



From Stonehenge to Witten – Some Further Details

Oporto Meeting on Geometry, Topology and Physics, July 2004

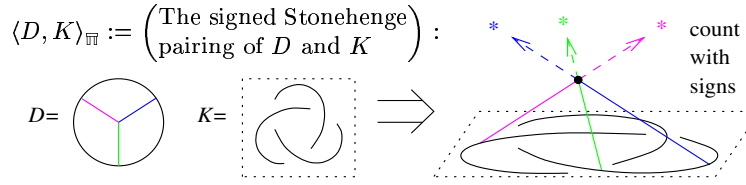
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Witten

We the generating function of all stellar coincidences:

$$Z(K) := \lim_{N \rightarrow \infty} \sum_{\substack{D \\ 3\text{-valent}}} \frac{1}{2^c c! \binom{N}{e}} \langle D, K \rangle_{\mathbb{R}} D \cdot \left(\begin{array}{l} \text{framing-} \\ \text{dependent} \\ \text{counter-term} \end{array} \right) \in \mathcal{A}(\odot)$$



Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

Dylan Thurston



$N := \#$ of stars
 $c := \#$ of chopsticks
 $e := \#$ of edges of D

$\mathcal{A}(\odot) := \text{Span} \left\langle \left(\begin{array}{c} \text{square} \\ \text{with} \\ \text{diagonals} \end{array} \right) \right\rangle / \text{oriented vertices AS: } \begin{array}{c} \text{Y} + \text{Y} = 0 \\ \text{and more relations} \end{array}$

When deforming, catastrophes occur when:

A plane moves over an intersection point –
 Solution: Impose IHX,

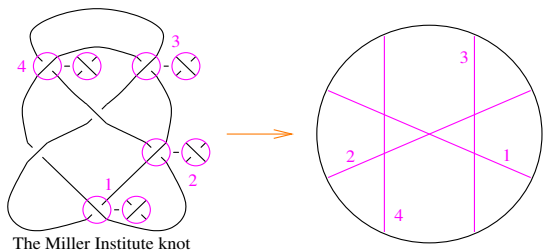
$$\text{I} = \text{H} - \text{X}$$

An intersection line cuts through the knot –
 Solution: Impose STU,

$$\text{Y} = \text{U} - \text{X}$$

The Gauss curve slides over a star –
 Solution: Multiply by a framing-dependent counter-term.

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A \text{hol}_K(A) \exp \left[\frac{ik}{4\pi} \int_{\mathbb{R}^3} \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right] \rightarrow \sum_{D: \text{Feynman diagram}} W_{\mathfrak{g}}(D) \int \mathcal{E}(D) \rightarrow \sum_{D: \text{Feynman diagram}} D \int \mathcal{E}(D)$$



Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

Conjecture. (Taylor's theorem) Finite type invariants separate knots.

Theorem. $Z(K)$ is a universal finite type invariant! (sketch: to dance in many parties, you need many feet).

Goussarov



Vassiliev



Related to Lie algebras

$$\begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ x \quad y \end{array} = \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \text{U} \\ \diagup \quad \diagdown \\ x \quad y \end{array} - \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ \text{X} \\ \diagup \quad \diagdown \\ x \quad y \end{array}$$

$$[x, y] = xy - yx$$

$$\begin{array}{c} x \quad y \quad z \\ \diagdown \quad \diagup \quad \diagdown \\ \text{I} \\ \diagup \quad \diagdown \quad \diagup \\ x \quad y \quad z \end{array} = \begin{array}{c} x \quad y \quad z \\ \diagdown \quad \diagup \quad \diagdown \\ \text{H} \\ \diagup \quad \diagdown \quad \diagup \\ x \quad y \quad z \end{array} - \begin{array}{c} x \quad y \quad z \\ \diagdown \quad \diagup \quad \diagdown \\ \text{X} \\ \diagup \quad \diagdown \quad \diagup \\ x \quad y \quad z \end{array}$$

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$



Sophus Lie

More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a, b], c \rangle \quad X_a v_\beta = \sum_{\gamma} r_{a\gamma}^{\beta} v_{\gamma}$$

and then

$$W_{\mathfrak{g}, R} : \begin{array}{c} \gamma \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ \alpha \end{array} \begin{array}{c} a \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ \beta \end{array} \begin{array}{c} b \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ \alpha \end{array} \begin{array}{c} c \\ \diagdown \quad \diagup \\ \text{Y} \\ \diagup \quad \diagdown \\ \alpha \end{array} \rightarrow \sum_{abc\alpha\beta\gamma} f_{abc} r_{a\gamma}^{\beta} r_{b\alpha}^{\gamma} r_{c\beta}^{\alpha}$$

Planar algebra and the Yang-Baxter equation

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ c \quad d \end{array} \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ h \quad i \quad j \\ \diagup \quad \diagdown \quad \diagup \\ d \quad e \quad f \end{array} = \begin{array}{c} a \quad b \quad c \\ \diagdown \quad \diagup \quad \diagdown \\ h \quad i \quad j \\ \diagup \quad \diagdown \quad \diagup \\ d \quad e \quad f \end{array}$$

$$R_{cd}^{ab} R_{ef}^{ih} R_{ij}^{fg} = R_{di}^{ah} R_{hj}^{bc} R_{ef}^{ij}$$



Yang



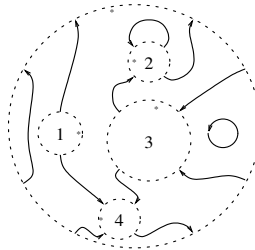
Baxter

$W_{\mathfrak{g}, R} \circ Z$ is often interesting:

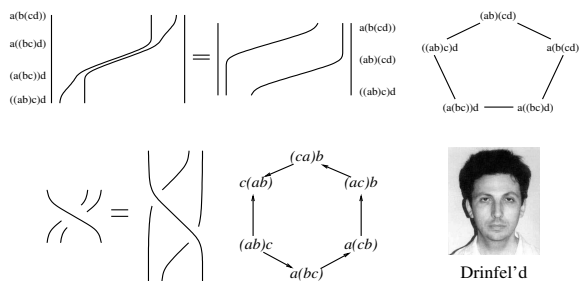
$\mathfrak{g} = \mathfrak{sl}(2)$ \rightarrow The Jones polynomial

$\mathfrak{g} = \mathfrak{sl}(N)$ \rightarrow The HOMFLYPT polynomial

$\mathfrak{g} = \mathfrak{so}(N)$ \rightarrow The Kauffman polynomial



Parenthesized tangles, the pentagon and hexagon



Kauffman's bracket and the Jones polynomial

claim $\hat{J}(\mathcal{D}) = \hat{J}(\mathcal{D}')$

$\langle X \rangle = \langle Y \rangle - q \langle Z \rangle$ (0-smoothing vs 1-smoothing)

$\langle O^k \rangle = (q + q^{-1})^k$

$\hat{J}(L) = (-1)^n q^{n+2n} \langle L \rangle$

(n_+, n_-) count (\nearrow, \searrow)

$\langle \mathcal{D}' \rangle = \langle \mathcal{D} \rangle - q \langle \mathcal{D}'' \rangle$

$-q \langle \mathcal{D}'' \rangle + q^2 \langle \mathcal{D}''' \rangle = -q \langle \mathcal{D} \rangle$



Reshetikhin



Turaev

"God created the knots, all else in topology is the work of man."

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