A Quick Reference Guide to Khovanov's Categorification of the Jones Polynomial

Dror Bar-Natan, June 12, 2002

The Kauffman Bracket: $\langle\emptyset\rangle = 1$; $\langle \bigcirc L\rangle = (q+q^{-1})\langle L\rangle$; $\langle \times \rangle = \langle \underset{0-\text{smoothing}}{\asymp} \rangle - q\langle \underset{1-\text{smoothing}}{)(} \rangle$.

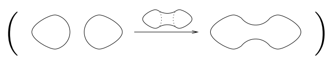
The Jones Polynomial: $\hat{J}(L) = (-1)^{n_-}q^{n_+-2n_-}\langle L\rangle$, where (n_+, n_-) count $(\overcrossing, \undercrossing)$ crossings.
Khovanov's construction: $[\![L]\!]$ — a chain complex of graded $\mathbb{Z}$-modules;

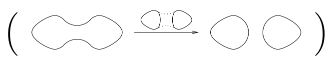
$$[\![\emptyset]\!] = 0 \to \underset{\text{height } 0}{\mathbb{Z}} \to 0; \qquad [\![\bigcirc L]\!] = V\otimes[\![L]\!]; \qquad [\![\times]\!] = \text{Flatten}\left(0 \to \underset{\text{height } 0}{[\![\asymp]\!]} \to \underset{\text{height } 1}{[\![)(]\!]\{1\}} \to 0\right);$$

$$\mathcal{H}(L) = \mathcal{H}\left(\mathcal{C}(L) = [\![L]\!][-n_-]\{n_+-2n_-\}\right)$$

$$V = \text{span}\langle v_+, v_-\rangle; \qquad \deg v_\pm = \pm 1; \qquad q\dim V = q+q^{-1} \quad \text{with} \quad q\dim\mathcal{O} := \sum_m q^m \dim\mathcal{O}_m;$$

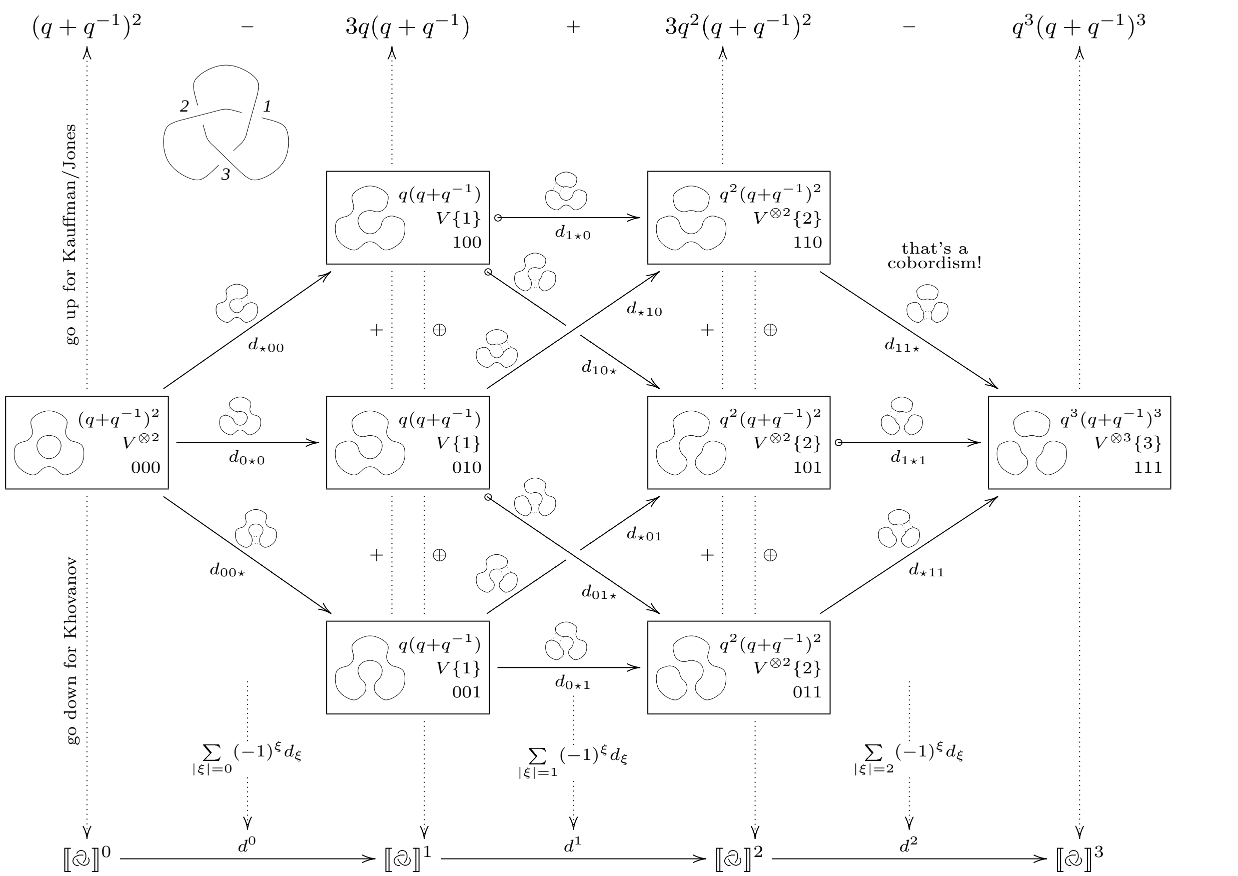
$$\mathcal{O}\{l\}_m := \mathcal{O}_{m-l} \quad \text{so} \quad q\dim\mathcal{O}\{l\} = q^l\, q\dim\mathcal{O}; \qquad \cdot[s]: \quad \text{height shift by } s;$$

 $\longrightarrow (V\otimes V \xrightarrow{m} V)$ $\qquad m: \begin{cases} v_+\otimes v_- \mapsto v_- & v_+\otimes v_+ \mapsto v_+ \\ v_-\otimes v_+ \mapsto v_- & v_-\otimes v_- \mapsto 0 \end{cases}$

 $\longrightarrow (V \xrightarrow{\Delta} V\otimes V)$ $\qquad \Delta: \begin{cases} v_+ \mapsto v_+\otimes v_- + v_-\otimes v_+ \\ v_- \mapsto v_-\otimes v_- \end{cases}$

That's a Frobenius Algebra! And a (1+1)-dimensional TQFT!

Example: $\quad q^{-2}+1+q^2-q^6 \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot(-1)^{n_-}q^{n_+-2n_-}} q+q^3+q^5-q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j = \star$) $\qquad = \quad [\![\,\cdot\,]\!] \xrightarrow[\text{(with } (n_+,n_-)=(3,0))]{\cdot[-n_-]\{n_+-2n_-\}} \mathcal{C}(\,\cdot\,).$

Theorem 1. The graded Euler characteristic of $\mathcal{C}(L)$ is $\hat{J}(L)$.
Theorem 2. The homology $\mathcal{H}(L)$ is a link invariant and thus so is $Kh_{\mathbb{F}}(L) := \sum_r t^r\, q\dim\mathcal{H}^r_{\mathbb{F}}(\mathcal{C}(L))$ over any field $\mathbb{F}$.
Theorem 3. $\mathcal{H}(\mathcal{C}(L))$ is strictly stronger than $\hat{J}(L)$: $\mathcal{H}(\mathcal{C}(\bar{5}_1)) \neq \mathcal{H}(\mathcal{C}(10_{132}))$ whereas $\hat{J}(\bar{5}_1) = \hat{J}(10_{132})$.
Conjecture 1. $Kh_{\mathbb{Q}}(L) = q^{s-1}\left(1+q^2+(1+tq^4)Kh'\right)$ and $Kh_{\mathbb{F}_2}(L) = q^{s-1}(1+q^2)\left(1+(1+tq^2)Kh'\right)$ for even $s = s(L)$ and non-negative-coefficients laurent polynomial $Kh' = Kh'(L)$.
Conjecture 2. For alternating knots s is the signature and Kh' depends only on tq^2.

References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and DBN's
http://www.ma.huji.ac.il/~drorbn/papers/Categorification/.

More at http://www.math.toronto.edu/~drorbn/Talks/UWO-040213/