ULTRAFILTERS, COMPACTNESS, AND THE STONE-ČECH COMPACTIFICATION

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1. The axiom of choice and Zorn's lemma

Axiom 1. *Whenever $\{X_\alpha\}_{\alpha\in I}$ is an arbitrary indexed collection of non-empty sets, their cartesian product*

$$\prod_{\alpha\in I} X_\alpha$$

is non-empty. In other words, whenever $\{X_\alpha\}_{\alpha\in I}$ is an arbitrary indexed collection of non-empty sets, there is a so-called **choice function** *$f : I \to \bigcup_{\alpha\in I} X_\alpha$ satisfying $f(\alpha) \in X_\alpha$ for every α in I.*

Warning: This axiom is far less innocent than it first seems!!!

Definition 1.1. A *partially ordered set* is a set $\mathcal{S}$ together with a binary relation $\leq$ on it, which is:

1. *Reflexive*: $s \leq s$ for every $s \in \mathcal{S}$.
2. *Anti-symmetric*: if $s \leq t$ and $t \leq s$ for $s, t \in \mathcal{S}$, then $t = s$.
3. *Transitive*: If $s \leq t$ and $t \leq u$ for $s, t, u \in \mathcal{S}$, then $s \leq u$.

A *chain* in a partially ordered set $\mathcal{S}$ is a subset $\mathcal{C}$ of $\mathcal{S}$ which is *simply ordered*, namely, a subset $\mathcal{C}$ for which whenever $s, t \in \mathcal{C}$, either $s \leq t$ or $t \leq s$. A chain $\mathcal{C}$ in a partially ordered set $\mathcal{S}$ is called *bounded from above* if there exists some $m \in \mathcal{S}$ for which $s \leq m$ whenever $s \in \mathcal{C}$.

Lemma 1.2. *(Zorn's lemma) If $\mathcal{S}$ is a partially ordered set in which every chain is bounded from above, then $\mathcal{S}$ contains (at least one)* **maximal element** *M — an element $M \in \mathcal{S}$ for which $s \in \mathcal{S}$ and $M \leq s$ implies $s = M$.*

Remark 1.3. Zorn's lemma is an equivalent and sometimes more convenient version of the axiom of choice. A proof of this equivalence can be found, for example, in [5].

2. Filters, ultrafilters, and compactness

Definition 2.1. A *filter* on a set X is a collection $\mathcal{F}$ of subsets of X satisfying:

1. $X \in \mathcal{F}$, but $\emptyset \notin \mathcal{F}$.
2. If $A \in \mathcal{F}$ and $A \subset B \subset X$, then $B \in \mathcal{F}$.
3. A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$: if $A_{1,2} \in \mathcal{F}$, then $A_1 \cap A_2 \in \mathcal{F}$.

Example 2.2. Let X be a set, x be a member of X, and $\mathcal{F}_x$ be the collection $\mathcal{F}_x = \{A \subset X : x \in A\}$. Then $\mathcal{F}_x$ is a filter on X, called "the *principal* filter on X at x".

Example 2.3. The collection of all sets containing some neighborhood of a fixed point in a topological space is a filter on that space.

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Example 2.4. Let $\mathbf{N}$ be the natural numbers, and let $\mathcal{F} = \{A \subset \mathbf{N} : \mathbf{N} - \mathbf{A} \text{ is finite}\}$. Then $\mathcal{F}$ is a filter on $\mathbf{N}$.

Definition 2.5. Let X be a topological space, $\mathcal{F}$ a filter on X, and x a point in X. We say that $\mathcal{F}$ *converges* to x and write $\mathcal{F} \to x$ if every neighborhood of x is in $\mathcal{F}$. If $\mathcal{F}$ converges to exactly one point x of X, we will call that point "the *limit* of $\mathcal{F}$" and write $x = \lim \mathcal{F}$.

Example 2.6. If X is a topological space, x is a point in X and $\mathcal{F}_x$ is the principal filter at x, then $\mathcal{F}_x \to x$.

Proposition 2.7. *A filter on a Hausdorff space X may converge to at most one point in X.*

Definition 2.8. Let X and Y be sets, $f : X \to Y$ be any function, and let $\mathcal{F}$ be a filter on X. The collection

$$f_*\mathcal{F} = \{A \subset Y : f^{-1}(A) \in \mathcal{F}\}$$

is a filter on Y, called "the *pushforward* of the filter $\mathcal{F}$ via the map f".

Example 2.9. Let $f : \mathbf{N} \to \mathbf{X}$ be an arbitrary sequence in a topological space X, let x be a point in X, and let $\mathcal{F}$ be the filter of example 2.4. Then $f_*\mathcal{F} \to x$ iff $f_n \to x$ as a sequence.

Theorem 1. *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous iff whenever a filter $\mathcal{F}$ on X converges to a point $x \in X$, the filter $f_*\mathcal{F}$ on Y converges to $f(x)$.*

Definition 2.10. An *ultrafilter* on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion. I.e., it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F}' \supset \mathcal{F}$ actually satisfies $\mathcal{F}' = \mathcal{F}$.

Example 2.11. Every principal filter is an ultrafilter. The filter of example 2.4 is not an ultrafilter.

Theorem 2. *Every filter is contained in some ultrafilter.*

Theorem 3. *The following are equivalent for a filter $\mathcal{F}$ on a set X:*

1. *$\mathcal{F}$ is an ultrafilter.*
2. *For every set $A \subset X$ either $A \in \mathcal{F}$ or $A^c = X - A \in \mathcal{F}$.*
3. *For every finite cover $\{A_i\}_{i=1}^n$ of a set $A \in \mathcal{F}$, $A_i \in \mathcal{F}$ for some i.*

Problem 2.12. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbf{N}$. Determine if the set

$$A_{\mathcal{F}} = \left\{ \sum_{n \in F} \frac{1}{2^n} \; : \; F \in \mathcal{F} \right\}$$

is Lebesgue measurable and if it is measurable, determine its Lebesgue measure. (Said differently, $A_{\mathcal{F}}$ is the collection of all numbers $x \in [0,1]$ for which the set of 1s in the binary expansion of x is in $\mathcal{F}$).

Theorem 4. *A topological space X is compact iff every ultrafilter on X is convergent.*

Proposition 2.13. *If $\mathcal{F}$ is an ultrafilter on a set X and $f : X \to Y$ is a function, then $f_*\mathcal{F}$ is also an ultrafilter.*

Tired of non-convergent sequences? You might like the following theorem: (Recall that l^∞ is the set of all bounded sequences of real numbers)

Theorem 5. *There exists a functional $l : l^\infty \to \mathbf{R}$ (called a* **generalized limit***) satisfying:*

1. *l is defined on* **all** *bounded sequences.*

(2) *If (x_n) is a sequence whose limit exists in the usual sense, then $l((x_n)) = \lim_{n\to\infty} x_n$.*
(3) *l is linear and multiplicative; whenever (x_n) and (y_n) are bounded sequences and a and b are real numbers, $l((ax_n+by_n)) = al((x_n))+bl((y_n))$ and $l((x_ny_n)) = l((x_n))l((y_n))$.*

Theorem 6. *Non-standard models of first order arithmetic (models containing infinite integers and like creatures) exist.*

Theorem 7. *(Tychonoff's theorem) If X_α is a compact topological space for every α in some arbitrary index set I, then $\prod_{\alpha\in I} X_\alpha$ is compact in the product topology.*

3. The Stone-Čech compactification

Definition 3.1. Let X be a T2 topological space. A *Stone-Čech compactification* of X is a compact T2 topological space βX containing X so that:

(1) The topology induced on X as a subset of βX is the original topology of X.
(2) Whenever $f : X \to Y$ is a continuous map of X into some compact T2 space Y, there exists a *unique* continuous map $\tilde{f} : \beta X \to Y$ whose restriction to X is f.

Remark 3.2. A rather non-trivial theorem (from our current perspective) says that if βX is a Stone-Čech compactification of X, then X is *dense* in βX, namely, the closure of X in βX is all of βX.

Theorem 8. *Any two Stone-Čech compactifications of the same topological space X are homeomorphic.*

For simplicity, we will work below only with the space $X = \mathbf{N}$ — the natural numbers with the discrete topology. The results in this section all have analogues for an arbitrary completely regular (whatever that is) topological space, and in particular, for an arbitrary metric space.

Definition 3.3. Let $\beta\mathbf{N}$ be the set of all ultrafilters on $\mathbf{N}$. We will identify $\mathbf{N}$ as a subset of $\beta\mathbf{N}$ by identifying every integer n with the principal ultrafilter μ_n at n.

Theorem 9. *There is a (naturally defined) topology on $\beta\mathbf{N}$ for which it is a Stone-Čech compactification of $\mathbf{N}$. A basis for that topology is given by $\mathcal{B} = \{U_A : A \subset \mathbf{N}\}$, where for any set $A \subset \mathbf{N}$,*

$$U_A = \{\mu \in \beta\mathbf{N} : A \in \mu\}$$

Remark 3.4. Notice that all the sets U_A are actually clopen in $\beta\mathbf{N}$!

Proposition 3.5. $\mathbf{N}$ *is dense in $\beta\mathbf{N}$.*

Exercise 3.6. Prove that $\beta\mathbf{N}$ is limit point compact but not sequentially compact.

4. Hindman's theorem

Definition 4.1. For a set $A \subset \mathbf{N}$ and a number $n \in \mathbf{N}$ define $A - n = \{k \in \mathbf{N} : \mathbf{k}+\mathbf{n} \in \mathbf{A}\}$. Let μ and ν be ultrafilters on $\mathbf{N}$. Define $\mu + \nu$ to be the collection

$$\mu + \nu = \{A \subset \mathbf{N} \; : \; \{\mathbf{n} \in \mathbf{N} : \mathbf{A} - \mathbf{n} \in \mu\} \in \nu\}.$$

Proposition 4.2. *If μ and ν are ultrafilters on $\mathbf{N}$, then so is $\mu + \nu$.*

Proposition 4.3. *The operation $+ : \beta\mathbf{N} \times \beta\mathbf{N} \to \beta\mathbf{N}$ just defined has the following three properties:*

(1) + *extends the usual addition of natural numbers. Namely, if* $m, n \in \mathbf{N}$, *then* $\mu_m + \mu_n = \mu_{m+n}$.
(2) + *is associative: if* $\mu, \nu, \rho \in \beta\mathbf{N}$, *then* $(\mu + \nu) + \rho = \mu + (\nu + \rho)$.
(3) + *is right-continuous. Namely, for each fixed* $\mu \in \beta\mathbf{N}$, *the function* $\beta\mathbf{N} \to \beta\mathbf{N}$ *defined by* $\nu \mapsto \mu + \nu$ *is continuous.*

Lemma 4.4. *If* X *is a non-empty compact space and* $+ : X \times X \to X$ *is associative and right continuous, then* X *contains (at least one)* **idempotent** *— an element* ι *of* X *for which* $\iota + \iota = \iota$.

Theorem 10. *(Hindman's theorem) Whenever the natural numbers are colored with finitely many colors (i.e., a function* $f : \mathbf{N} \to \{$*a finite set of colors*$\}$ *is specified), one can find an infinite subset* $A \subset \mathbf{N}$ *and a color* c, *so that whenever* $F \subset A$ *is finite, the color of the sum of the members of* F *is* c.

Remark 4.5. Hindman's theorem was proven by N. Hindman [4] in 1974. A simpler combinatorial proof was later found by Baumgartner [1]. The proof presented here was found by Glazer, and appears in print in [2]. A topological proof of a somewhat different flavor was found by H. Furstenberg [3].

References

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