

Facts and Dreams About v-Knots and Etingof-Kazhdan, 1

Dror Bar-Natan at Swiss Knots 2011

<http://www.math.toronto.edu/~drorbn/Talks/SwissKnots-1105/>
Foots & refs on PDF version, page 3.

Abstract. I will describe, to the best of my understanding, the relationship between virtual knots and the Etingof-Kazhdan [EK] quantization of Lie bialgebras, and explain why, IMHO, both topologists and algebraists should care. I am not happy yet about the state of my understanding of the subject but I haven't lost hope of achieving happiness, one day.

Abstract Generalities. (K, I) : an algebra and an "augmentation ideal" in it. $\hat{K} := \varprojlim K/I^m$ the " I -adic completion". $\text{gr}_I K := \widehat{\bigoplus} I^m/I^{m+1}$ has a product μ , especially, $\mu_{11}: (C = I/I^2)^{\otimes 2} \rightarrow I^2/I^3$. The "quadratic approximation" $\mathcal{A}_I(K) := \overline{FC}/\langle \ker \mu_{11} \rangle$ of K surjects using μ on $\text{gr } K$.

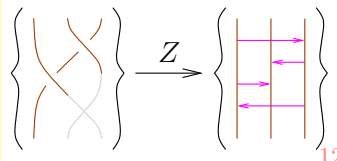


Peter Lee

The Prized Object. A "homomorphic \mathcal{A} -expansion": a homomorphic filtered $Z: K \rightarrow \mathcal{A}$ for which $\text{gr } Z: \text{gr } K \rightarrow \mathcal{A}$ inverts μ .

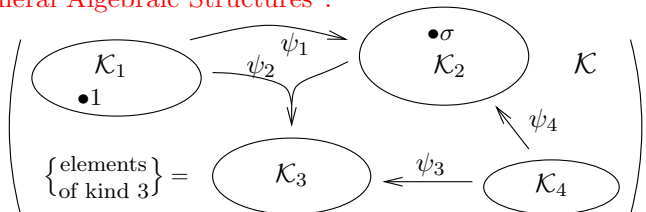
Dror's Dream. All interesting graded objects and equations, especially those around quantum groups, arise this way. 6

Example 2. For $K = \mathbb{Q}PvB_n =$ "braids when you look", [Lee] shows that a non-homomorphic Z exists. [BEER]: there is no homomorphic one.



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General Algebraic Structures¹.



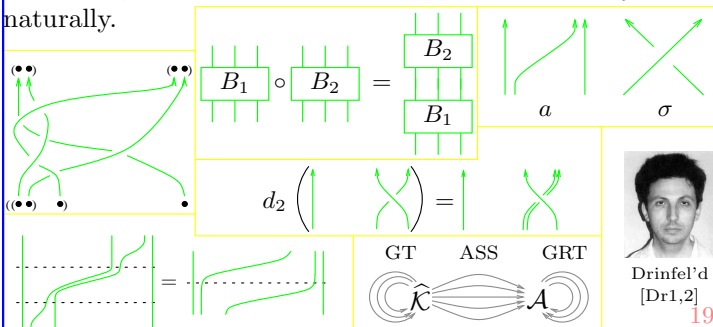
- Has kinds, elements, operations, and maybe constants. All still
- Must have "the free structure over some generators".
- We always allow formal linear combinations. 14 works!

Example 3. Quandle: a set K with an op \wedge s.t.
 $1 \wedge x = 1, \quad x \wedge 1 = x = x \wedge x, \quad (\text{appetizers})$
 $(x \wedge y) \wedge z = (x \wedge z) \wedge (y \wedge z). \quad (\text{main})$

$\mathcal{A}(K)$ is a graded Leibniz² algebra: Roughly, set $\bar{v} := (v-1)$ (these generate I !), feed $1 + \bar{x}, 1 + \bar{y}, 1 + \bar{z}$ in (main), collect the surviving terms of lowest degree:

$$(\bar{x} \wedge \bar{y}) \wedge \bar{z} = (\bar{x} \wedge \bar{z}) \wedge \bar{y} + \bar{x} \wedge (\bar{y} \wedge \bar{z}). \quad 15$$

Example 4. Parenthesized braids make a category with some extra operations. An expansion is the same thing as an A_n -associator, and the Grothendieck-Teichmüller story³ arises naturally.

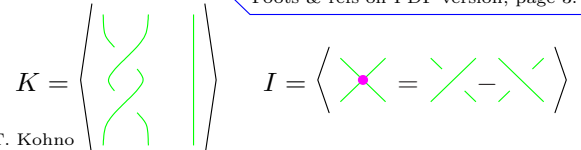


Drinfel'd [Dr1,2] 19

Example 1.



T. Kohno



$$(K/I^{m+1})^* = (\text{invariants of type } m) =: \mathcal{V}_m$$

$$(I^m/I^{m+1})^* = \mathcal{V}_m/\mathcal{V}_{m-1} \quad C = \langle t^{ij} | t^{ij} = t^{ji} \rangle = \langle \text{HH} \rangle$$

$$\ker \mu_{11} = \langle [t^{ij}, t^{kl}] = 0 = [t^{ij}, t^{ik} + t^{jk}] \rangle = \langle 4T \text{ relations} \rangle$$

$$\mathcal{A} = A_n = \left(\text{horizontal chord diagrams mod } 4T \right) = \langle \text{HHHH} \rangle / 4T$$

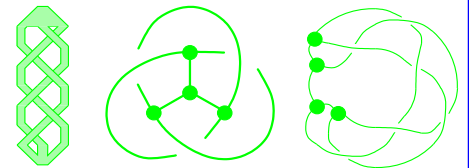
Z : universal finite type invariant, the Kontsevich integral. 9

Why Prized? Sizes K and shows it "as big" as \mathcal{A} ; reduces "topological" questions to quadratic algebra questions; gives life and meaning to questions in graded algebra; universalizes those more than "universal enveloping algebras" and allows for richer quotients. 11

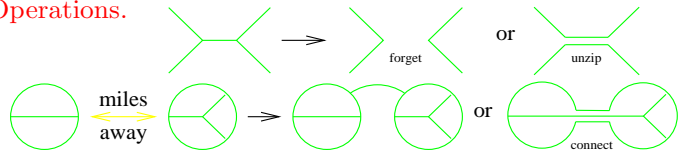
Example 5 - Knotted Trivalent Graphs.



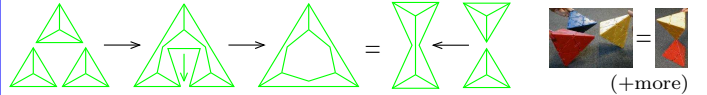
D. Thurston [Th]



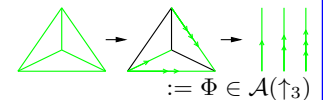
Operations.



Presentation. KTG is generated by ribbon twists and the tetrahedron Δ , modulo the relation(s):

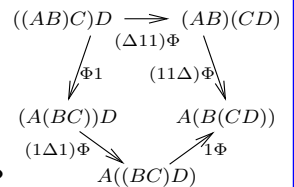


Claim. With $\Phi := Z(\Delta)$, the above relation becomes equivalent to the Drinfel'd's pentagon of the theory of quasi-Hopf algebras.



A $\mathcal{U}(\mathfrak{g})$ -Associator:

$$(AB)C \xrightarrow{\Phi \in \mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$



satisfying the "pentagon",
 $\Phi \cdot (1\Delta 1)\Phi \cdot 1\Phi = (\Delta 11)\Phi \cdot (11\Delta)\Phi$

$$\mathcal{A}(\uparrow_2) := \langle \text{trivalent graphs} \rangle / \text{AS, } (\text{deg} = \frac{1}{2} \# \{ \text{trivalent vertices} \}) \xrightarrow[\mathfrak{g} = \langle X_a \rangle]{\text{Given a metrized } \mathfrak{g}} \mathcal{U}(\mathfrak{g})^{\otimes 2}$$

$$\sum_{a,b,c,d,e,f=1}^{\dim \mathfrak{g}} f_{abcfdce} X_a X_d X_f \otimes X_b X_f X_e$$



Penrose



Cvitanovic 25