Some dimensions of $\mathcal{A}$

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Abstract

We compute the dimensions of $\mathcal{A}_1$ thru $\mathcal{A}_5$ and quote the dimensions of $\mathcal{A}_6$ thru $\mathcal{A}_{12}$.

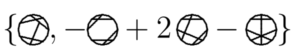
Starting up mathematica [Wo], loading a definitions file and testing the $4T$ relation:

```
Mathematica 4.1 for Linux
Copyright 1988-2000 Wolfram Research, Inc.
 -- Motif graphics initialized --
```

In[1]:= `<< ChordDiagrams.m`

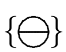
```
Loading ChordDiagrams...
```

In[2]:= `{d=Diagram[Chord[1,3],Chord[4,6],D4T[5,2,7]], b[d]}`

Out[2]= 

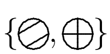
There is only one way to place a single chord on a circle...

In[3]:= `Place[Chord]`

Out[3]= 

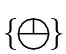
and there can be no $4T$ relations in degree 1. Therefore $\dim \mathcal{A}_1 = 1$. Now, there are two ways to place two chords...

In[4]:= `Place[2*Chord]`

Out[4]= 

and one way to place a $4T$ relation symbol and no chords...

In[5]:= `RelationSymbol = Place[D4T]`

Out[5]= 

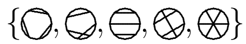
but the actual relation that corresponds to this symbol is 0...

In[6]:= `Relation = b /@ RelationSymbol`

Out[6]= $\{0\}$

and therefore $\dim \mathcal{A}_1 = 2$. Likewise, there are 5 ways to place three chords...

In[7]:= `Place[3*Chord]`

Out[7]= 

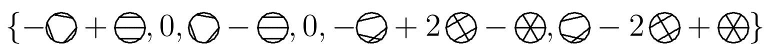
and 6 relations symbols made of one $4T$ symbol and one chord:

In[8]:= `RelationSymbols = Place[D4T+Chord]`

Out[8]= 

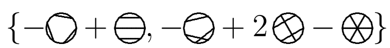
The corresponding relations are...

In[9]:= `Relations = b /@ RelationSymbols`

Out[9]= 

and their linear span is...

In[10]:= `LinearSpan[Relations]`

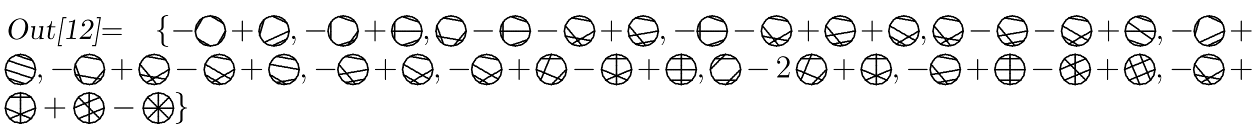
Out[10]= 

As this span is 2 dimensional, we find that $\dim \mathcal{A}_3 = 5-2 = 3$. We now repeat this procedure in degree 4...

In[11]:= `CDs = Place[4*Chord]`

Out[11]= 

In[12]:= `Rels = LinearSpan[b /@ Place[D4T+2*Chord]]`

Out[12]= 

In[13]:= `Length[CDs]-Length[Rels]`

Out[13]= 6

and find that $\dim \mathcal{A}_4 = 6$. Finally,

In[14]:= `Length[Place[5*Chord]]`

Out[14]= 105

In[15]:= `Length[LinearSpan[b /@ Place[D4T+3*Chord]]]`

Out[15]= 95

In[16]:= `105-95`

Out[16]= 10

and thus $\dim \mathcal{A}_5 = 10$.

Working harder and with better programs (see [Ba, Kn]), we learn that

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim \mathcal{A}_n$	1	1	2	3	6	10	19	33	60	104	184	316	548

A conjectured generating function for the sequence $\dim \mathcal{A}_n$ is at [Br]. At present, the computation of $\dim \mathcal{A}_n$ for general n seems to be beyond our reach.

References

[Ba] D. Bar-Natan, *On the Vassiliev knot invariants,* Topology **34** (1995) 423–472.

[Br] D. J. Broadhurst, *Conjectured enumeration of Vassiliev invariants,* Open University UK preprint, September 1997, arXiv:q-alg/9709031.

[Kn] J. A. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve,* University of Bonn preprint, June 1997, arXiv:q-alg/9706022.

[Wo] S. Wolfram, *The Mathematica Book,* Cambridge University Press, 1999 and `http://documents.wolfram.com/framesv4/frames.html`.

This handout and the program used in it are available at
`http://www.ma.huji.ac.il/~drorbn/classes/0001/KnotTheory`.