A BIT ON MAXWELL’S EQUATIONS

Prerequisites:

- Poincaré’s Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists $\eta$ with $d\eta = \omega$.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star \omega = |\omega|^2 dx_1 \cdots dx_n$ and $\omega \wedge \star \eta = \eta \wedge \star \omega$ whenever $\omega$ and $\eta$ are of the same degree.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.

The Action Principle: The Vector Field is a compactly supported 1-form $A$ on $\mathbb{R}^4$ which extremizes the action

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} ||dA||^2 dt dx dy dz + J \wedge A$$

where the 3-form $J$ is the charge-current.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there’s no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré’s Lemma!) change variables to $F := dA$. We thus get

$$dJ = 0 \quad dF = 0 \quad d \star F = J$$

These are the Maxwell equations! Indeed, writing $F = (E_x dx dt + E_y dy dt + E_z dz dt) + (B_x dy dz + B_y dz dx + B_z dx dy)$ and $J = \rho dx dy dz - j_x dy dz dt - j_y dz dx dt - j_z dx dy dt$, we find:

$$dJ = 0 \implies \frac{\partial \rho}{\partial t} + \text{div} \, j = 0 \quad \text{“conservation of charge”}$$

$$dF = 0 \implies \text{div} \, B = 0 \quad \text{“no magnetic monopoles”}$$

$$d \star F = J \implies \text{curl} \, E = -\frac{\partial B}{\partial t} \quad \text{that’s how generators work!}$$

$$\text{div} \, E = -\rho \quad \text{“electrostatics”}$$

$$d \star F = J \implies \text{curl} \, B = -\frac{\partial E}{\partial t} + j \quad \text{that’s how electromagnets work!}$$

Exercise: Use the Lorentz metric to fix the sign error!