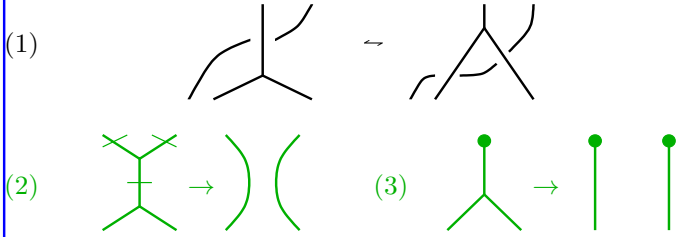
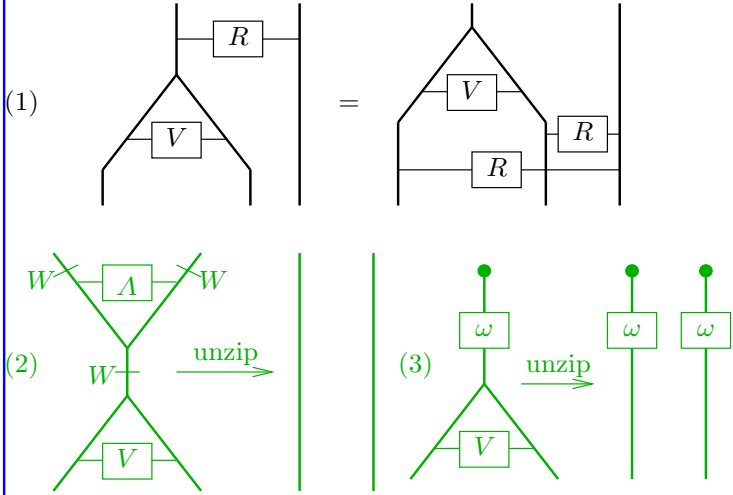


Convolutions on Lie Groups and Lie Algebras and Ribbon 2-Knots, Page 2

Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R = \exp \uparrow \in \mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega \in \mathcal{A}^w(\uparrow)$ and $V \in \mathcal{A}^w(\uparrow\uparrow)$ so that



Algebraic statement. With $I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}$, with $c : \hat{U}(I\mathfrak{g}) \rightarrow \hat{U}(I\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$ the obvious projection, with S the antipode of $\hat{U}(I\mathfrak{g})$, with W the automorphism of $\hat{U}(I\mathfrak{g})$ induced by flipping the sign of \mathfrak{g}^* , with $r \in \mathfrak{g}^* \otimes \mathfrak{g}$ the identity element and with $R = e^r \in \hat{U}(I\mathfrak{g}) \otimes \hat{U}(\mathfrak{g})$ there exist $\omega \in \hat{S}(\mathfrak{g}^*)$ and $V \in \hat{U}(I\mathfrak{g})^{\otimes 2}$ so that

(1) $V(\Delta \otimes 1)(R) = R^{13}R^{23}V$ in $\hat{U}(I\mathfrak{g})^{\otimes 2} \otimes \hat{U}(\mathfrak{g})$
 (2) $V \cdot SWV = 1$ (3) $(c \otimes c)(V\Delta(\omega)) = \omega \otimes \omega$

Unitary statement. There exists $\omega \in \text{Fun}(\mathfrak{g})^G$ and an (infinite order) tangential differential operator V defined on $\text{Fun}(\mathfrak{g}_x \times \mathfrak{g}_y)$ so that

(1) $V\widehat{e^{x+y}} = \widehat{e^x e^y} V$ (allowing $\hat{U}(\mathfrak{g})$ -valued functions)
 (2) $VV^* = I$ (3) $V\omega_{x+y} = \omega_x \omega_y$

Group-Algebra statement. There exists $\omega^2 \in \text{Fun}(\mathfrak{g})^G$ so that for every $\phi, \psi \in \text{Fun}(\mathfrak{g})^G$ (with small support), the following holds in $\hat{U}(\mathfrak{g})$:

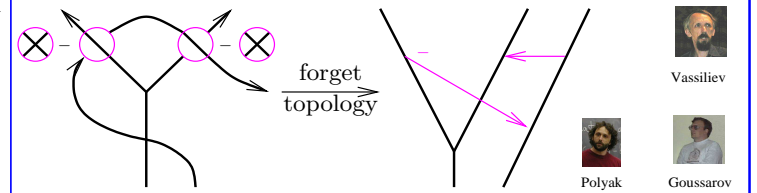
$$\iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_{x+y}^2 e^{x+y} = \iint_{\mathfrak{g} \times \mathfrak{g}} \phi(x)\psi(y)\omega_x^2 \omega_y^2 e^x e^y. \quad (\text{shhh, } \omega^2 = j^{1/2})$$

(shhh, this is Duflo)

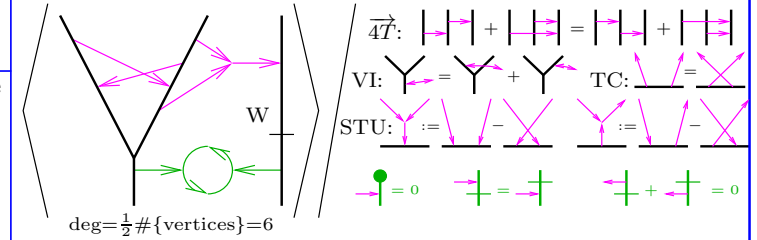
Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let G be a finite dimensional Lie group and let \mathfrak{g} be its Lie algebra, let $j : \mathfrak{g} \rightarrow \mathbb{R}$ be the Jacobian of the exponential map $\exp : \mathfrak{g} \rightarrow G$, and let $\Phi : \text{Fun}(G) \rightarrow \text{Fun}(\mathfrak{g})$ be given by $\Phi(f)(x) := j^{1/2}(x)f(\exp x)$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then

$$\Phi(f) \star \Phi(g) = \Phi(f \star g).$$

From wTT to \mathcal{A}^w . $\text{gr}_m \text{wTT} := \{m\text{-cubes}\} / \{(m+1)\text{-cubes}\}$:



w-Jacobi diagrams and \mathcal{A} . $\mathcal{A}^w(Y \uparrow) \cong \mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



Diagrammatic to Algebraic. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via

$$\begin{matrix} b_{ji}^k & k & b_{kl}^m \\ \begin{matrix} i & j & n & m & l \\ \varphi^i & \varphi^j & x_n & x_m & \varphi^n & \varphi^l \end{matrix} \end{matrix} \xrightarrow{\dim \mathfrak{g}} \sum_{i,j,k,l,m,n=1} b_{ij}^k b_{kl}^m \varphi^i \varphi^j x_n x_m \varphi^l \in \mathcal{U}(I\mathfrak{g})$$

Unitary \iff Algebraic. The key is to interpret $\hat{U}(I\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad } x$: $(x\varphi)(y) := \varphi([x, y])$.
- $c : \hat{U}(I\mathfrak{g}) \rightarrow \hat{U}(I\mathfrak{g})/\hat{U}(\mathfrak{g}) = \hat{S}(\mathfrak{g}^*)$ is "the constant term".

Unitary \implies Group-Algebra. $\iint \omega_{x+y}^2 e^{x+y} \phi(x)\psi(y)$
 $= \langle \omega_{x+y}, \omega_{x+y} e^{x+y} \phi(x)\psi(y) \rangle = \langle V\omega_{x+y}, V e^{x+y} \phi(x)\psi(y)\omega_{x+y} \rangle$
 $= \langle \omega_x \omega_y, e^x e^y V \phi(x)\psi(y)\omega_{x+y} \rangle = \langle \omega_x \omega_y, e^x e^y \phi(x)\psi(y)\omega_x \omega_y \rangle$
 $= \iint \omega_x^2 \omega_y^2 e^x e^y \phi(x)\psi(y).$

Convolutions and Group Algebras (ignoring all Jacobians). If G is finite, A is an algebra, $\tau : G \rightarrow A$ is multiplicative then $(\text{Fun}(G), \star) \cong (A, \cdot)$ via $L : f \mapsto \sum f(a)\tau(a)$. For Lie (G, \mathfrak{g}) ,

$$\begin{matrix} (\mathfrak{g}, +) \ni x \xrightarrow{\tau_0 = \exp_S} e^x \in \hat{S}(\mathfrak{g}) & \text{Fun}(\mathfrak{g}) \xrightarrow{L_0} \hat{S}(\mathfrak{g}) \\ \downarrow \exp_G & \searrow \exp_U & \downarrow \chi & \text{so} & \downarrow \Phi^{-1} & \downarrow \chi \\ (G, \cdot) \ni e^x \xrightarrow{\tau_1} e^x \in \hat{U}(\mathfrak{g}) & & \text{Fun}(G) \xrightarrow{L_1} \hat{U}(\mathfrak{g}) \end{matrix}$$

with $L_0\psi = \int \psi(x)e^x dx \in \hat{S}(\mathfrak{g})$ and $L_1\Phi^{-1}\psi = \int \psi(x)e^x \in \hat{U}(\mathfrak{g})$. Given $\psi_i \in \text{Fun}(\mathfrak{g})$ compare $\Phi^{-1}(\psi_1) \star \Phi^{-1}(\psi_2)$ and $\Phi^{-1}(\psi_1 \star \psi_2)$ in $\hat{U}(\mathfrak{g})$: (shhh, $L_{0/1}$ are "Laplace transforms")

$$\star \text{ in } G : \iint \psi_1(x)\psi_2(y)e^x e^y \quad \star \text{ in } \mathfrak{g} : \iint \psi_1(x)\psi_2(y)e^{x+y}$$

- We skipped...**
- The Alexander polynomial and Milnor numbers.
 - v-Knots, quantum groups and Etingof-Kazhdan.
 - u-Knots, Alekseev-Torossian, BF theory and the successful and Drinfel'd associators.
 - The simplest problem hyperbolic geometry solves.