

A Sample Term Exam 2

University of Toronto, March 3, 2005

Math 1300Y Students: Make sure to write “1300Y” in the course field on the exam notebook. Solve one of the two problems in part A and three of the four problems in part B. Each problem is worth 25 points. If you solve more than the required 1 in 2 and 3 in 4, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have an hour and 50 minutes. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write “427S” in the course field on the exam notebook. Solve the four problems in part B, do not solve anything in part A. Each problem is worth 25 points. You have an hour and 50 minutes. No outside material other than stationary is allowed.

Part A

Problem 1. Consider $S^1 := \{z \in \mathbb{C} : |z| = 1\}$.

1. Use $\pi_1(S^1)$ to define the degree $\deg f$ of a map $f : S^1 \rightarrow S^1$.
2. A map $f : S^1 \rightarrow S^1$ is *even* if $f(-z) = f(z)$ for all $z \in S^1$. Show that if $f : S^1 \rightarrow S^1$ is even then $\deg f$ is even.
3. A map $f : S^1 \rightarrow S^1$ is *odd* if $f(-z) = -f(z)$ for all $z \in S^1$. Show that if $f : S^1 \rightarrow S^1$ is odd then $\deg f$ is odd.

Problem 2. State Van-Kampen’s theorem and compute the fundamental group of the Klein bottle (a square with two pairs opposite edges identified, one pair in a parallel manner and one pair in an anti-parallel manner).

Part B

Problem 3. Let B be a connected, locally connected and semi-locally simply connected topological space with basepoint b .

1. Explicitly construct “the universal covering of B ” as a set U with a map $p : U \rightarrow B$ and a basepoint u .
2. Explicitly describe the topology of U . You don’t need to show that the topology you have described is indeed a topology or that p is a covering map, or even that p is continuous.

- Without referring to the general classification of covering spaces show that if $q : (X, x) \rightarrow (B, b)$ is a connected covering of B (with basepoint x) then there is a unique basepoint-preserving $r : (U, u) \rightarrow (X, x)$ such that $p = q \circ r$.

Problem 4.

- Define “a morphism between two chain complexes”.
- Show that a morphism f between two chain complexes induces a map f_\star between their homologies.
- Define “a homotopy between two morphisms of chain complexes”.
- Show that if f and g are homotopic morphisms of chain complexes, then $f_\star = g_\star$.

Problem 5. A chain complex A is said to be “acyclic” if its homology vanishes (i.e., if it is an exact sequence). Let C be a subcomplex of some chain complex B .

- Show that if C is acyclic then the homology of B is isomorphic to the homology of B/C (so C “doesn’t matter”).
- Show that if B/C is acyclic then the homology of B is isomorphic to the homology of C (so “the part of B out of C ” doesn’t matter).
- If B is acyclic, can you say anything about the relation between the homology of C and the homology of B/C ?

Problem 6. Let P be a wedge of 5 lines, $P = \{z \in \mathbb{C} : z^5 \in \mathbb{R} \text{ and } |z| \leq 1\}$, let Q be the result of gluing the ends of $P \times I$ to each other with a $1/5$ twist, $Q = P \times I / (z, 0) \sim (\eta z, 1)$, with $\eta = \exp 2\pi i/5$. The “boundary” of Q is a single circle that “wraps around five times”. Let X be the result of identifying that circle with the boundary of some disk D .

- Describe X as a polygon with some edges identified.
- Describe X as the geometric realization of some Δ -complex.
- Compute the homology of X .

Warning: The real exam will be similar to this sample, to my taste. Your taste may be significantly different.