Some Hints for Solutions

1 Exercise #1 in handout #2

(1) Suppose that \( \varphi : \mathbb{R} \to X \) is a diffeomorphism, and let \( \psi : X \to \mathbb{R} \) be its inverse. Suppose that \( \varphi(0) = 0 \).

Let \( j : X \to \mathbb{R}^2 \) be the inclusion map. Then \( j \) is smooth (why?). By the definition of \( F_X \), there exists a smooth map \( \Psi \) extending \( \psi \) and rendering the following diagram commutative

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\varphi} & X \\
\downarrow{j} & & \downarrow{\Psi} \\
\mathbb{R} & \xrightarrow{i} & \mathbb{R}
\end{array}
\]

(2) \( \Psi \varphi = \Psi j \varphi = \psi \varphi = \text{id} \) implies \( \Phi_0 \neq 0 \), namely \( \Phi(0) \neq 0 \).

(3) Connectedness consideration of \( X - \{0\} \) and \( \mathbb{R} - \{0\} \) implies that we may assume \( \Phi((\infty, 0)) \subseteq (\infty, 0) \times \mathbb{R} \) and \( \Phi((0, \infty)) \subseteq (0, \infty) \times \mathbb{R} \). Therefore (explain this)

\[
\Phi'(0) = \lim_{t \to 0} \frac{\Phi(t) - \Phi(0)}{t} = (a, -a)
\]

\[
\Phi'(0) = \lim_{t \to 0} \frac{\Phi(t) - \Phi(0)}{t} = (b, b),
\]

It follows that \( a = b = 0 \) hence \( \Phi'(0) = 0 \), a contradiction.

2 Exercise #2 in handout #3

The following argument is due to Yariv. Use paths to represent derivations. The inclusion \( j : U \to M \) induces \( j_* : T_pU \to T_pM \) via \( j_* (D\gamma) = D\gamma \).

Define \( \pi : T_pM \to T_pU \) as follows. For \( \gamma : (a, a) \to M \) there exists \( \epsilon \) such that \( \gamma((\infty, \epsilon)) \subseteq U \). Define

\[
\pi (D\gamma) = D\gamma|_{(a, a)}.
\]
This is independent in the choice of $\epsilon$ and of $\gamma$ (why?). It is almost trivial that $j_\ast$ and $\pi$ are inverses of one another.

Another easy proof can be obtained using the definition of the tangent space as in exercise #1 in this handout. Surjectivity of $j_\ast$ is obvious and dimensional argument concludes.

What I said in class was rubbish. It will make sense when we talk about derivations $C^\infty(M) \to C^\infty(M)$ which correspond to vector fields. The best is yet to come.

3 Exercise #3 in handout #3

Identify $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ via

$$\frac{\partial}{\partial x_i} \mapsto e_i.$$  

Under this identification, if a path $\gamma:(-\alpha,\alpha) \to \mathbb{R}^n$ represents a tangent vector in $T_p\mathbb{R}^n$, then $\gamma$ corresponds to $\gamma'(0)$ in $\mathbb{R}^n$. To see this recall that using the trivial chart on $\mathbb{R}^n$,

$$D_\gamma = \sum_{i=1}^{n} \frac{d\gamma^i}{dt}(0) \frac{\partial}{\partial x_i}.$$  

In our example

$$\Psi_\ast \frac{\partial}{\partial \theta} \equiv D_{(\sin \phi \cos(\theta+t), \sin \phi \sin(\theta+t), \cos \phi)} \subset S^2 \mapsto j_\ast$$

$$D_{(\sin \phi \cos(\theta+t), \sin \phi \sin(\theta+t), \cos \phi)} \subset \mathbb{R}^3 \sim (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)$$

A similar computation shows that

$$\Psi_\ast \frac{\partial}{\partial \varphi} \sim (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi).$$

One checks directly that these are nonzero orthogonal vectors and both are orthogonal to the vector $\Psi(\theta, \varphi)$.

This argument also shows that the inclusion map $i:S^2 \to \mathbb{R}^3$ is an immersion because $S^2$ is a 2-dimensional manifold and $i_\ast$ has a 2 dimensional space as its image. Of course, you may (and better) use the theorem proved in class about the inverse image of a regular value.