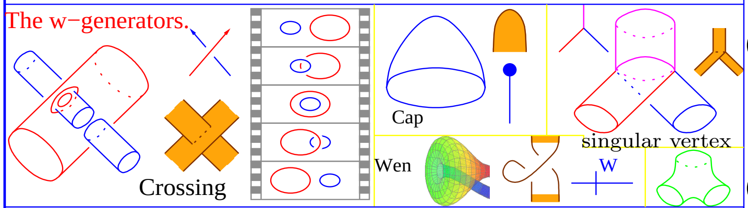
I understand Drinfel'd and Alekseev−Torossian, I don't understand Etingof−Kazhdan yet, and I'm clueless about Kontsevich

Dror Bar−Natan, Montpellier, June 2010, http://www.math.toronto.edu/~drorbn/Talks/Montpellier−1006/

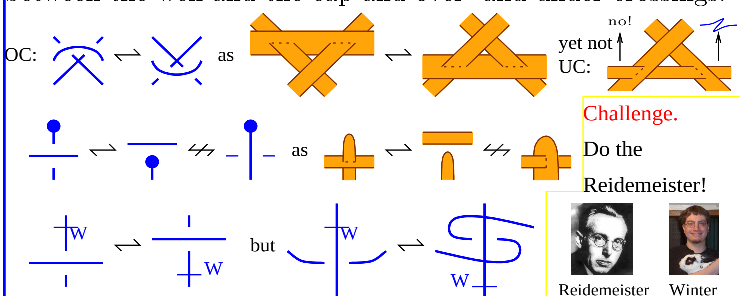
Trivalent w-Tangles.

$$\mathrm{wTT} = \mathrm{CA}\left\langle \begin{array}{c}\text{w-}\\ \text{generators}\end{array} \middle| \begin{array}{c}\text{w-}\\ \text{relations}\end{array} \middle| \begin{array}{c}\text{unary w-}\\ \text{operations}\end{array}\right\rangle$$

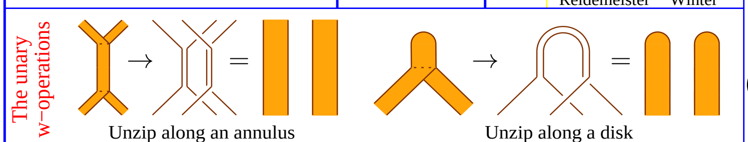
The w−generators.



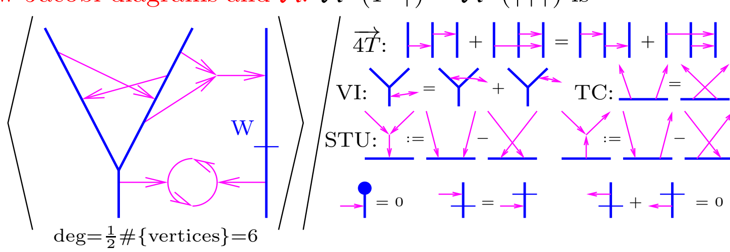
The w-relations include R234, VR1234, D, Overcrossings Commute (OC) but not UC, $W^2=1$, and funny interactions between the wen and the cap and over- and under-crossings:



The unary w−operations



w-Jacobi diagrams and $\mathcal{A}$. $\mathcal{A}^w(Y\uparrow)\cong\mathcal{A}^w(\uparrow\uparrow\uparrow)$ is



An Associator:

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

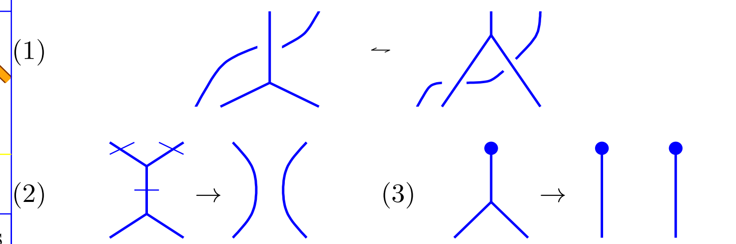
satisfying the "pentagon",

$$\begin{tikzcd} ((AB)C)D \arrow[rr, "(\Delta 11)\Phi"] \arrow[d, "\Phi 1"'] & & (AB)(CD) \arrow[d, "(11\Delta)\Phi"] \\ (A(BC))D \arrow[dr, "(1\Delta 1)\Phi"'] & & A(B(CD)) \\ & A((BC)D) \arrow[ur, "1\Phi"'] & \end{tikzcd}$$

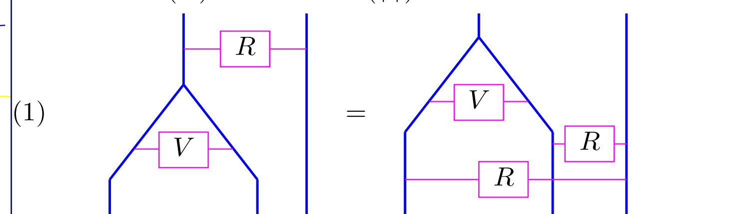
$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$

The hexagon? Never heard of it.

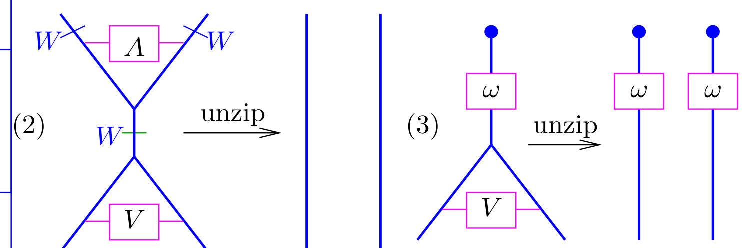
Knot-Theoretic statement. There exists a homomorphic expansion Z for trivalent w-tangles. In particular, Z should respect $R4$ and intertwine annulus and disk unzips:



Diagrammatic statement. Let $R=\exp \mapsto\ \in\mathcal{A}^w(\uparrow\uparrow)$. There exist $\omega\in\mathcal{A}^w(\uparrow)$ and $V\in\mathcal{A}^w(\uparrow\uparrow)$ so that



(1) $V\cdot(\Delta\otimes 1)(R) = R^{13}R^{23}V$ in $\mathcal{A}^w(\uparrow\uparrow\uparrow)$



(2) $VV^*=I$ in $\mathcal{A}^w(\uparrow\uparrow)$ (3) $V\cdot\Delta(\omega)=\omega\otimes\omega$ in $\mathcal{A}^w(\uparrow\uparrow)$

Alekseev-Torossian statement. There are elements $F\in\mathrm{TAut}_2$ and $a\in\mathfrak{tr}_1$ such that

$$F(x+y)=\log e^xe^y \quad\text{and}\quad jF=a(x)+a(y)-a(\log e^xe^y).$$

Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.

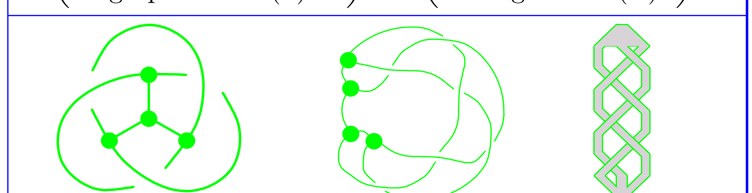
Proof. Write $V=e^ce^{uD}$ with $c\in\mathfrak{tr}_2$, $D\in\mathfrak{tder}_2$, and $\omega=e^b$ with $b\in\mathfrak{tr}_1$. Then (1) $\Leftrightarrow e^{uD}(x+y)e^{-uD}=\log e^xe^y$, (2) $\Leftrightarrow I=e^ce^{uD}(e^{uD})^*e^c=e^{2c}e^{jD}$, and (3) $\Leftrightarrow e^ce^{uD}e^{b(x+y)}=e^{b(x)+b(y)} \Leftrightarrow e^ce^{b(\log e^xe^y)}=e^{b(x)+b(y)}$ $\Leftrightarrow c=b(x)+b(y)-b(\log e^xe^y)$.

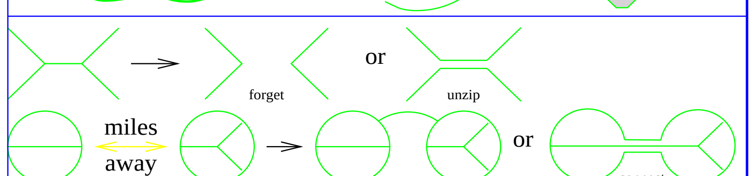
The Alekseev-Torossian Correspondence.

$$\{\text{Drinfel'd Associators}\}\leftrightarrows\{\text{Solutions of KV}\}.$$

We need an even bigger algebraic structure!

$$\left(\begin{array}{c}\text{green knotted trivalent}\\ \text{graphs in } \mathbb{R}^3\ (u)\end{array}\right)\xrightarrow{\alpha_e}\left(\begin{array}{c}\text{blue tubes and red}\\ \text{strings in } \mathbb{R}^4\ (\overline{w})\end{array}\right)$$





Video and more at http://www.math.toronto.edu/~drorbn/Talks/Montpellier-1006/