

Homomorphic Expansions and w-Knots

Dror Bar-Natan, UWO February 2010,

<http://www.math.toronto.edu/~drorbn/Talks/UWO-100225/>

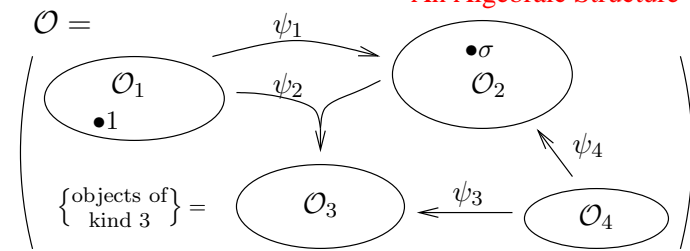
Abstract Even though little known, the notion of a “homomorphic expansion” is extremely general; it makes sense in the context of practically *any* algebraic structure, be it a group, or a group homomorphism, or a quandle, or a planar algebra, or a circuit algebra with unzip operations, or whatever.

Even though little known, w-knots make a cool generalization of ordinary knots. They contain ordinary knots and are contained in 2-knots in 4-space and are easier than the latter. They are a quotient of “virtual knots” and are easier than those.

My talk will be about these two notions, homomorphic expansions and w-knots, and about what happens when the two are put together. Lie algebras arise, and Lie groups, and the Kashiwara-Vergne statement, which is one of the deeper statements about the relationship between Lie groups and Lie algebras.

There are also u-knots, and v-knots, and f-knots, and other things which are not knots at all, and there are equally nifty things to say about homomorphic expansions for all those. But not today.

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Homomorphic expansions for a filtered algebraic structure \mathcal{K} :

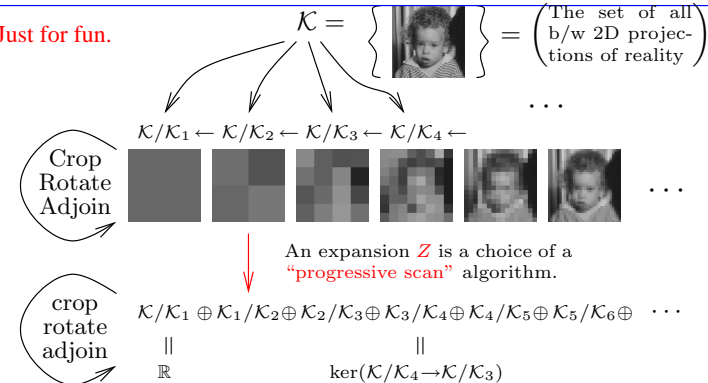
$$\text{ops} \supset \mathcal{K} = \mathcal{K}_0 \supset \mathcal{K}_1 \supset \mathcal{K}_2 \supset \mathcal{K}_3 \supset \dots$$

$$\downarrow \qquad \qquad \qquad \downarrow z$$

$$\text{ops} \supset \text{gr } \mathcal{K} := \mathcal{K}_0/\mathcal{K}_1 \oplus \mathcal{K}_1/\mathcal{K}_2 \oplus \mathcal{K}_2/\mathcal{K}_3 \oplus \mathcal{K}_3/\mathcal{K}_4 \oplus \dots$$

An **expansion** is a filtration respecting $Z : \mathcal{K} \rightarrow \text{gr } \mathcal{K}$ that “covers” the identity on $\text{gr } \mathcal{K}$. A **homomorphic expansion** is an expansion that respects all relevant “extra” operations.

Just for fun.



Filtered algebraic structures are cheap and plenty. In any \mathcal{K} , allow formal linear combinations, let \mathcal{K}_1 be the ideal generated by differences (the “augmentation ideal”), and let $\mathcal{K}_m := \langle (\mathcal{K}_1)^m \rangle$ (using all available “products”).

[1] <http://qlink.queensu.ca/~4lb11/interesting.html>

25/2/10, 2:52pm

Also see <http://www.math.toronto.edu/~drorbn/papers/WKO/>

Examples. 1. The projectivization of a group is a graded associative algebra. 2. Quandle: a set Q with an $\text{op } \wedge \text{ s.t.}$

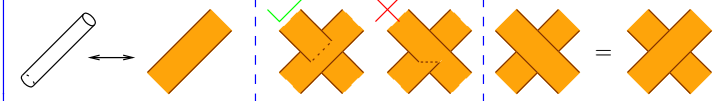
$$1 \wedge x = 1, \quad x \wedge 1 = x \wedge x = x, \quad (\text{appetizers})$$

$$(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad (\text{main})$$

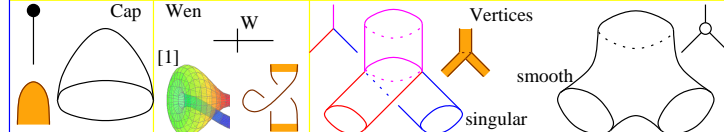
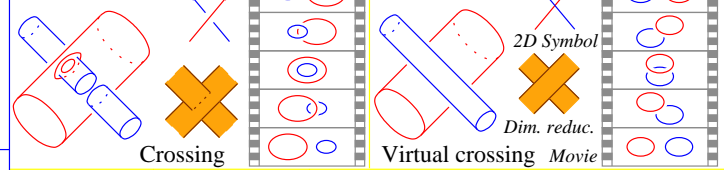
$\text{proj } Q$ is a graded Lie algebra: set $\bar{v} := (v - 1)$ (these generate $I!$), feed $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}).$$

A **Ribbon 2-Knot** is a surface S embedded in \mathbb{R}^4 that bounds an immersed handlebody B , with only “ribbon singularities”; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



The w-generators.



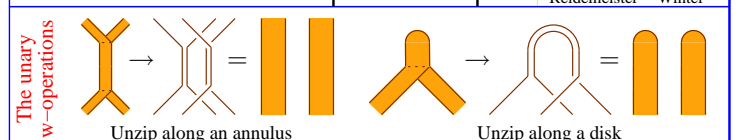
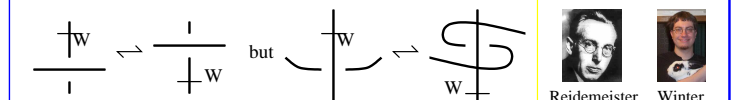
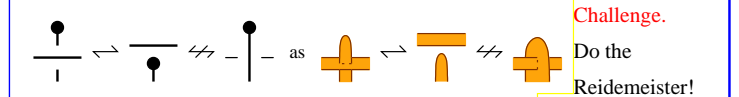
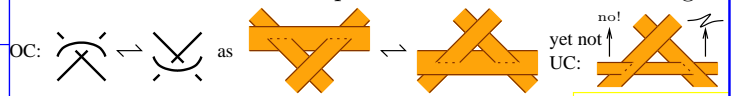
What are w-Trivalent Tangles?

$$\{\text{knots \& links}\} = \text{PA} \langle \text{trivalent symbols} \mid R123 : \text{relations} \rangle_{0 \text{ legs}}$$

$$\{\text{trivalent tangles}\} = \text{PA} \langle \text{trivalent symbols} \mid R23, R4 : \text{relations} \rangle$$

$$\text{wTT} = \{\text{trivalent w-tangles}\} = \text{PA} \langle \text{w-generators} \mid \text{w-relations} \mid \text{unary w-operations} \rangle$$

The **w-relations** include R234, VR1234, M, Overcrossings Commute (OC) but not UC, $W^2 = 1$, and funny interactions between the wen and the cap and over- and under-crossings:



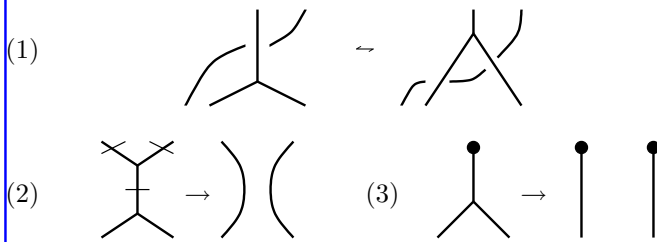
Our case(s).

$$\mathcal{K} \xrightarrow[\text{solving finitely many equations in finitely many unknowns}]{Z: \text{high algebra}} \mathcal{A} := \text{gr } \mathcal{K} \xrightarrow[\text{low algebra: pictures represent formulas}]{\text{given a "Lie" algebra } \mathfrak{g}} "U(\mathfrak{g})"$$

\mathcal{K} is knot theory or **topology**; $\text{gr } \mathcal{K}$ is finite **combinatorics**: bounded-complexity diagrams modulo simple relations.

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Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



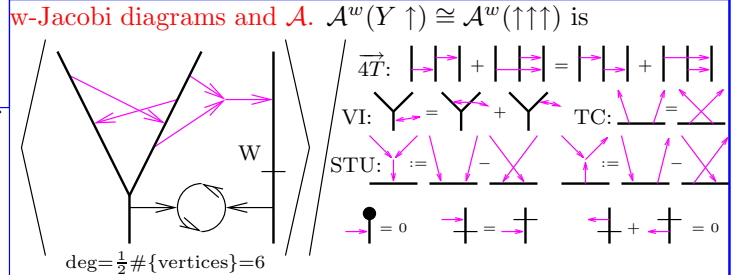
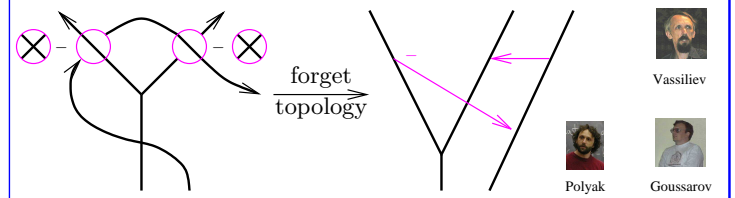
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$



Measure theoretic statement. Ignoring all j 's, there exists a measure preserving and orbit preserving transformation $T : \mathfrak{g}_x \times \mathfrak{g}_y \rightarrow \mathfrak{g}_x \times \mathfrak{g}_y$ for which $e^{x+y} \circ T = e^x e^y$.

Top. to Comb. $\text{gr}_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}$:



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via

$$\sum_{i,j,k,l,m,n=1} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^n \in \mathcal{U}(\mathfrak{g})$$

The u-v-w Story

| | u-Knots | v-Knots | w-Knots |
|----------------------|--|--|--|
| Topology | Ordinary (usual) knotted objects in 3D — braids, knots, links, tangles, knotted graphs, etc. | Virtual knotted objects — “algebraic” knotted objects, or “not specifically embedded” knotted objects; knots drawn on a surface, modulo stabilization. | Ribbon knotted objects in 4D; “flying rings”. Like v, but also with “overcrossings commute”. |
| Combinatorics | Chord diagrams and Jacobi diagrams, modulo $4T, STU, IHX$, etc. | Arrow diagrams and v-Jacobi diagrams, modulo $6T$ and various “directed” $STUs$ and $IHXs$, etc. | Like v, but also with “tails commute”. Only “two in one out” internal vertices. |
| Low Algebra | Finite dimensional metrized Lie algebras, representations, and associated spaces. | Finite dimensional Lie bi-algebras, representations, and associated spaces. | Finite dimensional co-commutative Lie bi-algebras (i.e., $\mathfrak{g} \times \mathfrak{g}^*$), representations, and associated spaces. |
| High Algebra | The Drinfel’d theory of associators. | Likely, quantum groups and the Etingof-Kazhdan theory of quantization of Lie bi-algebras. | The Kashiwara-Vergne-Alekseev-Torossian theory of convolutions on Lie groups and Lie algebras. |

Some Propaganda

“God created the knots, all else in topology is the work of mortals.”
Leopold Kronecker (modified)



w-Search “Knotted Candies”