggested to study knot invariants by studying the space of all knots. The first new ingredient thus added is the notion of "neighboring" knots, knots that differ in only one crossing, and the idea that one should study how knot invariants change in the "neighborhood" of a given knot. This idea led to a definition of a certain class of knot invariants, now known as "Vassiliev" or "finite type" invariants. By now more than 350 papers have been written on the subject; see [B-N2]. Vassiliev's original definition (independently discovered by Goussarov [Go1, Go2] at roughly the same time) was formalized in a different way by Birman and Lin [BL], and later re-interpreted by analogy with multi-variable calculus by Bar-Natan [B-N1].

The basic idea in Bar-Natan [B-N1] was that differences are "cousins" of derivatives, and hence one should think of the difference between the values of a knot invariant V on two neighboring knots as a "derivative" of the original invariant. Repeating this, we find that iterated differences (as considered by [Va1, Va2, Go1, Go2, BL, B-N1]) of values of V in the neighborhood of some knot should be thought of as multiple derivatives. A "Vassiliev" or "finite type" invariant of type m is then the analog of a polynomial — an invariant whose $m+1$st derivatives, or $m+1$st iterated differences, vanishes.

Clearly, these ideas are very general, and knots (and even topology in general) are just a particular case. Whenever an appropriate notion of "neighborhood of an object" exists, one can talk about finite type invariants of such objects. This leads to many different "species" of finite type invariants. Let us mention just a few:

- The usual notion of nearness of knots, knots that differ at only a single crossing, leads to the usual Vassiliev invariants.
- Similarly, one can define "Vassiliev" invariants of braids, links, tangles, knotted graphs, etc.

- Goussarov [Go3] has also an alternative notion of a neighborhood of a knot (or link), defined by "interdependent modifications". This notion leads to a *different* (though at the end, equivalent) theory of finite type invariants of knots and links.
- Two algebraically split links (links whose linking numbers all vanish) can be considered neighboring if they differ by the simultaneous flip of two opposite crossings between two given components (such a double flip preserves linking numbers, whereas a single flip doesn't). See Figure 1. This leads to a little known but probably interesting theory of finite type invariants of algebraically split links.

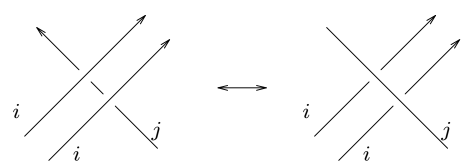


Figure 1. A move that preserves linking numbers.

- Several authors [] have considered several notions of finite type invariants of plane curves.
- Ohtsuki [Oh1] considers two integral homology spheres to be neighboring (roughly) if they differ by a single surgery. This leads to a notion of finite type invariants of integral homology spheres. Several variants of his definition were considered in [Ga1, GL1, GO1, GL2, GO2, Ga2, GL3]

A finite type theory automatically comes bundled with several spaces that play a significant role in it. In the best known case of knots, these are the spaces of chord diagrams, $4T$ relations, weight systems, chord diagrams modulo $4T$ relations, relations between $4T$ relations, and a few lesser known spaces that should probably be better known. There is a "general theory of finite type invariants", defined in terms of these spaces, in which one attempts to classify finite type invariants by first classifying their potential mth derivatives, and then by studying which of those potential derivatives can actually be integrated to an honest invariant. I should say that though this "general theory" is rather small, it is also rather interesting (with the most interesting parts developed by M. Hutchings [Hu] and private communication), and insufficiently well known *even in the case of the usual finite type invariants of knots.*

The purpose of this paper is twofold:

1. To state (and propagate) this general theory of finite type invariants of anything. Namely, to construct, name, and study the relationships between those spaces that come automatically with every finite type theory, especially from the perspective of the integration theory of "weight systems". We first do it in Section 1 on a well known example, the original finite type theory of knots. We then extract some general features from this example and give them general names; this is done in the rather short Section 2.
2. To list many of the currently known finite type theories, and figure out (to the degree that is now possible) what these associated spaces are on a species by species basis. Our list takes the form of a "classification"[1]. The top subdivision is into the classes of

[1] classification (klàse-fî-kîshen), in biology, the systematic categorization of organisms. One aim of modern classification, or systematics, is to show the evolutionary relationships among organisms. The broadest

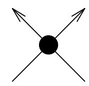


Figure 2. A singular point.

"Knotted Objects", "3-Manifolds" and "Plane Curves", and these classes are described in Sections 4, 5 and 6, respectively.

Acknowledgement: I wish to thank Mike Hutchings for the inspiration to write this paper and for telling me his ideas about integration theory. I also wish to thank E. Appleboim and H. Scolnicov for many ideas and conversations.

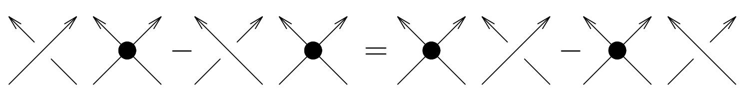
Part 2. Background Material

1. The case of knots

1.1. **Singular knots, the co-differential δ, and finite type invariants.** As we have already indicated in the introduction, the finite type theory for knots (Vassiliev theory) is built around the notions of n-singular knots, and differences between overcrossings and undercrossings. Let us make those notions precise:

Definition 1.1. An n-singular knot is an oriented knot in an oriented $\mathbb{R}^3$, which is allowed to have n singular points that locally look like the image in Figure 2. For simplicity in the later parts of this section, we only consider framed (singular or not) knots, and always use blackboard framing when a knot projection or a part of a knot projection is drawn.

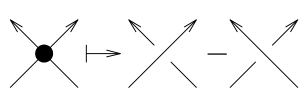
Definition 1.2. Let $\mathcal{K}_n$ be the $\mathbb{Z}$-module freely generated by all n-singular knots, modulo the following "co-differentiability relation":



Notice that $\mathcal{K}_0 = \mathcal{K}$ is simply the free $\mathbb{Z}$-module generated by all (framed) knots.

Definition 1.3. Let $\delta : \mathcal{K}_{n+1} \to \mathcal{K}_n$ be defined by "resolving" any one of the singular points in an $(n+1)$-singular knot in $\mathcal{K}_{n+1}$:

(1)



Note that thanks to the co-differentiability relation, δ is well defined. It is called "the co-derivative". We denote the adjoint of δ by ∂ and call it "the derivative". It is a map $\partial : \mathcal{K}_n^\star \to \mathcal{K}_{n+1}^\star$.

division of organisms is into kingdoms, traditionally two-Animalia (animals) and Plantae (plants). Widely accepted today are three additional kingdoms: the Protista, comprising protozoans and some unicellular algae; the Monera, bacteria and blue-green algae; and the Fungi. From most to least inclusive, kingdoms are divided into the following categories: phylum (usually called division in botany), class, order, family, genus, and species. The species, the fundamental unit of classification, consists of populations of genetically similar, interbreeding or potentially interbreeding individuals that share the same gene pool (collection of inherited characteristics whose combination is unique to the species).

The name "derivative" is justified by the fact that $(\partial V)(K)$ for some $V \in \mathcal{K}_n^\star$ and $K \in \mathcal{K}_{n+1}$ is by definition the difference of the values of V on two "neighboring" n-singular knots, in harmony with the usual definition of derivative for functions on $\mathbb{R}^d$.

Definition 1.4. An invariant of knots V (equivalently, a $\mathbb{Z}$-linear functional on $\mathcal{K}$) is said to be of finite type n if its $(n+1)$-st derivative vanishes, that is, if $\partial^{n+1}V \equiv 0$. (This definition is the analog of one of the standard definitions of polynomials on $\mathbb{R}^d$).

When thinking about finite type invariants, it is convenient to have in mind the following ladders of spaces and their duals, printed here with the names of some specific elements that we will use later:

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \longrightarrow & \ldots & \longrightarrow \mathcal{K}_1 & \xrightarrow{\delta} & \mathcal{K}_0 = \mathcal{K} \\
\ldots \longleftarrow & \mathcal{K}_{n+1}^\star & \xleftarrow{\delta} & \mathcal{K}_n^\star & \xleftarrow{\delta} & \mathcal{K}_{n-1}^\star & \longleftarrow & \ldots & \longleftarrow \mathcal{K}_1^\star & \xleftarrow{\delta} & \mathcal{K}_0^\star = \mathcal{K}^\star \\
& \Cup & & \Cup & & & & & & & \Cup \\
& \partial^{n+1}V \equiv 0 & & \partial^n V = W & & & & & & & V
\end{array} \tag{2}$$

One may take the definition of a general "theory of finite type invariants" to be the data in (2), with arbitrary "n-singular objects" replacing the n-singular knots. Much of what we will say below depends only on the existance of the ladders (2), or on the existance of certain natural extensions thereof, and is therefore quite general.

1.2. **Constancy conditions, $\mathcal{K}_n/\delta\mathcal{K}_{n+1}$, and chord diagrams.** As promised in the introduction, we study invariants of type n by studying their nth derivatives. Clearly, if V is of type n and $W = \partial^n V$, then $\partial W = 0$ ("W is a constant"). Glancing at (2), we see that W descends to a linear functional, also called W, on $\mathcal{K}_n/\delta\mathcal{K}_{n+1}$:

Definition 1.5. We call $\bar{\mathcal{K}}_n := \mathcal{K}_n/\delta\mathcal{K}_{n+1}$ the space of "n-symbols" associated with the ladders in (2). (The name is inspired by the theory of differential operators, where the "symbol" of an operator is essentially its equivalence class modulo lower order operators. The symbol is responsible for many of the properties of the original operator, and for many purposes, two operators that have the same symbol are "the same".) We denote the projection mapping $\mathcal{K}_n \to \bar{\mathcal{K}}_n$ that maps every singular knot to its symbol by π.

The following classical proposition (see e.g. [B-N1, Bi, BL, Go1, Go2, Ko1, Va1, Va2] identifies the space of n-symbols in our case:

Proposition 1.6. *The space $\bar{\mathcal{K}}_n$ of n-symbols for* (2) *is canonically isomorphic to the space $\mathcal{D}_n$ of n-chord diagrams, defined below.* □

Definition 1.7. An n-chord diagram is a choice of n pairs of distinct points on an oriented circle, considered up to orientation preserving homeomorphisms of the circle. Usually an n-chord diagram is simply drawn as a circle with n chords (whose ends are the n pairs), as in the 5-chord example in Figure 3. The space $\mathcal{D}_n$ is the space of all formal $\mathbb{Z}$-linear combinations of n-chord diagrams.

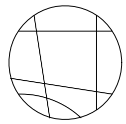


Figure 3. A chord diagram.

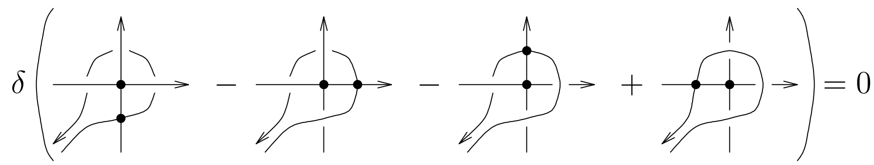


Figure 4. A Topological 4-Term ($T4T$) relation. Each of the four graphics in the picture represents a part of an n-singular knot (so there are $n-2$ additional singular points not shown), and, as usual in knot theory, the 4 singular knots in the equation are the same outside the region shown.

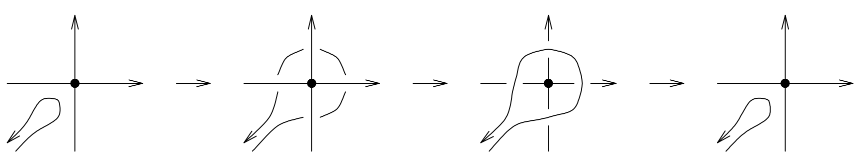


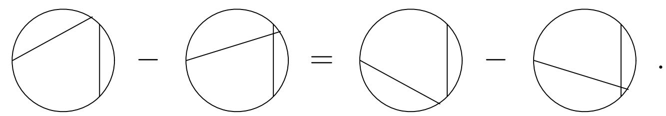
Figure 5. Lassoing a singular point: Each of the graphics represents an $(n-1)$-singular knot, but only one of the singularities is explicitly displayed. Start from the left-most graphic, pull the "lasso" under the displayed singular point, "lasso" the singular point by crossing each of the four arcs emenating from it one at a time, and pull the lasso back out, returning to the initial position. Each time an arc is crossed, the difference between "before" and "after" is the co-derivative of an n-singular knot (up to signs). The four n-singular knot thus obtained are the ones making the Topological 4-Term relation, and the co-derivative of their signed sum is the difference between the first and the last $(n-1)$-singular knot shown in this figure; namely, it is 0.

1.3. **Integrability conditions, $\ker\delta$, lassoing singular points, and four-term relations.** Next, we wish to find conditions that a "potential top derivative" has to satisfy in order to actually be a top derivative. More precisely, we wish to find conditions that a functional $W\in\bar{\mathcal{K}}_n^\star$ has to satisfy in order to be $\partial^n V$ for some invariant V. A first condition is that W must be "integrable once"; namely, there has to be some $W^1\in\mathcal{K}_{n-1}^\star$ with $W=\partial W^1$. Another quick glance at (2), and we see that W is integrable once iff it vanishes on $\ker\delta$, which is the same as requiring that W descends to $\mathcal{A}_n=\mathcal{A}_n(\mathcal{K}):=\bar{\mathcal{K}}_n/\pi(\ker\delta)=\mathcal{K}_n/(\operatorname{im}\delta+\ker\delta)$ (there should be no confusion regarding the identities of the δ's involved). Often elements of $\mathcal{A}_n^\star$ are refered to as "weight systems". A more accurate name would be "once-integrable weight systems".

We see that it is necessary to understand $\ker\delta$. In Figure 4 we show a family of members of $\ker\delta$, the "Topological 4-Term" ($T4T$) relations. Figure 5 explains how they arise from "lassoing a singular point". The following theorem says that this is all:

Theorem 1. *(Stanford* [St1]*) The $T4T$ relations of Figure 4 span* $\ker\delta$. □

Pushing the $T4T$ relations down to the level of symbols, we get the well-known $4T$ relations, which span $\pi(\ker\delta)$: (see e.g. [B-N1])

$4T$: 

We thus find that $\mathcal{A}_n$ = (chord diagrams)/($4T$ relations), as usual in the theory of finite type invariants of knots.

1.4. **Hutchings' theory of integration.** We have so far found that if V is a type-n invariant, then $W = \partial^n V$ is a linear functional on $\mathcal{A}_n$. A question arises whether every linear functional on $\mathcal{A}_n$ arises in this way. At least if the ground ring is extended to $\mathbb{Q}$, the answer is positive:

Theorem 2. *(The Fundamental Theorem of Finite Type Invariants, Kontsevich* [Ko1]*) Over $\mathbb{Q}$, for every $W \in \mathcal{A}_n^\star$ there exists a type n invariant V with $W = \partial^n V$. In other words, every once-integrable weight system is fully integrable.*

The problem with the Fundamental Theorem is that all the proofs we have for it are somehow "transcendental", using notions from realms outside the present one, and none of the known proofs settles the question over the integers (see [BS]). In this section we describe what appears to be the most natural and oldest approach to the proof, having been mentioned already in [Va1, BL]. Presently, we are stuck and the so-called "topological" approach does not lead to a proof. But it seems to me that it's worth studying further; when something natural fails, there ought to be a natural reason for that, and it would be nice to know what it is.

The idea of the topological approach is simple: To get from W to V, we need to "integrate" n times. Let's do this one integral at a time. By the definition of $\mathcal{A}_n$, we know that we can integrate once and find $W^1 \in \mathcal{K}_{n-1}^\star$ so that $\partial W^1 = W$. Can we work a bit harder, and find a "good" W^1, so that there would be a $W^2 \in \mathcal{K}_{n-2}^\star$ with $\partial W^2 = W^1$? Proceeding like that and assuming that all goes well along the way, we would end with a $V = W^n \in \mathcal{K}_0^\star$ with $\partial^n V = W$, as required. Thus we are naturally lead to the following conjecture, which implies the Fundamental Theorem by the backward-inductive argument just sketched:

Conjecture 1. *Every once-integrable invariant of n-singular knots also twice integrable. Glancing at* (2)*, we see that this is the same as saying that* $(\ker\delta^2)/(\ker\delta) = 0$.

This conjecture is somewhat stronger than Theorem 2. Indeed, Theorem 2 is equivalent to Conjecture 1 restricted to the case when the given invariant has some (possibly high) derivative identically equal to 0 (exercise!). But it is hard to imagine a topological proof of the restricted form of Conjecture 1 that would not prove it in full.

The difficulty in Conjecture 1 is that it's hard to say much about $\ker\delta^2$. In [Hu], Michael Hutchings was able to translate the statement $(\ker\delta^2)/(\ker\delta) = 0$ to an easier-looking combinatorial-topological statement, which is implied by and perhaps equivalent to an even simpler fully combinatorial statement. Furthermore, Hutchings proved the fully combinatorial statement in the analogous case of finite type braid invariants, thus proving Conjecture 1 and Theorem 2 (over $\mathbb{Z}$) in that case, and thus proving the viability of his technique.

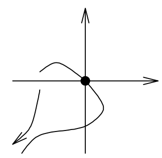


Figure 6. The "Topological Relator" singularity.

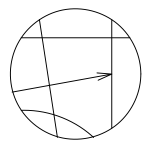


Figure 7. A "relator symbol".

Hutchings' first step was to write a chain of isomorphisms reducing $(\ker\delta^2)/(\ker\delta)$ to something more manageable. Our next step will be to introduce all the spaces participating in Hutchings' chain. First, let us consider the space of all $T4T$ relations:

Definition 1.8. Let $\mathcal{K}^1_n$ be the $\mathbb{Z}$-module generated by all (framed) knots having $n-2$ singularities as in Definition 1.1, and plus one additional "Topological Relator" singularity that locally looks like the image in Figure 6, modulo the same co-differentiability relations as in Definition 1.2. Define $\delta : \mathcal{K}^1_{n+1} \to \mathcal{K}^1_n$ in the same way as for knots, using equation (1). Finally, define $b : \mathcal{K}^1_n \to \mathcal{K}_n$ by mapping the topological relator to the topological 4-term relation, the 4-term alternating sum inside the paranthesis in Figure 4.

The spaces $\mathcal{K}^1_n$ form a ladder similar to the one in (2), and, in fact, they combine with the ladder in (2) to a single commutative diagram:

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\delta} & \mathcal{K}^1_{n+1} & \xrightarrow{\delta} & \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\delta} & \dots \\ & & \downarrow b & & \downarrow b & & \downarrow b & & \\ \dots & \xrightarrow{\delta} & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\delta} & \dots , \end{array} \tag{3}$$

In this language, Stanford's theorem (Theorem 1) says that all L shapes in the above diagram (compositions $\delta \circ b$ of "down" followed by "right") are exact.

Just like singular knots had symbols which were simplar combinatorial objects (chord diagrams), so do toplogical relators have combinatorial symbols:

Definition 1.9. Let $\bar{\mathcal{K}}^1_n := \mathcal{K}^1_n/\delta\mathcal{K}^1_{n+1}$, and let $\pi : \mathcal{K}^1_n \to \bar{\mathcal{K}}^1_n$ be the projection map.

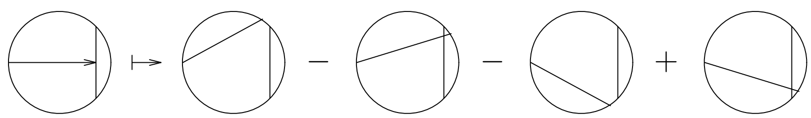
The following proposition is proved along the same lines as the standard proof of Proposition 1.6.

Proposition 1.10. *$\bar{\mathcal{K}}^1_n$ is canonically isomorphic to the space spanned by all "relator symbols", chord diagrams with $n-2$ chords and one $\rightarrow\!|$ piece corresponding to the special singularity of Definition 1.8. An example appears in Figure 7.*

We need to display one additional commutative diagram before we can come to Hutchings' chain of isomorphisms:

$$(4)\qquad \begin{array}{ccccccc} \mathcal{K}_n^1 & \xrightarrow{\delta} & \mathcal{K}_{n-1}^1 & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1}^1 & \longrightarrow & 0 \\ \downarrow b & & \downarrow b & & \downarrow \bar{b} & & \\ \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}_{n-1} & \longrightarrow & 0 \end{array} \qquad \text{(exact rows).}$$

In this diagram, $\bar{b}$ is the "symbol level" version of b, and is induced by $b : \mathcal{K}_{n-1}^1 \to \mathcal{K}_{n-1}$ in the usual manner. It can be described combinatorially by

$\bar{b}$:  .

Hutchings' chain of isomorphisms is the following chain of equalities and maps: (here the symbol $\cup$ means that the space below is a subspace of the space above, and the symbol $\uplus$ means that the space below is a sub-quotient of the space above)

$$\begin{array}{ccccccccccccc} \mathcal{K}_n & & \mathcal{K}_{n-1} & & \mathcal{K}_{n-1} & & \mathcal{K}_{n-1}^1 & & \mathcal{K}_{n-1}^1 & & \mathcal{K}_{n-1}^1 & & \bar{\mathcal{K}}_{n-1}^1 \\ \uplus & & \cup & & \cup & & \uplus & & \uplus & & \uplus & & \uplus \\ \frac{\ker \delta^2}{\ker \delta} & \xrightarrow{\delta} & \ker\delta \cap \operatorname{im}\delta & = & \operatorname{im} b \cap \ker \pi & \xleftarrow{b} & \frac{\ker \pi \circ b}{\ker b} & = & \frac{\ker \bar{b} \circ \pi}{\ker b} & = & \frac{\pi^{-1}(\ker \bar{b})}{\ker b} & \xrightarrow{\pi} & \frac{\ker \bar{b}}{\pi(\ker b)}. \end{array}$$

Theorem 3. *(Hutchings* [Hu]*) All maps in the above chain are isomorphisms. In particular,* $(\ker \delta^2)/(\ker \delta) \simeq (\ker \bar{b})/(\pi(\ker b))$.

Proof. Immediate from diagrams (3) and (4). □

It doesn't look like we've achieved much, but in fact we did, as it seems that $(\ker \bar{b})/(\pi(\ker b))$ is easier to digest than the original space of interest, $(\ker \delta^2)/(\ker \delta)$. The point is that $\ker \bar{b}$ lives fully in the combinatorial realm, being essentially the space of all relations between $4T$ relations at the symbol level. Similarly, $\pi(\ker b)$ is the space of projections to the symbol level of relation between $4T$ relations, and hence we have shown

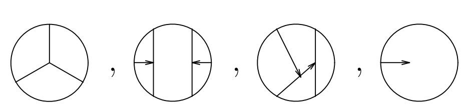
Corollary 1.11. *Conjecture 1 is equivalent to the statement "every relation between* $4T$ *relations at the symbol level has a lift to the topological level".*

An obvious approach to proving Conjecture 1 thus emerges:

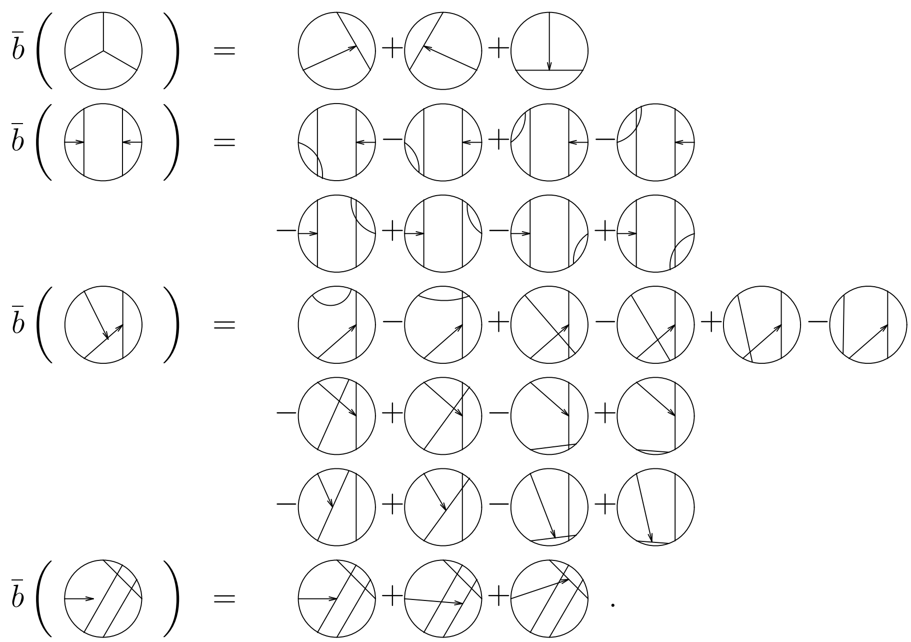
- Combinatorial step: Find all relations between $4T$ relations at the symbols level; that is, find a generating set for $\ker \bar{b}$.
- For every relation found in the combinatorial step, show that it lifts to the topological level.

So far, the problem with this approach appears to be in the combinatorial step. There is a conjectural generating set $\bar{\mathcal{K}}_{n-1}^2$ for $\ker \bar{b}$. Every element in $\bar{\mathcal{K}}_{n-1}^2$ indeed has a lifting to $\ker b$, but we still don't know if $\bar{\mathcal{K}}_{n-1}^2$ indeed generates $\ker \bar{b}$. We state these facts very briefly; more information can be found in [Hu] and in [BS].

Definition 1.12. Define $\bar{\mathcal{K}}_{n-1}^2$ by

$\bar{\mathcal{K}}_{n-1}^2 = \operatorname{span}\Big\{$  $\Big\}$.

As usual, each graphic in the above formula represents a large number of elements of $\bar{\mathcal{K}}^2_{n-1}$, obtained from the graphic by the addition of $n-3$ chords (first graphic), or $n-5$ chords (second graphic), or $n-4$ chords (third graphic), or $n-2$ chords (fourth graphic). Define also $\bar{b} : \bar{\mathcal{K}}^2_{n-1} \longrightarrow \bar{\mathcal{K}}^1_{n-1}$ by



Conjecture 2. *The sequence* $\bar{\mathcal{K}}^2_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}^1_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}_{n-1}$ *is exact.*

A parallel of Conjecture 2 for braids was proven by Hutchings in [Hu].

Exercise 1.13. Find a space $\mathcal{K}^2_n$ and maps $\delta : \mathcal{K}^2_n \to \mathcal{K}^2_{n-1}$ and $b : \mathcal{K}^2_n \to \mathcal{K}^1_n$ that fit into a commutative diagram,

$$\begin{array}{ccccccc} \mathcal{K}^2_n & \xrightarrow{\delta} & \mathcal{K}^2_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}^2_{n-1} & \longrightarrow & 0 \\ \downarrow b & & \downarrow b & & \downarrow \bar{b} & & \\ \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\pi} & \bar{\mathcal{K}}^1_{n-1} & \longrightarrow & 0 \end{array} \qquad \text{(exact rows)},$$

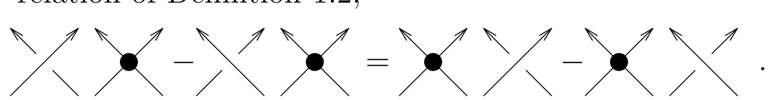
and hence show that the relations in $\ker \bar{b}$ all lift to $\ker b$.

Question 1.14. Is the sequence $\bar{\mathcal{K}}^2_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}^1_{n-1} \xrightarrow{\bar{b}} \bar{\mathcal{K}}_{n-1}$ related to Kontsevich's graph cohomology [Ko2]?

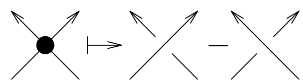
1.5. **Summary.** In summary, we have introduced and studied the following objects, spaces, and maps:

Objects: $\mathcal{O} = \mathcal{K}$ is the space of all framed oriented knots in an oriented $\mathbb{R}^3$. More precisely, it is the free $\mathbb{Z}$-module generated by framed oriented knots in an oriented $\mathbb{R}^3$.

The n-Cubes: $\mathcal{O}_n = \mathcal{K}_n$ is the free $\mathbb{Z}$-module generated by framed oriented knots in an oriented $\mathbb{R}^3$, that have precisely n double point singularities (⤫) as in Figure 2, modulo the co-differentiability relation of Definition 1.2,



The Co-Derivative: The co-derivative $\delta : \mathcal{O}_{n+1} \to \mathcal{O}_n$ is the map $\delta : \mathcal{K}_{n+1} \to \mathcal{K}_n$ defined by


.

The Cube Ladder and Finite Type Invariants:
The n-Symbols: The space of n-symbols $\mathcal{O}_n/\delta\mathcal{O}_{n+1}$ is the space $\mathcal{K}_n/\delta\mathcal{K}_{n+1}$ of n-chord diagrams, as in Figure 3.
The Relator Ladder: The relator ladder is the ladder

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\delta} & \mathcal{K}^1_{n+1} & \xrightarrow{\delta} & \mathcal{K}^1_n & \xrightarrow{\delta} & \mathcal{K}^1_{n-1} & \xrightarrow{\delta} & \dots \\ & & \downarrow b & & \downarrow b & & \downarrow b & & \\ \dots & \xrightarrow{\delta} & \mathcal{K}_{n+1} & \xrightarrow{\delta} & \mathcal{K}_n & \xrightarrow{\delta} & \mathcal{K}_{n-1} & \xrightarrow{\delta} & \dots , \end{array}$$

(see Equation 3), of singular knots with exactly one "Topological Relator" singularity as in Figure 6.
The Primary Integrability Constraints: The primary integrability constraints are the images of the relators via the map b; that is, they are the Topological 4-Term relations of Figure 4.
The Relator Symbols and the Symbol-Level Relations: The relator symbols are diagrams of the kind appearing in Figure 7.
The Once-Reduced Symbol Space and Once Integrable Weight Systems:
The Inductive Problem:
The Lifting Problem:
Generic Symbol-Level Redundencies:
The Object-Level Redundencies:
The Redundency Problem:

2. The General Theory of Finite Type Invariants

2.1. **Cubical complexes.** TBW.

2.2. **Polynomials on an affine space.** TBW.

3. The case of integral homology spheres

3.1. **The definition.**

Definition 3.1. An n-singular integral homology sphere is a pair (M, L) where M is an integral homology sphere and $L = \bigcup_{i=1}^n L_i$ is a unit-framed algebraically split ordered n-component link in M. Namely, the components L_i of M are numbered 1 to n ("ordered"), framed with ± 1 framing ("unit framed"), and the pairwise linking numbers between the different components of L are 0 ("algebraically split"). We think of L as marking n sites for performing small modifications of M, each modification being the surgery on one of the components of L. Let us temporarily define $\mathcal{M}_n$ to be the $\mathbb{Z}$-module of all formal $\mathbb{Z}$-linear combinations of n-singular integral homology spheres. A correction to the definition of $\mathcal{M}_n$ will be given in Definition 3.2 below. Notice that $\mathcal{M}_0$, which we often simply denote by $\mathcal{M}$, is simply the space of all $\mathbb{Z}$-linear combinations of integral homology spheres.

If $L = L^1 \cup L^2$ is a framed link (presented as a union of two sublinks L^1 and L^2) in some 3-manifold M, we denote by $(M, L^1)_{L^2}$ the result of surgery[2] of (M, L^1) along L^2. Namely, $(M, L^1)_{L^2}$ is a pair $(M', L^{1'})$, in which M' is the result of surgery of M along L^2, and $L^{1'}$ is the image in M' of L^1. Notice that if (M, L) is an $(n+1)$-singular integral homology sphere, then $(M, L - L_i)_{L_i}$ is again an n-singular integral homology sphere for any component L_i of L.

We now wish to define the co-derivative map $\delta : \mathcal{M}_{n+1} \to \mathcal{M}_n$, whose adjoint will be the differentiation map for invariants:

Definition 3.2. Define δ_i on generators by $\delta_i(M, L) = (M, L - L_i) - (M, L - L_i)_{L_i}$, and extend it to be a $\mathbb{Z}$-linear map $\mathcal{M}_{n+1} \to \mathcal{M}_n$. For later convinience, we want to set $\delta = \delta_i$ for any i, but the different i's may give different answers. We resolve this by redefining $\mathcal{M}_n$. Set

$$\mathcal{M}_n = (\text{old } \mathcal{M}_n) \Bigg/ \left(\begin{array}{c} \delta_i(M, L) = \delta_j(M, L) \\ \text{for all } 1 \leq i, j \leq n+1 \text{ and all} \\ (n+1)\text{-singular integral homology spheres} \end{array} \right). \tag{5}$$

We can now set (in the new $\mathcal{M}_n$)

$$\delta(M, L) = (M, L - L_i) - (M, L - L_i)_{L_i} \quad \text{for } \textit{any } i.$$

The relations in equation (5) are called "the co-differentiability relations".

We can finally differentiate invariants using the adjoint $\partial = \delta^\star : \mathcal{M}_n^\star \to \mathcal{M}_{n+1}^\star$. That is, if $I \in \mathcal{M}_n^\star$ is a differentiable invariant of n-singular integral homology spheres (namely, which vanishes on the co-differentiability relations), let its derivative $I' \in \mathcal{M}_{n+1}^\star$ be $\partial I = I \circ \delta$. Iteratively, one can define multiple derivatives such us $I^{(k)}$ for any $k \geq 0$.

Definition 3.3. (Ohtsuki [Oh1] We say that an invariant I of integral homology spheres is of type n if its $n+1$st derivative vanishes. We say that it is of finite type if it is of type n for some natural number n.

Unravelling the definitions, we find that I is of type n precisely when for all integral homology spheres M and all unit-framed algebraically split $(n+1)$-component links L in M,

$$\sum_{L' \subset L} (-1)^{|L'|} I\left(M_{L'}\right) = 0, \tag{6}$$

where the sum runs on all sublinks L' of L (including the empty and full sublinks), $|L'|$ is the number of components of L', and $M_{L'}$ is the result of surgery of M along L'. We will not use equation (6) in this paper.

3.2. **Preliminaries.**

3.2.1. *Surgery and the Kirby calculus.*

3.2.2. *The Borromean rings.* TBW

3.2.3. *The triple linking numbers μ_{ijk}.* TBW

3.3. **Constancy conditions or $\mathcal{M}_n / \delta\mathcal{M}_{n+1}$.**

[2] We recall some basic facts about surgery in Section 3.2.1.

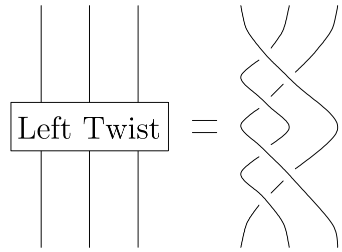


Figure 8. The Left Twist (LT).

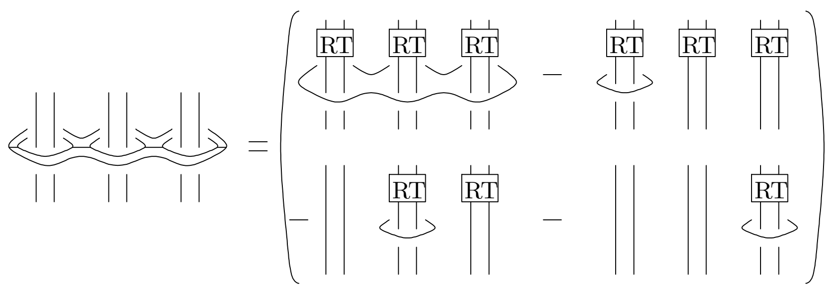


Figure 9. A 3-mask.

3.3.1. *Statement of the result.*

Definition 3.4. Let $\mathcal{Y}_n$ be the unital commutative algebra over $\mathbb{Z}$ generated by symbols Y_{ijk} for distinct indices $1 \leq i, j, k \leq n$, modulo the anti-cyclicity relations $Y_{ijk} = Y_{jik}^{-1} = Y_{jki}$.

Warning 3.5. Below we will mostly regard $\mathcal{Y}_n$ as an $\mathbb{Z}$-module, and not as an algebra. Thus we will only use the product of $\mathcal{Y}_n$ as a convenient way of writing certain elements and linear combinations of elements. The subspaces of $\mathcal{Y}_n$ that we will consider will be subspaces in the linear sense, but not ideals or subalgebras, and similarly for quotients and maps from or to $\mathcal{Y}_n$.

It is easy to define a map $\mu : \mathcal{M}_n/\delta\mathcal{M}_{n+1} \to \mathcal{Y}_n$. For an n-link L set

$$\mu(L) = \prod_{1 \leq i < j < k \leq n} Y_{ijk}^{\mu_{ijk}(L)}.$$

It follows from Section 3.2.3 that this definition descends to the quotient of $\mathcal{M}_n$ by the co-derivatives of $(n+1)$-links.

Theorem 4. *The thus defined map $\mu : \mathcal{M}_n/\delta\mathcal{M}_{n+1} \to \mathcal{Y}_n$ is an isomorphism.*

3.3.2. *On a connected space, polynomials are determined by their values at any given point.*

3.3.3. *Homotopy invariance and pure braids.*

3.3.4. *The mask and the interchange move.*

3.3.5. *Reducing third commutators.* TBW.

3.4. **Integrability conditions or** $\ker \delta$.

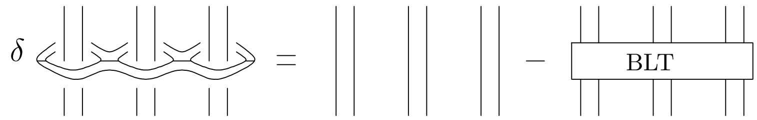


Figure 10. The co-derivative of a 3-mask.

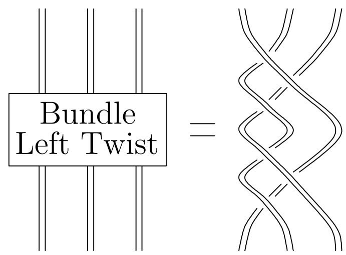


Figure 11. The Bundle Left Twist (BLT) is the same as the Left Twist, only that the strands within each "bundle" are not twisted internally.

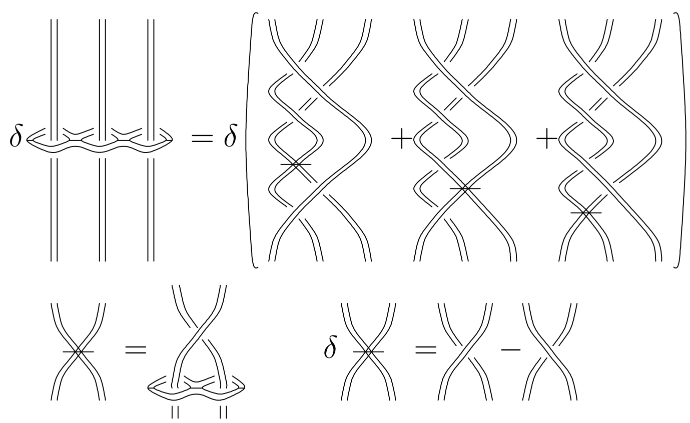


Figure 12. Undoing a Bundle Left Twist one crossing at a time.

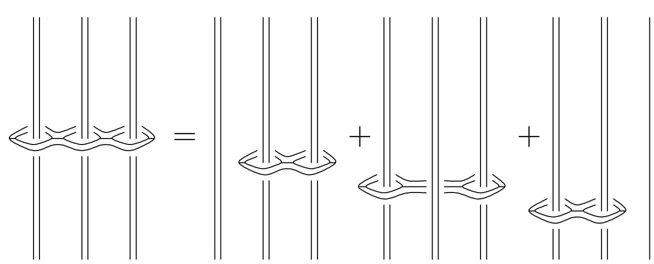


Figure 13. The Total Twist Relation (TTR).

3.4.1. $+1$ *and* -1 *surgeries are opposites.*

3.4.2. *A total twist is a composition of many little ones.*

3.4.3. *The two ways of building an interchange.*

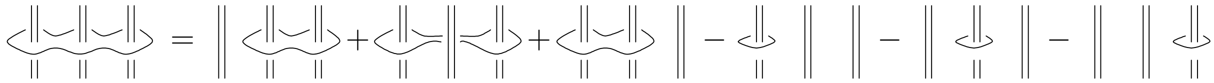


Figure 14. The Total Twist Relation (TTR).

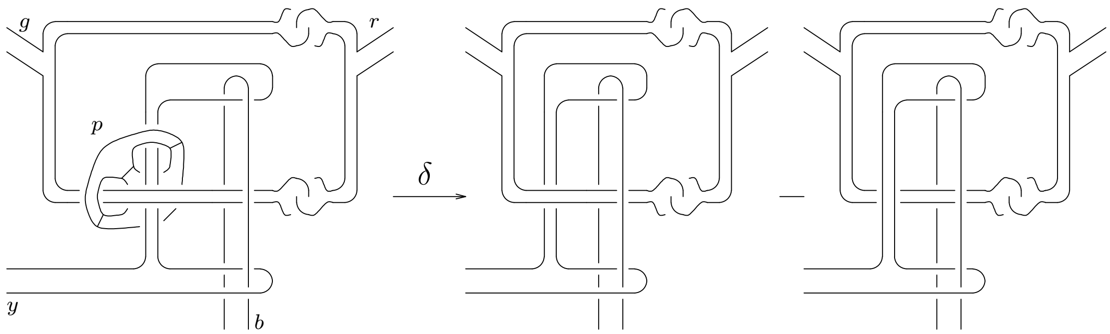


Figure 15. The Monster

3.4.4. *Lassoing a Borromean link and the IHX relation.*

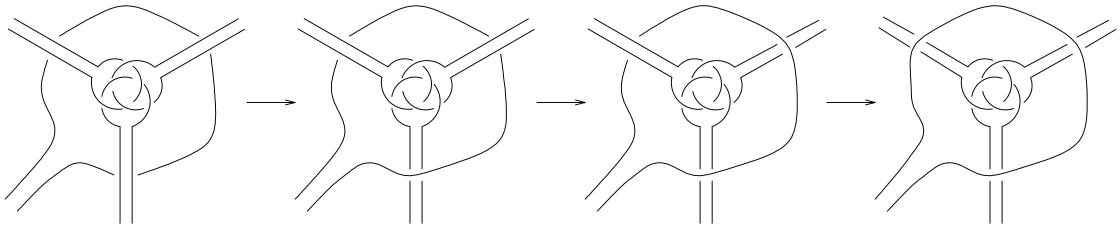


Figure 16. Lassoing a Borromean link.

$$\tilde{Y}_{rab}Y_{rgb}(Y_{rgp}Y_{pyb} - Y_{rgp} - Y_{pyb}) = \tilde{Y}_{rab}Y_{rgb}(Y_{rgp}\tilde{Y}_{pyb} - Y_{pyb})$$

$$= Y_{rab}Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} - Y_{rab}Y_{rgb}Y_{pyb} - Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} + Y_{rgb}Y_{pyb}$$

$$= Y_{rab}Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} - Y_{rab}Y_{rgb}\tilde{Y}_{pyb} - Y_{rgb}Y_{rgp}\tilde{Y}_{pyb} + Y_{rgb}\tilde{Y}_{pyb}$$

(The last equality holds because in the two error terms, $Y_{rab}Y_{rgb}$ and Y_{rgb}, the component p is unknotted). Now reduce the component r using the total twist relation. Only the first term is affected, and 3 of the 6 terms that are produced from its reduction cancel against the 3 remaining terms of the above equation. The result is:

$$= (Y_{rab}Y_{rgp} - Y_{rab} - Y_{rgp})\tilde{Y}_{pyb} = \tilde{Y}_{rab}\tilde{Y}_{rgp}\tilde{Y}_{pyb} - \tilde{Y}_{pyb}.$$

The last term here drops out because in it the component r is unknotted, and so the end result is $\tilde{Y}_{rab}\tilde{Y}_{rgp}\tilde{Y}_{pyb}$. In graphical terms, this is precisely the graph I! Cyclically permuting the roles of r, g, and b, we find that we have proven the IHX relation.

Part 3. **The Classification**

4. The class of Knotted Objects

4.1. **The Order of Braided Objects.**

4.2. **The Order of** $1 \hookrightarrow 3$ **embeddings.**

4.2.1. *The Crossing Change family.*

4.2.2. *The Multiple Crossing Change family.*

4.2.2.1. The Wedge genus. TBW

4.2.2.2. The Double Dating genus. TBW

4.2.3. *The Interdependent Modifications family.* TBW

5. The class of 3-Manifolds

6. The class of Plane Curves

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