Quantum Probability

University of Toronto, October 2, 2003

This is http://www.math.toronto.edu/~drorbn/Talks/UofT-031002/ [index.html | QuantumProbability.pdf | QuantumProbability.nb]

We start by loading a necessary *Mathematica* package, by defininig the tensor product **A⊗B** of two matrices A and B and the 2×2 identity matrix I_2:

```
In[1]:= << LinearAlgebra`MatrixManipulation`
```

```
In[2]:= A_ ⊗ B_ := BlockMatrix[Outer[Times, A, B]]; I2 = IdentityMatrix[2];
```

Next we define the unit "probability vector" **v**, and our observables ("random variables") A_α and B_β as tensor products of I_2 with some prescribed S_γ:

In[3]:= $\mathbf{v} = \frac{\sqrt{2}}{2}\{0, 1, -1, 0\};\ S_{\gamma\_} := \begin{pmatrix} \mathrm{Cos}[2\gamma] & \mathrm{Sin}[2\gamma] \\ \mathrm{Sin}[2\gamma] & -\mathrm{Cos}[2\gamma] \end{pmatrix};\ A_{\alpha\_} := I_2 \otimes S_\alpha;\ B_{\beta\_} := -S_\beta \otimes I_2$

In[4]:= $\{A_\alpha$ // MatrixForm, B_β // MatrixForm$\}$

Out[4]= $\left\{\begin{pmatrix} \mathrm{Cos}[2\alpha] & \mathrm{Sin}[2\alpha] & 0 & 0 \\ \mathrm{Sin}[2\alpha] & -\mathrm{Cos}[2\alpha] & 0 & 0 \\ 0 & 0 & \mathrm{Cos}[2\alpha] & \mathrm{Sin}[2\alpha] \\ 0 & 0 & \mathrm{Sin}[2\alpha] & -\mathrm{Cos}[2\alpha] \end{pmatrix}, \begin{pmatrix} -\mathrm{Cos}[2\beta] & 0 & -\mathrm{Sin}[2\beta] & 0 \\ 0 & -\mathrm{Cos}[2\beta] & 0 & -\mathrm{Sin}[2\beta] \\ -\mathrm{Sin}[2\beta] & 0 & \mathrm{Cos}[2\beta] & 0 \\ 0 & -\mathrm{Sin}[2\beta] & 0 & \mathrm{Cos}[2\beta] \end{pmatrix}\right\}$

We check that both A_α and B_β are (±1)-valued and have zero mean, hence both attain +1 and -1 with 50-50 chance:

In[5]:= $\{\{\mathrm{Eigenvalues}[A_\alpha], \mathbf{v}.A_\alpha.\mathbf{v}\}, \{\mathrm{Eigenvalues}[B_\beta], \mathbf{v}.B_\beta.\mathbf{v}\}\}$

```
Out[5]= {{{-1, -1, 1, 1}, 0}, {{-1, -1, 1, 1}, 0}}
```

The A_α's and the B_β's commute, hence they have a joint distribution! Indeed,

In[6]:= $A_\alpha.B_\beta == B_\beta.A_\alpha$

```
Out[6]= True
```

The A_α's and the B_β's are both (±1)-valued, so the probability that they are equal is the expectation value (mean) of $\frac{1+A_\alpha B_\beta}{2}$:

In[7]:= $p_{eq}[\alpha\_, \beta\_] := \mathrm{Simplify}\left[\frac{1 + \mathbf{v}.A_\alpha.B_\beta.\mathbf{v}}{2}\right]$

Finally, the following is stricktly impossible, classically speaking:

In[8]:= $\{p_{eq}[\alpha, \beta], \mathrm{Outer}[p_{eq}, \{-60°, 0, 60°\}, \{-60°, 0, 60°\}]$ // MatrixForm$\}$

Out[8]= $\left\{\mathrm{Cos}[\alpha-\beta]^2, \begin{pmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 \end{pmatrix}\right\}$

See also N. D. Mermin, Physics Today 39(4) 38 (1985) and D. Bar-Natan, Foundations of Physics 19(1) 97 (1989).