

Motivation Homology measures our failure to construct all solutions of a given equation:



$$\mathbb{R}_\theta \xrightarrow{d^1} \mathbb{R}_{x,y}^2 \xrightarrow{d^2} \mathbb{R}_r^1$$

$$\theta \mapsto \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad (x,y) \mapsto \sqrt{x^2+y^2}$$

$$d^2 \circ d^1 = \cos \sin \theta$$

$$V \xrightarrow{d_1} W \xrightarrow{d_2} Z$$

$$\text{im } d_1 \subset \ker d_2 \Leftrightarrow d_2 \circ d_1 = 0$$

$$H(W) := \ker d_2 / \text{im } d_1$$

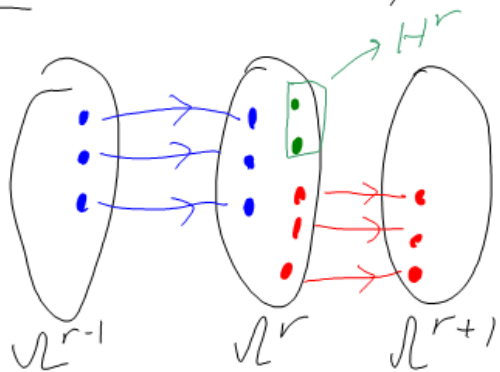
Euler Characteristic

Theorem If everything is finite, then

$$\sum (-1)^r \dim \mathcal{N}^r = \sum (-1)^r \dim H^r$$

$$=: \chi(\mathcal{N})$$

Proof (more or less)



Definition A "complex" is a long chain of "parametrization problems":

$$\mathcal{N} = (\dots \rightarrow \mathcal{N}^{r-1} \xrightarrow{d^{r-1}} \mathcal{N}^r \xrightarrow{d^r} \mathcal{N}^{r+1} \rightarrow \dots)$$

s.t. $d^2 = 0$ or $\text{im}(d) \subset \ker(d)$

Homology:

$$H^r(\mathcal{N}) := \ker d^r / \text{im } d^{r-1}$$

The "parametrization failure" at step r .

[I don't understand why "long" complexes are so common 😊]

Morphisms and Homotopy

Morphisms:

$$\begin{array}{ccccccc} \dots & \rightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \rightarrow & \dots \\ & & F^{r-1} \downarrow & & F^r \downarrow & & F^{r+1} \downarrow & & \\ \dots & \rightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \rightarrow & \dots \end{array}$$

Homotopies:

$$\begin{array}{ccccc} \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} \\ F^{r-1} \downarrow \parallel G^{r-1} & \swarrow h^r & F^r \downarrow \parallel G^r & \swarrow h^{r+1} & F^{r+1} \downarrow \parallel G^{r+1} \\ \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} \end{array}$$

$$F^r - G^r = h^{r+1} d^r + d^{r-1} h^r$$

If there are $\mathcal{N}_0 \xrightleftharpoons[h]{f} \mathcal{N}_1$,
s.t. $f \circ g \sim I_{\mathcal{N}_0}$ and $g \circ f \sim I_{\mathcal{N}_1}$,
then " \mathcal{N}_0 & \mathcal{N}_1 are homotopy equivalent" [and they have equal homology]