Do not turn this page until instructed.

Math 1100 Core Algebra I

Term Test

University of Toronto, October 26, 2010

Solve the 4 problems on the other side of this page.
Each problem is worth 25 points.
You have an hour and fifty minutes to write this test.

Notes.

- No outside material other than stationary is allowed.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and made of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.
- Advance apology: It may take us a while to grade this exam; sorry.

Good Luck!

Solve the following 4 problems. Each problem is worth 25 points. You have an hour and fifty minutes. **Neatness counts! Language counts!**

Problem 1. Let n be a natural number and let F be a subset of the set $\{(i,j)\colon 1 \le i < j \le n\}$. For each $(i,j) \in F$ you are given an element $\sigma_{i,j}$ of the permutation group S_n having the property that $\sigma_{i,j}(\alpha) = \alpha$ if $\alpha < i$, and $\sigma_{i,j}(i) = j$. Let M_1 be the set of all "monotone products":

$$M_1 := \{\sigma_{i_1,j_1}\sigma_{i_2,j_2}\cdots\sigma_{i_t,j_t}\colon i_1 < i_2 < \cdots < i_t, \text{ and } \forall\alpha\ (i_\alpha, j_\alpha) \in F\}.$$

1. It is also given that for every $(i,j) \in F$ and every $(k,l) \in F$, we have $\sigma_{i,j}\sigma_{k,l} \in M_1$. Prove that M_1 is a subgroup of S_n.
2. In one or two paragraphs, explain why we cared about this statement in class. What did it give us that we could not have had without it?

Problem 2. Let G be a group of odd order. Show that $x \in G$ is not conjugate to x^{-1} unless $x = e$.

Problem 3. Let G be a finite group, let p be a prime number, and let P be a Sylow-p subgroup of G.

1. Suppose that $x \in G$ is an element whose order is a power of p, and suppose that x normalizes P. Show that $x \in P$.
2. Prove that the number of conjugates of P in G is 1 modulo p. (You are not allowed to use the Sylow theorems, of course).

Problem 4. Let A_4 be the alternating subgroup of S_4, the permutation group of $\{1,2,3,4\}$, let $C = \langle x\colon x^3 = e\rangle$ be the cyclic group of order 3, and let $P = (\mathbb{Z}/2) \times (\mathbb{Z}/2)$ be the direct product of the cyclic group of order 2 with itself.

1. Explain how C and P can be viewed as subgroups of A_4.
2. Is C normal in A_4? Is P normal in A_4?
3. Find an action $\phi\colon P \times C \to P$ so that $A_4 \cong C \ltimes_\phi P$.

Good Luck!