

Quantum Probability

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We start by loading a necessary *Mathematica* package, by defining the tensor product $\mathbf{A} \otimes \mathbf{B}$ of two matrices A and B and the 2×2 identity matrix \mathbf{I}_2 :

```
In[1]:= << LinearAlgebra`MatrixManipulation`
In[2]:= A_ \otimes B_ := BlockMatrix[Outer[Times, A, B]]; I2 = IdentityMatrix[2];
```

Next we define the unit "probability vector" \mathbf{v} , and our observables ("random variables") A_α and B_β as tensor products of \mathbf{I}_2 with some prescribed S_γ :

```
In[3]:= v =  $\frac{\sqrt{2}}{2}$  {0, 1, -1, 0}; S_\gamma :=  $\begin{pmatrix} \text{Cos}[2 \gamma] & \text{Sin}[2 \gamma] \\ \text{Sin}[2 \gamma] & -\text{Cos}[2 \gamma] \end{pmatrix}$ ; A_\alpha := I2 \otimes S_\alpha; B_\beta := -S_\beta \otimes I2
```

```
In[4]:= {A_\alpha // MatrixForm, B_\beta // MatrixForm}
```

$$\text{Out}[4] = \left\{ \begin{pmatrix} \text{Cos}[2 \alpha] & \text{Sin}[2 \alpha] & 0 & 0 \\ \text{Sin}[2 \alpha] & -\text{Cos}[2 \alpha] & 0 & 0 \\ 0 & 0 & \text{Cos}[2 \alpha] & \text{Sin}[2 \alpha] \\ 0 & 0 & \text{Sin}[2 \alpha] & -\text{Cos}[2 \alpha] \end{pmatrix}, \begin{pmatrix} -\text{Cos}[2 \beta] & 0 & -\text{Sin}[2 \beta] & 0 \\ 0 & -\text{Cos}[2 \beta] & 0 & -\text{Sin}[2 \beta] \\ -\text{Sin}[2 \beta] & 0 & \text{Cos}[2 \beta] & 0 \\ 0 & -\text{Sin}[2 \beta] & 0 & \text{Cos}[2 \beta] \end{pmatrix} \right\}$$

We check that both A_α and B_β are (± 1) -valued and have zero mean, hence both attain +1 and -1 with 50-50 chance:

```
In[5]:= {{Eigenvalues[A_\alpha], v.A_\alpha.v}, {Eigenvalues[B_\beta], v.B_\beta.v}}
Out[5]= {{{-1, -1, 1, 1}, 0}, {{-1, -1, 1, 1}, 0}}
```

The A_α 's and the B_β 's commute, hence they have a joint distribution! Indeed,

```
In[6]:= A_\alpha.B_\beta == B_\beta.A_\alpha
Out[6]= True
```

The A_α 's and the B_β 's are both (± 1) -valued, so the probability that they are equal is the expectation value (mean) of $\frac{1+A_\alpha B_\beta}{2}$:

```
In[7]:= p_eq[\alpha_, \beta_] := Simplify[ $\frac{1 + \mathbf{v} \cdot \mathbf{A}_\alpha \cdot \mathbf{B}_\beta \cdot \mathbf{v}}{2}$ ]
```

Finally, the following is strictly impossible, classically speaking:

```
In[8]:= {p_eq[\alpha, \beta], Outer[p_eq, {-60 \text{ }^\circ, 0, 60 \text{ }^\circ}, {-60 \text{ }^\circ, 0, 60 \text{ }^\circ}]} // MatrixForm
```

$$\text{Out}[8] = \left\{ \text{Cos}[\alpha - \beta]^2, \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix} \right\}$$

See also N. D. Mermin, *Physics Today* 39(4) 38 (1985) and D. Bar-Natan, *Foundations of Physics* 19(1) 97 (1989).