The Devil is in the Details

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Let $f : [0, 3] \to \mathbb{R}$ be the function given by the rule $f(x) = 2$ whenever $x \neq 1$ and define $f(1) = 7$. So f is constant (and hence continuous) except at one point, namely $x = 1$. We would like to show f is integrable.

We will show that f is integrable by showing that for every $\epsilon > 0$, there is some partition P of $[0, 1]$ that satisfies the inequality:

$$U(f, P) - L(f, P) < \epsilon$$

To that end, let $\epsilon > 0$, and let's try to find some partition P that will make the above inequality true. Consider the partition $P = \{0, 1 - \frac{\epsilon}{20}, 1 + \frac{\epsilon}{20}, 3\}$. On the subintervals $[0, 1 - \frac{\epsilon}{20}]$, $[1 + \frac{\epsilon}{20}, 3]$ the function is constant, and therefore the $m_i = M_i$ for these intervals. So the difference $U(f, P) - L(f, P) = 7(\frac{\epsilon}{10}) - 2(\frac{\epsilon}{10}) = \epsilon/2 < \epsilon$. Therefore f is indeed integrable.

Remark. Notice that on the interval containing the discontinuity, the difference $M_i - m_i = 5$. So the difference $U(f, P) - L(f, P)$ would have been $5\times$(the width of the subinterval containing 1). So the only way to make that term small, was to make that width small, and that's what we did.

Let's consider now a more complicated example. Let $f : [0, 1] \to \mathbb{R}$ be the (nasty) function given by:

$$f(x) = \begin{cases} \frac{1}{[\frac{1}{x}]} & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$

It may be helpful, in the discussion that follows, to draw yourself a sketch of the graph of the function. Before showing f is integrable, let's make the following observation. Notice that f is constant on the sub-intervals of the form $(\frac{1}{n+1}, \frac{1}{n}]$, and that constant value is $\frac{1}{n}$. We now show that f is integrable by using the integrability criterion mentioned above.

Let $\epsilon > 0$ be given. We seek some partition P of $[0, 1]$ that makes the difference of upper and lower sums small. To that end, let's take a positive integer N so large that $1/N < \sqrt{\epsilon/2}$, and let $r > 0$ be a small number that satisfies

$$2r < \frac{1}{N} - \frac{1}{N+1} = \frac{1}{N(N+1)}$$

Now let P be the partition $P = \{0 < \frac{1}{N} - r < \frac{1}{N} + r < \frac{1}{N-1} - r < \frac{1}{N-1} + r < \ldots < \frac{1}{2} - r < \frac{1}{2} + r < 1\}$. (Notice that all the inequalities are indeed true, since we chose r to be small enough. Namely, we can be sure that for any positive integer $n \leq N$, we have $\frac{1}{n+1} + r < \frac{1}{n} - r$, since $2r < \frac{1}{N(N+1)} < \frac{1}{n(n+1)}$.)

Okay, so what is $U(f,P) - L(f,P)$? Since f is constant on the sub-intervals of the form $(\frac{1}{n+1}, \frac{1}{n}]$, it is therefore constant on sub-intervals of the form $[\frac{1}{n+1} + r, \frac{1}{n} - r]$, and thus the $m_i = M_i$ on these sub-intervals. And on the sub-intervals of the form $[\frac{1}{n} - r, \frac{1}{n} + r]$ we have that $M_i = \frac{1}{n-1}$, and $m_i = \frac{1}{n}$. There are two remaining sub-intervals to talk about: the first and last ones. On $[0, \frac{1}{N} - r]$ $M_i = \frac{1}{N}$, and $m_i = 0$. On $[\frac{1}{2} + r, 1]$, f is constant, so $M_i = m_i$ for this interval as well. Therefore, if we were to calculate the difference $U(f,P) - L(f,P)$, we would only have to worry about the terms coming from the first interval, and the intervals of type $[\frac{1}{n} - r, \frac{1}{n} + r]$, since the other terms would be zero.

Since these sub-intervals have width equal to $2r$, then

$$\begin{aligned}
U(f,P) - L(f,P) &= \frac{1}{N}\left(\frac{1}{N} - r\right) + \sum (M_i - m_i)(2r) \\
&= \frac{1}{N}\left(\frac{1}{N} - r\right) + (2r)\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= \frac{1}{N}\left(\frac{1}{N} - r\right) + 2r\left(1 - \frac{1}{N}\right) \\
&\leq \frac{1}{N^2} + 2r \\
&\leq \frac{1}{N^2} + \frac{1}{N(N+1)} \\
&\leq \frac{1}{N^2} + \frac{1}{N^2} < \epsilon
\end{aligned}$$

Therefore, the partition P satisfies the desired inequality, and therefore f is integrable. Phew!

It is worth trying to understand why this approach worked. Namely, think about why one might have difficulty making the difference $U(f,P) - L(f,P)$ small. What could make it big? There were basically two forces working to make the difference big. First, the size of the jumps in the function f, and second, the number of jumps (even a lot of little jumps can *add up* to make the difference of upper and lower sums large!) To control the first force (the values of the jumps in the functions), we chose a partition that included the jumps, but with very small width. That way, the corresponding rectangle may be tall, but at least its thin! And second, there were infinitely many jumps, so we combined an infinite number of jumps in one interval (the first one!), but made that interval small.