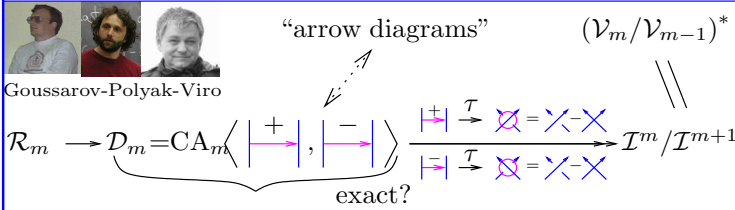
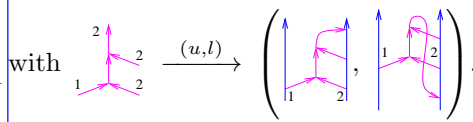


1. $\text{proj } \mathcal{K}^w(\uparrow_n) \cong \mathcal{U}((\mathfrak{a}_n \oplus \mathfrak{tder}_n) \ltimes \mathfrak{tr}_n)$, continued.



Wheels and Trees. With \mathcal{P} for Primitives,

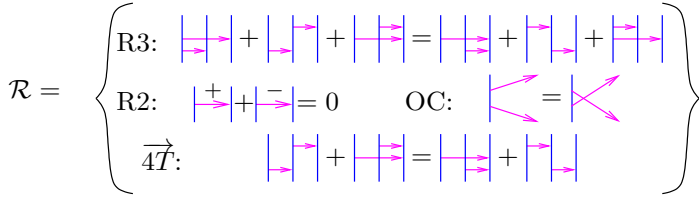
$$0 \rightarrow \langle \text{wheels} \rangle \xrightarrow{l} \mathcal{P}A^w(\uparrow_n) \xrightleftharpoons[\pi]{u} \langle \text{trees} \rangle \rightarrow 0,$$



trees atop a wheel, and a little prince.

So $\text{proj } \mathcal{K}^w(\uparrow_n) \cong \mathcal{U}(\langle \text{trees} \rangle \ltimes \langle \text{wheels} \rangle)$.

Imperfect Thumb-Rule. Take R3 (say), substitute $\times \rightarrow \times +$, keep the lowest degree terms that don't immediately die:

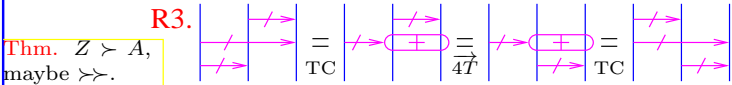
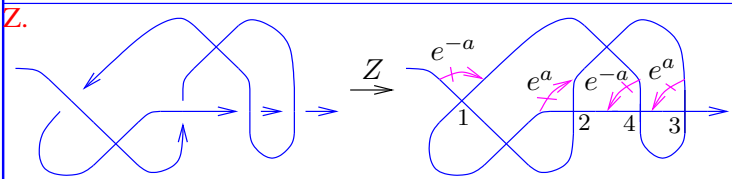


Some A-T Notions. \mathfrak{a}_n is the vector space with basis x_1, \dots, x_n , $\mathfrak{lie}_n = \mathfrak{lie}(\mathfrak{a}_n)$ is the free Lie algebra, $\text{Ass}_n = \mathcal{U}(\mathfrak{lie}_n)$ is the free associative algebra “of words”, $\text{tr} : \text{Ass}_n^+ \rightarrow \mathfrak{tr}_n = \text{Ass}_n^+ / (x_{i_1}x_{i_2} \cdots x_{i_m} = x_{i_2} \cdots x_{i_m}x_{i_1})$ is the “trace” into “cyclic words”, $\mathfrak{der}_n = \mathfrak{der}(\mathfrak{lie}_n)$ are all the derivations, and

$$\mathfrak{tder}_n = \{D \in \mathfrak{der}_n : \forall i \exists a_i \text{ s.t. } D(x_i) = [x_i, a_i]\}$$

are “tangential derivations”, so $D \leftrightarrow (a_1, \dots, a_n)$ is a vector space isomorphism $\mathfrak{a}_n \oplus \mathfrak{tder}_n \cong \bigoplus_n \mathfrak{lie}_n$. Finally, $\text{div} : \mathfrak{tder}_n \rightarrow \mathfrak{tr}_n$ is $(a_1, \dots, a_n) \mapsto \sum_k \text{tr}(x_k(\partial_k a_k))$, where for $a \in \text{Ass}_n^+$, $\partial_k a \in \text{Ass}_n$ is determined by $a = \sum_k (\partial_k a)x_k$, and $j : \text{TAut}_n = \exp(\mathfrak{tder}_n) \rightarrow \mathfrak{tr}_n$ is $j(e^D) = \frac{e^D - 1}{D} \cdot \text{div } D$.

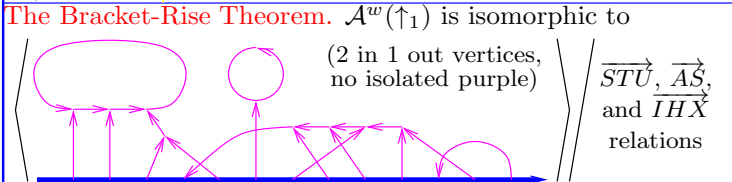
Theorem. Everything matches. $\langle \text{trees} \rangle$ is $\mathfrak{a}_n \oplus \mathfrak{tder}_n$ as Lie algebras, $\langle \text{wheels} \rangle$ is \mathfrak{tr}_n as $\langle \text{trees} \rangle / \mathfrak{tder}_n$ -modules, $\text{div } D = \iota^{-1}(u-l)(D)$, and $e^{uD}e^{-lD} = e^{jD}$.



Differential Operators. Interpret $\dot{\mathcal{U}}(\mathfrak{g})$ as tangential differential operators on $\text{Fun}(\mathfrak{g})$:

- $\varphi \in \mathfrak{g}^*$ becomes a multiplication operator.
- $x \in \mathfrak{g}$ becomes a tangential derivation, in the direction of the action of $\text{ad } x$: $(x\varphi)(y) := \varphi([x, y])$.

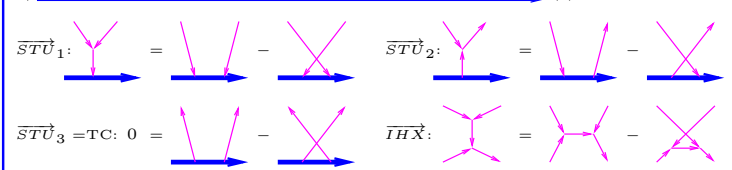
Trees become vector fields and $uD \mapsto lD$ is $D \mapsto D^*$. So $\text{div } D$ is $D - D^*$ and $jD = \log(e^D(e^D)^*) = \int_0^1 dt e^{tD} \text{div } D$.



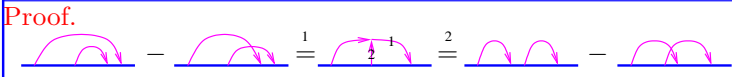
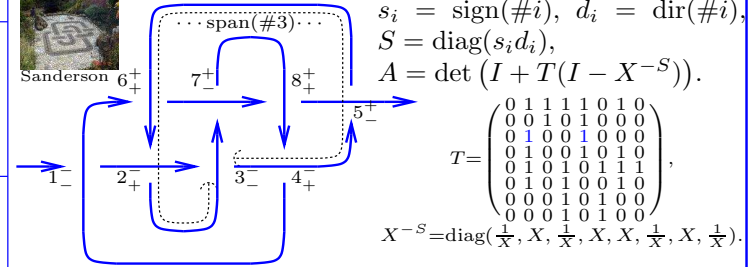
Special Derivations. Let $\mathfrak{sder}_n = \{D \in \mathfrak{tder}_n : D(\sum x_i) = 0\}$.

Theorem. $\mathfrak{sder}_n = \pi\alpha(\text{proj u-tangles})$, where α is the obvious map $\text{proj u-tangles} \rightarrow \text{proj w-tangles}$.

Proof. After decoding, this becomes Lemma 6.1 of Drinfel'd's amazing $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ paper.

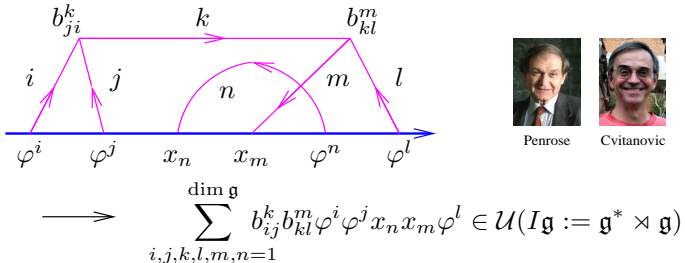


The Alexander Theorem.



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

To Lie Algebras. With (x_i) and (φ^j) dual bases of \mathfrak{g} and \mathfrak{g}^* and with $[x_i, x_j] = \sum b_{ij}^k x_k$, we have $\mathcal{A}^w \rightarrow \mathcal{U}$ via



Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k = (\text{the } k\text{-wheel})$,

$$Z = N \exp_{\mathcal{A}^w} \left(-w \left(\log_{\mathbb{Q}[\![x]\!]} A(e^x) \right) \right) \pmod{w_k w_l = w_{k+l}, Z = N \cdot A^{-1}(e^x)}$$

This is the **ultimate Alexander invariant!** computable in polynomial time, local, composes well, behaves under cabling. Seems to significantly generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers. But it's ugly, and much work remains.

Theorem (PBW, “ $\mathcal{U}(I\mathfrak{g})^{\otimes n} \cong \mathcal{S}(I\mathfrak{g})^{\otimes n}$ ”). As vector spaces, $\mathcal{A}^w(\uparrow_n) \cong \mathcal{B}_n$, where

