Midterm Examination

Math 131, March 22 1993

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You have 120 minutes to answer the following 8 questions, each worth 25 points. It is a good idea to read the entire exam before starting to solve it. Notice that the maximal possible score is 200, but for the purpose of the final grade grades higher than 150 will count as 150. You may not use any material other than your pen or pencil. You may use any lemmas proven in class, provided that you quote them in full. Don't forget to sign your name on anything you submit.

(1) Give a precise definition of each of the following:
   (a) A *basis* for a topology.
   (b) An *embedding* of a topological space Y in a topological space X.
   (c) The *order topology* on a simply ordered set X.
   (d) A *locally compact* topological space.
   (e) The *pushforward* of a filter $\mathcal{F}$.

(2) Prove that if for every $\alpha \in I$ a connected topological space X_α is given, then their product $\prod_{\alpha\in I} X_\alpha$ is also connected, in the case when I is a finite set.

(3) Prove that if $f : X \to Y$ is a continuous function defined on a compact metric space X with values in a metric space Y, then f is uniformly continuous.

(4) (a) For a natural number $k \in \mathbf{N}$ define $k\mathbf{N} = \{\mathbf{kn} : \mathbf{n} \in \mathbf{N}\}$. Show that the collection $\mathcal{F} = \{\mathcal{A} \subset \mathbf{N} : \text{for some } \mathbf{k} \in \mathbf{N},\ \mathbf{kN} \subset \mathbf{A}\}$ is a filter on $\mathbf{N}$.
   (b) Define $f : \mathbf{N} \to [\mathbf{0}, \mathbf{1}]$ by
$$f(n) = \frac{1}{\text{no. of prime factors of } n}.$$
   Does $\lim f_*\mathcal{F}$ exist? What is it?
   (c) Does the sequence $f(n)$ converge? What is its limit?

(5) A topological space X is called *regular* if whenever F is a closed subset of X and y is a point not in F, there exist disjoint open subsets U and V of X such that $F \subset U$ and $y \in V$.
   (a) Prove that a compact Hausdorff space is always regular,
   (b) Prove that all metric spaces are regular.

(6) Let X be an arbitrary topological space. Show that the diagonal $\{(x, x) : x \in X\}$, in the topology induced from $X \times X$, is homeomorphic to X.

(7) Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\mathbf{N}$ consisting of all sequences that are "eventually zero", that is, all $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\mathbf{N}$ in the box and product topologies? Justify your answer.

(8) Let $A \subset X$. Show that if C is a connected subset of X that intersects both A and $X - A$, then C intersects Bd A. (Recall that Bd $A = \overline{A} \cap \overline{X - A}$).

GOOD LUCK!!