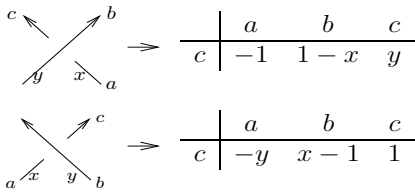


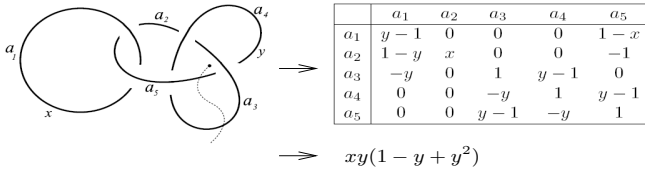
# The Penultimate Alexander Invariant

A Definition of the MVA (From [Ar])



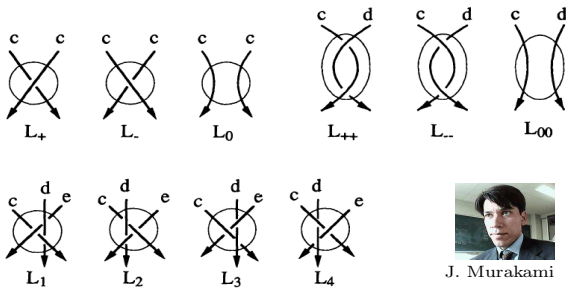
Joint with  
Jana Archibald

$$A = \frac{(-)^{i+j} \det(M_i^j)}{w_i(t_i-1)} \prod_k t_k^{\frac{\text{rot}(k)-\mu(k)}{2}}$$

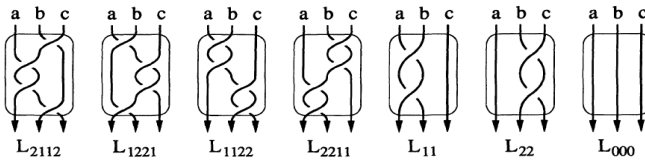


$$\Rightarrow xy(1-y+y^2)$$

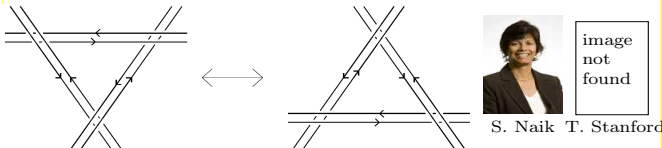
Relations by J. Murakami (From [MJ])



J. Murakami



The Naik-Stanford Double Delta Relation (From [NS])



S. Naik T. Stanford  
image not found

$$1h_j \{ W_n \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ i \quad j \quad k \end{array} \right) - W_n \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ i \quad j \quad k \end{array} \right) \} + 2(h_i - h_k) \{ W_n \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ i \quad j \quad k \end{array} \right) + W_n \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ i \quad j \quad k \end{array} \right) \} + (h_k - h_i)(h_i h_k + h_j^2) W_n \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ i \quad j \quad k \end{array} \right) = 0$$

(From [MH])



H. Murakami

$$C \left( \begin{array}{c} \uparrow \uparrow \uparrow \\ \uparrow \uparrow \uparrow \end{array} \right) = 0$$

A Relation by A. Vaintrob (From [Va])

A Relation by H. Murakami

There's Lots More!

"God created the knots,  
all else in topology  
is the work of mortals"  
Leopold Kronecker (paraphrased)



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This handout and further links are at

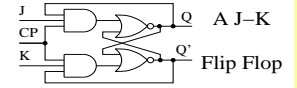
<http://www.math.toronto.edu/~drorbn/Talks/Sandbjerg-0810/>

**Our Goal.** Prove all these relations uniformly, at maximal confidence and minimal brain utilization.

⇒ We need an "Alexander Invariant" for arbitrary tangles, easy to define and compute and well-behaved under tangle compositions; better, "virtual tangles".

## Circuit Algebras

\* Have "circuits" with "ends",  
\* Can be wired arbitrarily.



\* May have "relations" - de-Morgan, etc.

Example  $VT = CA \langle \times, \times, \times \rangle / R23 = PA \langle \times, \times, \times \rangle / R23, VR123, MR3$

Reminders from linear algebra. If  $X$  is a (finite) set,

$$\Lambda^k(X) := \langle k\text{-tuples in } X, \text{ modulo anti-symmetry} \rangle$$

$$\Lambda^{\text{top}}(X) := \langle |X|\text{-tuples in } X, \text{ modulo anti-symmetry} \rangle$$

$$\Lambda^{1/2}(X) := \langle (|X|/2)\text{-tuples in } X, \text{ modulo anti-symmetry} \rangle.$$

If  $Y \subset X^m$ , the "interior multiplication"  $i_Y : \Lambda^k(X) \rightarrow \Lambda^{k-m}(X)$  is anti-symmetric in  $Y$ .

**Definition.** An "Alexander half density with input strands  $X^{\text{in}}$  and output strands  $X^{\text{out}}$ " is an element of

$$\text{AHD}(X^{\text{in}}, X^{\text{out}}) := \Lambda^{\text{top}}(X^{\text{out}}) \otimes \Lambda^{1/2}(X^{\text{in}} \cup X^{\text{out}}).$$

Often we extend the coefficients to some polynomial ring without warning.

**Definition.** If  $\alpha_i \otimes p_i \in \text{AHD}(X_i^{\text{in}}, X_i^{\text{out}})$  (for  $i = 1, 2$ ), and  $G = (X_1^{\text{in}} \cup X_2^{\text{in}}) \cap (X_1^{\text{out}} \cup X_2^{\text{out}})$  is the set of "gluable legs", the "gluing" in  $\text{AHD}(X_1^{\text{in}} \cup X_2^{\text{in}} - G, X_1^{\text{out}} \cup X_2^{\text{out}} - G)$  is

$$i_G(\alpha_1 \wedge \alpha_2) \otimes i_G(p_1 \wedge p_2).$$

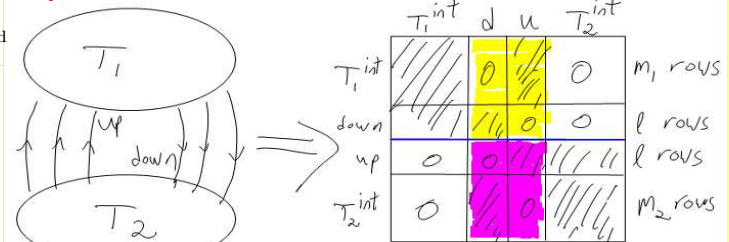
**Claim.** This makes AHD a circuit algebra.

**Definition.** The "Penultimate Alexander Invariant" is defined using

$$pA : \begin{array}{c} k \\ \times \\ i \quad j \end{array} \mapsto (j \wedge k) \otimes \left( \begin{array}{l} l \wedge i + (t_i - 1)l \wedge j - t_l l \wedge k \\ + i \wedge j + t_{ij} l \wedge k \end{array} \right)$$

$$pA : \begin{array}{c} l \\ \times \\ i \quad j \end{array} \mapsto (k \wedge l) \otimes \left( \begin{array}{l} t_{ji} l \wedge j - t_{ji} l \wedge l + j \wedge k \\ + (t_i - 1)j \wedge l + k \wedge l \end{array} \right)$$

Why Works?



Every "rook arrangement" in the above picture must have exactly  $l$  rooks in the yellow zone and  $l$  rooks in the purple zone. So for  $T_1$  we only care about the minors in which exactly  $l$  of the  $2l$  middle columns are dropped, and the rest is signs...

**Weaknesses.** Exponential, no understanding of cablings, no obvious "meaning". The ultimate Alexander invariant should address all that...

**Challenge.** Can you categorify this?