A BIT ON MAXWELL'S EQUATIONS

Prerequisites:

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\omega = ||\omega||^2 dx_1 \cdots dx_n$ and $\omega \wedge \star\eta = \eta \wedge \star\omega$ whenever ω and η are of the same degree.
- Integration by parts: $\int \omega \wedge d\eta = -(-1)^{\deg \omega} \int (d\omega) \wedge \eta$ on domains that have no boundary.

The Action Principle: The *Vector Field* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} ||dA||^2 dt dx dy dz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations is this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\frac{\partial \rho}{\partial t} + \operatorname{div} j = 0$	"conservation of charge"
	$\operatorname{div} B = 0$	"no magnetic monopoles"
$dF = 0 \Longrightarrow$	and	
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
	$\operatorname{div} E = -\rho$	"electrostatics"
$d * F = J \Longrightarrow$	and	
	$\operatorname{curl} B = -\frac{\partial E}{\partial t} + j$	that's how electromagnets work!

Exercise: Use the Lorentz metric to fix the sign error!

November 29, 2007; http://www.math.toronto.edu/~drorbn/classes/0708/GeomAndTop and http://www.math.toronto.edu/~drorbn/classes/0708/GeomAndTop/Maxwell.pdf