1. Show that \( \wedge \) is a commutative and associative operation on \( A^*(V) \), where \( V \) is a finite dimensional vector space.

2. Show that the two definitions given in class for the smoothness of a differential form \( \omega \in \Omega^p(M) \) are equivalent.

3. (The Hodge operator.) Let \( V \cong \mathbb{R}^n \). Suppose that \( V \) is equipped with the additional data of:
   (i) A non degenerate symmetric bilinear form \( B: V \times V \to \mathbb{R} \).
   (ii) A nonzero top form \( \nu \in A^n(V) \).

   It is a standard fact that \( B \) induces an isomorphism \( V \cong V^* \), hence \( V^* \) is also naturally equipped with a non degenerate symmetric bilinear form also denoted \( B \), for obvious reasons.

   (a) Show that \( B \) induces a symmetric bilinear form \( B^{\otimes k} \) on \( V^{\ast \otimes k} \) which is unique under the requirement that for 1-forms \( \omega_1, \ldots, \omega_k \) and \( \eta_1, \ldots, \eta_k \)

   \[
   B^{\otimes k}(\omega_1 \otimes \cdots \otimes \omega_k, \eta_1 \otimes \cdots \otimes \eta_k) = \prod_{i=1}^k B(\omega_i, \eta_i).
   \]

   Recall that \( A^k(V) \) is a subspace of \( V^{\ast \otimes k} \), hence \( B^{\otimes k} \) induces a bilinear form on \( A^k(V) \) by

   \[
   B^{\otimes k} = \frac{1}{k!} B^{\otimes k} |_{A^k(V)}
   \]

   Show that \( B^{\wedge k} \) is non degenerate (and symmetric). Show further, that if \( x_1, \ldots, x_n \) is a basis for \( V \), then \( B^{\otimes n}(dx_1 \wedge \cdots \wedge dx_n, dx_1 \wedge \cdots \wedge dx_n) = \det \{ B(dx_i, dx_j) \} \). For this reason, we usually assume that \( B^{\otimes n}(\nu, \nu) = \pm 1 \).

   (b) Use \( \nu \) to canonically identify \( A^n(V) \) with \( \mathbb{R} \). Then, every \( \alpha \in A^{n-k}(V) \) induces a linear functional \( \bar{\alpha}^{\otimes k} \to \mathbb{R} \) via the assignment \( \eta^\otimes k \mapsto \eta^\otimes k \wedge \alpha \). Show that this induces a canonical isomorphism \( A^{n-k}(V) \cong A^k(V)^* \).

   (c) Deduce that for every \( \alpha \in A^k(V) \) there exists a unique \((n-k)\)-form \( \star \alpha \) satisfying the following equation for every \( k \)-form \( \eta \):

   \[
   \eta \wedge \star \alpha = B^{\otimes k} (\eta, \alpha) \nu.
   \]

   Show that \( \star \) is a linear isomorphism \( A^k(V) \cong A^{n-k}(V) \). Show also that if one chooses \( \nu \) as in the end of (a), then, up to sign

   \[
   B^{\wedge k}(\alpha, \beta) = B^{n-k}(\alpha \wedge \beta).
   \]

   (d) Let \( V = \mathbb{R}^3 \) be equipped with the usual inner product and top form. Let \( \alpha, \beta \) be 1-forms and compute \( \star (\alpha \wedge \beta) \). Is this result familiar?

   (e) Let \( V = \mathbb{R}^4 \) be equipped with the Lorenz form \( B(x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4 \) and the obvious top form. Compute \( \star \).