From Stonehenge to Witten Skipping all the Details

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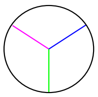
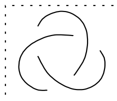
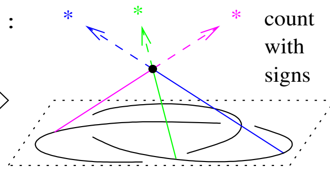
It is well known that when the Sun rises on midsummer's morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

$$\langle D,K\rangle_{\text{⩨}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}:$$

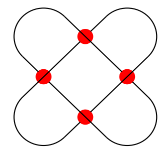
$D=$  $K=$  $\Longrightarrow$ 

count with signs

The Gaussian linking number

$$lk(\text{◯◯}) = \frac{1}{2}\sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs})$$

Carl Friedrich Gauss



$lk=2$

Thus we consider the generating function of all stellar coincidences:

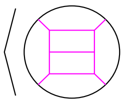
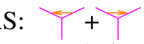
Dylan Thurston

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D,K\rangle_{\text{⩨}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

N := # of stars

c := # of chopsticks

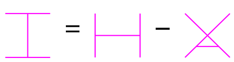
e := # of edges of D

$\mathcal{A}(\circlearrowleft)$:=Span $\left\langle \right.$  $\left.\right\rangle$ / oriented vertices, AS:  =0, & more relations

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,



(see below)

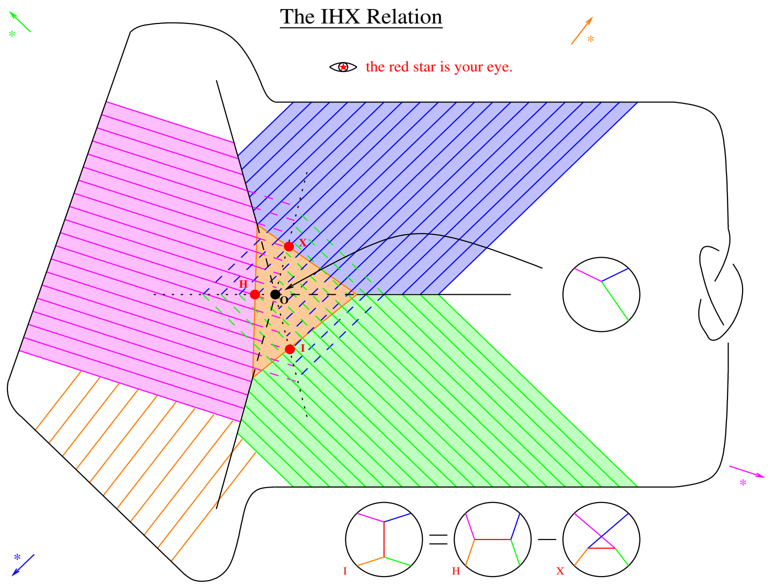
An intersection line cuts through the knot – Solution: Impose STU,

(similar argument)

The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.

(not shown here)

The IHX Relation

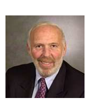


It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to \sum_{D:\text{ Feynman diagram}} W_{\mathfrak{g}}(D) \sum\!\!\!\!\!\!/\ \mathcal{E}(D) \to \sum_{D:\text{ Feynman diagram}} D \sum\!\!\!\!\!\!/\ \mathcal{E}(D)$$

Shiing–shen Chern



James H Simons

V: Vector space

dV: Lebesgue's measure on V.

Q: A quadratic form on V; $Q(V)=\langle LV,V\rangle$ where $L:V\to V^*$ is linear

Compute $I=\int_V dv\, e^{\frac{1}{2}Q+P}$

$$\text{"="} \sum_{m=0}^{\infty} \frac{1}{m!}\int_V dv\, P^m e^{Q/2}$$

$$\sim \sum_{m=0}^{\infty} \frac{1}{m!} P^m(\partial_\varphi) e^{-\frac{1}{2}Q^{-1}(\varphi)}\Big|_{\varphi=0}$$

$\pi=2=i=1$

$$= \sum_{m,n=0}^{\infty} \frac{(-1)^n}{2^n m!\, n!} P^m(\partial)(Q^{-1})^n\Big|_{\varphi=0}$$

The Fourier Transform:

$(F:V\to\mathbb{C}) \Longrightarrow (\tilde{F}:V^*\to\mathbb{C})$

via $\tilde{F}(\varphi)=\int_V F(v)e^{-i\langle\varphi,v\rangle}dv$.

Simple Facts:

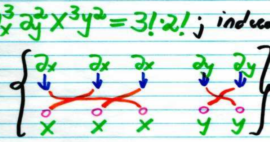
1. $\tilde{F}(0)=\int_V F(v)dv$.
2. $\frac{\partial}{\partial\varphi_i}\tilde{F} \sim \widetilde{v^i F}$.
3. $\widetilde{(e^{Q/2})} \sim e^{-Q^{-1}/2}$

where $Q^{-1}(\varphi) := \langle\varphi, L^{-1}\varphi\rangle$

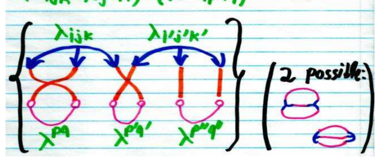
(that's the heart of the Fourier Inversion Formula).

Differentiation and Pairings:

$\partial_x^3\partial_y^2 x^3y^2 = 3!\cdot 2!$; indeed,

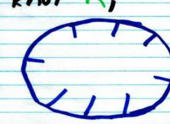


$(\lambda_{ijk}\partial_i\partial_j\partial_k)^2(\lambda^{pq}\varphi_p\varphi_q)^3$ is



In our case,

* Q is d, so Q^{-1} is an integral operator.
* P is $\frac{2}{3}A\wedge A\wedge A$
* H is the holonomy, itself a sum of integrals along the knot K,

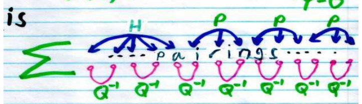


& when the dust settles, we get $Z(K)$!

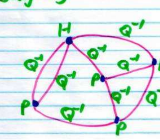
So $\int_V H(v)e^{\frac{1}{2}Q+P}dv$

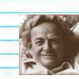
$$\sim H(\partial)e^{P(\partial)}e^{-Q^{-1}(\varphi)/2}\Big|_{\varphi=0}$$

is



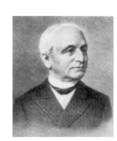
$$= \sum_{\text{Diagrams}} c(D)\begin{pmatrix}\text{Products of}\\ Q^{-1}\text{'s, } P\text{'s}\\ \text{and one } H\end{pmatrix}$$





Richard Feynman

"God created the knots, all else in topology is the work of man."



Leopold Kronecker

(modified)

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

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