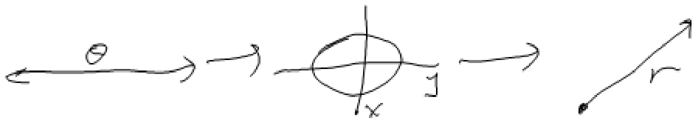
10 Minutes on Homology

Dror Bar-Natan, Bogota, February 23 2009
http://www.math.toronto.edu/~drorbn/Talks/Bogota-0902/
and http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2009-02/

Motivation

Homology measures our failure to construct all solutions of a given equation:



$$\mathbb{R}^1_\theta \xrightarrow{d^1} \mathbb{R}^2_{x,y} \xrightarrow{d^2} \mathbb{R}^1_r$$

$$\theta \mapsto \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} \qquad \begin{pmatrix}x \\ y\end{pmatrix} \mapsto \sqrt{x^2+y^2}$$

$$d^2 \circ d^1 = \text{const}$$

$$V \xrightarrow{d_1} W \xrightarrow{d_2} Z$$

$$\operatorname{im} d_1 \subset \ker d_2 \iff d_2 \circ d_1 = 0$$

$$H(W) := \ker d_2 / \operatorname{im} d_1$$

Definition

A "complex" is a long chain of "parametrization problems":

$$\Omega = \left(\cdots \to \Omega^{r-1} \xrightarrow{d^{r-1}} \Omega^r \xrightarrow{d^r} \Omega^{r+1} \to \cdots \right)$$

s.t. $d^2 = 0$ or $\operatorname{im}(d) \subset \ker(d)$

Homology:

$$H^r(\Omega) := \ker d^r / \operatorname{im} d^{r-1}$$

The "parametrization failure" at step r.

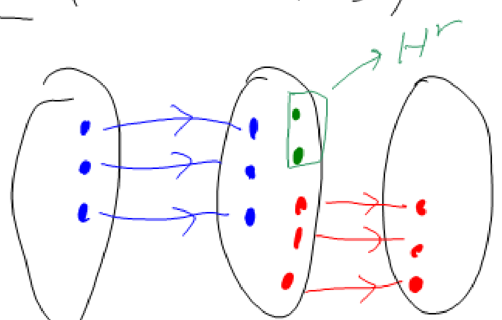
[I don't understand why "long" complexes are so common ☹]

Euler Characteristic

Theorem If everything is finite, then

$$\sum (-1)^r \dim \Omega^r = \sum (-1)^r \dim H^r =: \chi(\Omega)$$

Proof (more or less)



Ω^{r-1} Ω^r Ω^{r+1}

Morphisms and Homotopy

Morphisms:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle F^{r-1}} & & \downarrow{\scriptstyle F^r} & & \downarrow{\scriptstyle F^{r+1}} & & \\
\cdots & \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow & \cdots
\end{array}$$

Homotopies:

$$\begin{tikzcd}
\Omega_0^{r-1} \arrow[r, "d^{r-1}"] \arrow[d, shift right, "F^{r-1}"'] \arrow[d, shift left, "G^{r-1}"] & \Omega_0^r \arrow[r, "d^r"] \arrow[d, shift right, "F^r"'] \arrow[d, shift left, "G^r"] \arrow[dl, "h^r"'] & \Omega_0^{r+1} \arrow[d, shift right, "F^{r+1}"'] \arrow[d, shift left, "G^{r+1}"] \arrow[dl, "h^{r+1}"'] \\
\Omega_1^{r-1} \arrow[r, "d^{r-1}"] & \Omega_1^r \arrow[r, "d^r"] & \Omega_1^{r+1}
\end{tikzcd}$$

$$F^r - G^r = h^{r+1} d^r + d^{r-1} h^r$$

If there are $\Omega_0 \underset{g}{\overset{f}{\rightleftarrows}} \Omega_1$ s.t. $f \circ g \sim I_{\Omega_1}$ and $g \circ f \sim I_{\Omega_2}$, then "$\Omega_0$ & Ω_1 are homotopy equivalent" [and they have equal homology]

More at http://www.math.toronto.edu/~drorbn/Talks/Bogota-0902/