(u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra)

Dror Bar-Natan, Kansas State April 7 2009, http://www.math.toronto.edu/~drorbn/Talks/KSU-090407

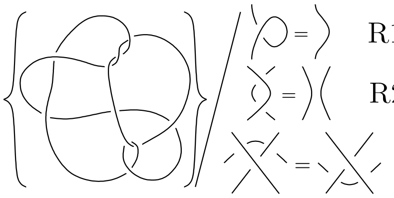
"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)

1 Witten, Chern–Simons — **u–knots** —1-1→ **v–knots** —onto→ **w–knots** — Cattaneo, BF Theory

topology

DBN, data by Brian Gilbert, http://www.math.utoronto.ca/~drorbn/Gallery/KnottedObjects/Candies/

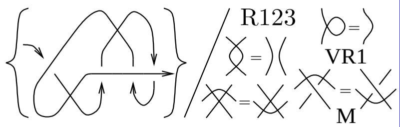
2 u–knots are usual knots:



$=\text{PA}\left\langle \;\middle|\; \text{R123}\right\rangle_{0\text{ legs}}$ Reidemeister

"Knots in $\mathbb{R}^3$"

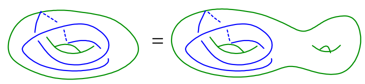
3 v–knots are virtual knots:



$=\text{PA}\left\langle \;\middle|\; \text{R123, VR123, M}\right\rangle_0$

$=\text{CA}\left\langle \;\middle|\; \text{R123}\right\rangle_0$ Kauffman

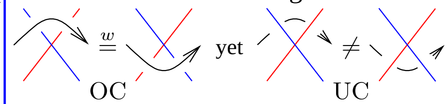
= Knots on surfaces, modulo stabilazation:

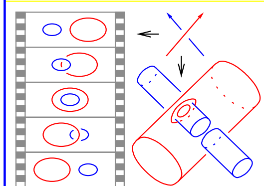


w is for welded, weakly v, and warmup:

4 {w–knots}={v–knots}/(OC)

where OC is Overcrossings Commute:





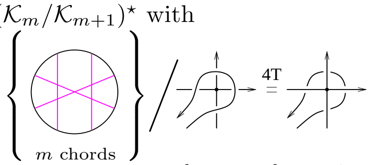
Related to "movies of flying rings" to knotted tubes in 4–space, and to "basis conjugating automorphisms of free groups".

McCool, Goldsmith, Fenn, Rimanyi, Rourke, Satoh, Brendle, Hatcher

combinatorics

Extend any V : {u-knots} $\to A$ to "singular u-knots" using $V(\times) := V(\times) - V(\times)$, and think "differentiation". Declare "V is of type m" iff $V^{(m+1)} \equiv 0$, think "polynomial of degree m". $W = V^{(m)}$ roughly determines V; $W \in \mathcal{A}^\star_m = (\mathcal{K}_m/\mathcal{K}_{m+1})^\star$ with

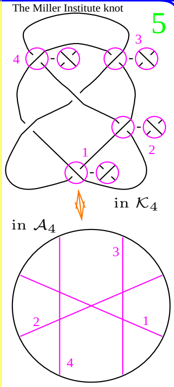
$\mathcal{A}_m := \left\{ \;\right\} \Big/$ (4T), m chords



Need an expansion Z : {u-knots} $\to \mathcal{A} = \bigoplus \mathcal{A}_m$.

Vassiliev, Goussarov

5 The Miller Institute knot



6 All the same, except

$V(\times) := V(\times) - V(\times)$

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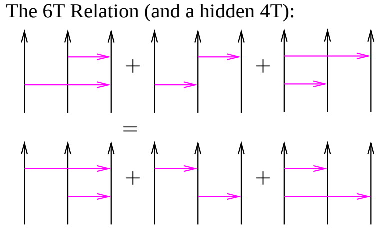
$\mathcal{A}^v := \{\text{"arrow diagrams"}\}/6T$

Need a Z : {v-knots} $\to \mathcal{A}^v$. — Polyak

The 6T Relation (and a hidden 4T):

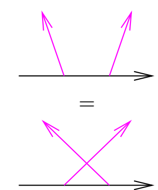


7 All the same, except

$\mathcal{A}^w := \mathcal{A}^v/TC$

Need a Z : {w-knots} $\to \mathcal{A}^w$.

"Tails Commute (TC)":



low algebra

10 Similar with metrized Lie algebras replacing arbitrary Lie algebras

Penrose, Cvitanovic, Vogel

So Far: Invariants for any $(\mathfrak{g}, R)$:

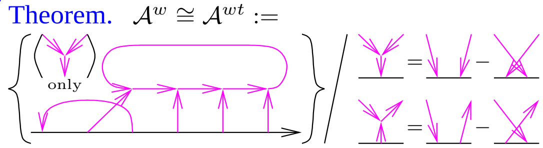
$\{\text{knots}\} \xrightarrow[\text{high}]{Z} \mathcal{A} \xrightarrow[\text{low}]{T_\mathfrak{g}} \mathcal{U}(\mathfrak{g}) \xrightarrow{\text{Tr}_R} \mathbb{C}$

9 Similar with Lie bi–algebras replacing arbitrary Lie algebras

Haviv, Leung

m	1	2	3	4
$\dim \mathcal{A}^v_m$	2	7	27	139
$\dim \mathcal{L}ie_m$	2	7	27	≥ 122

8 **Theorem.** $\mathcal{A}^w \cong \mathcal{A}^{wt} :=$

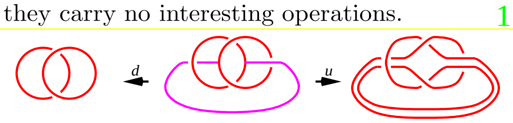


This screams, if you speak the language, LIE ALGEBRAS. And indeed we have

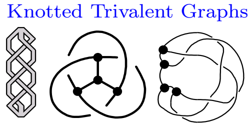
Theorem. Given a finite dimensional Lie algebra $\mathfrak{g}$, there is $T : \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g}) := \mathcal{U}(\mathfrak{g} \ltimes \mathfrak{g}^\star_{ab})$.

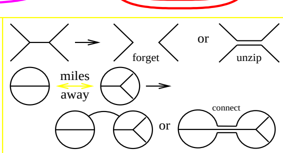
high algebra

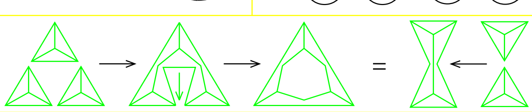
11 Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations.



Knotted Trivalent Graphs







Theorem (∼). A homomorphic Z is the same as a "Drinfel'd Associator". — Drinfel'd

13 Z is a Quantum Group?

More precisely, a homomorphic Z ought to be equivalent to the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras.

Etingof, Kazhdan

Dror's Dream: Straighten and fatten this column.

An Idle Question.
Is there physics in this column?

12 Switch to w-knotted trivalent tangles,

$$\text{wKTT} := CA\langle \times, \times, Y \rangle.$$

Theorem (∼). A homomorphic Z is equivalent to proving the Kashiwara-Vergne statement.

Statement (∼, KV, 1978) (proven Alekseev-Meinrenken, 2006). Convolutions of invariant functions on a group match with convolutions of invariant functions on its Lie algebra: for any finite dim. Lie group G with Lie algebra $\mathfrak{g}$,

$$(\text{Fun}(G)^{\text{Ad}\,G}, \star) \cong (\text{Fun}(\mathfrak{g})^{\text{Ad}\,G}, \star).$$

(Closely related to the "orbit method" of representation theory).

Alekseev, Torossian

More at http://www.math.toronto.edu/~drorbn/Talks/KSU-090407/