

Let K be a unital algebra over a field \mathbb{F} with $\text{char } \mathbb{F} = 0$, and let $I \subset K$ be an “augmentation ideal”; so $K/I \xrightarrow{\sim} \mathbb{F}$.

Definition. Say that K is **quadratic** if its associated graded $\text{gr } K = \bigoplus_{p=0}^{\infty} I^p/I^{p+1}$ is a quadratic algebra. Alternatively, let $A = q(K) = \langle V = I/I^2 \rangle / \langle R_2 = \ker(\bar{\mu}_2 : V \otimes V \rightarrow I^2/I^3) \rangle$ be the “quadratic approximation” to K (q is a lovely functor). Then K is quadratic iff the obvious $\mu : A \rightarrow \text{gr } K$ is an isomorphism. If G is a group, we say it is quadratic if its group ring is, with its augmentation ideal.

The Overall Strategy. Consider the “singularity tower” of (K, I) (here “ \cdot ” means \otimes_K and μ is (always) multiplication):

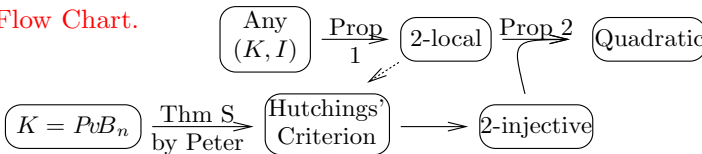
$$\dots \rightarrow I^{p+1} \xrightarrow{\mu_{p+1}} I^p \xrightarrow{\mu_p} I^{p-1} \longrightarrow \dots \longrightarrow K$$

We care as $\text{im}(\mu^p = \mu_1 \circ \dots \circ \mu_p) = I^p$, so $I^p/I^{p+1} = \text{im } \mu^p / \text{im } \mu^{p+1}$. Hence we ask:

- What’s $I^p/\mu(I^{p+1})$? • How injective is this tower?

Lemma. $I^p/\mu(I^{p+1}) \simeq (I/I^2)^{\otimes p} = V^{\otimes p}$; set $\pi : I^p \rightarrow V^{\otimes p}$.

Flow Chart.



Proposition 1. The sequence

$$\mathfrak{R}_p := \bigoplus_{j=1}^{p-1} (I^{j-1} : \mathfrak{R}_2 : I^{p-j-1}) \xrightarrow{\partial} I^p \xrightarrow{\mu_p} I^{p-1}$$

is exact, where $\mathfrak{R}_2 := \ker \mu : I^2 \rightarrow I$; so (K, I) is “2-local”.

The Free Case. If J is an augmentation ideal in $K = F = \langle x_i \rangle$, define $\psi : F \rightarrow F$ by $x_i \mapsto x_i + \epsilon(x_i)$. Then $J_0 := \psi(J)$ is $\{w \in F : \text{deg } w > 0\}$. For J_0 it is easy to check that $\mathfrak{R}_2 = \mathfrak{R}_p = 0$, and hence the same is true for every J .

The General Case. If $K = F/\langle M \rangle$ (where M is a vector space of “moves”) and $I \subset K$, then $I = J/\langle M \rangle$ where $J \subset F$. Then $I^p = J^p / \sum J^{j-1} \langle M \rangle : J^{p-j}$ and we have

$$\begin{array}{ccc} J^p & \xrightarrow[\text{1-1}]{\mu_F} & J^{p-1} \\ \text{onto } \downarrow \pi_p & & \downarrow \text{onto } \pi_{p-1} \\ I^p = J^p / \sum J^{j-1} \langle M \rangle : J^j & \xrightarrow{\mu} & I^{p-1} = J^{p-1} / \sum J^{j-1} \langle M \rangle : J^j \end{array}$$

So $\ker(\mu) = \pi_p(\mu_F^{-1}(\ker \pi_{p-1})) = \pi_p(\sum \mu_F^{-1}(J^j : \langle M \rangle : J^j)) = \sum \pi_p(J^j : \mu_F^{-1} \langle M \rangle : J^j) = \sum I^j : \mathfrak{R}_2 : I^j =: \sum_{j=1}^{p-1} \mathfrak{R}_{p,j}$.

\mathfrak{R}_2 is simpler than may seem! It’s an “augmentation bimodule” ($I\mathfrak{R}_2 = 0 = \mathfrak{R}_2 I$ thus $xr = \epsilon(x)r = r\epsilon(x) = rx$ for $x \in K$ and $r \in \mathfrak{R}_2$), and hence $I^2 \xrightarrow{\mu} I = J/\langle M \rangle$ $\mathfrak{R}_2 = \pi_2(\mu_F^{-1}M)$.

\mathfrak{R}_p is simpler than may seem! In $\mathfrak{R}_{p,j} = I^{j-1} : \mathfrak{R}_2 : I^{p-j-1}$ the I factors may be replaced by $V = I/I^2$. Hence

$$\mathfrak{R}_p \simeq \bigoplus_{j=1}^{p-1} V^{\otimes j-1} \otimes \pi_2(\mu_F^{-1}M) \otimes V^{\otimes p-j-1}.$$

Claim. $\pi(\mathfrak{R}_{p,j}) = R_{p,j}$; namely,

$$\pi(I^{j-1} : \mathfrak{R}_2 : I^{p-j-1}) = V^{\otimes j-1} \otimes R_2 \otimes V^{\otimes p-j-1}.$$

Why Care?

- In abstract generality, $\text{gr } K$ is a simplified version of K and if it is quadratic it is as simple as it may be without being silly.
- In some concrete (somewhat generalized) knot theoretic cases, A is a space of “universal Lie algebraic formulas” and the “primary approach” for proving (strong) quadraticity, constructing an appropriate homomorphism $Z : K \rightarrow \hat{A}$, becomes wonderful mathematics:

K	u-Knots and Braids	v-Knots	w-Knots
A	Metrized Lie algebras [BN1]	Lie bialgebras [Hav]	Finite dimensional Lie algebras [BN3]
Z	Associators [Dri, BND]	Etingof-Kazhdan quantization [EK, BN2]	Kashiwara-Vergne-Alekseev-Torossian [KV, AT]

2-Injectivity. A (one-sided infinite) sequence

$$\dots \longrightarrow K_{p+1} \xrightarrow{\delta_{p+1}} K_p \xrightarrow{\delta_p} \dots \longrightarrow K_0 = K$$

is “injective” if for all $p > 0$, $\ker \delta_p = 0$. It is “2-injective” if its “1-reduction”

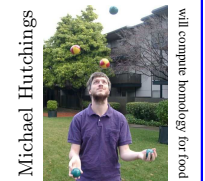
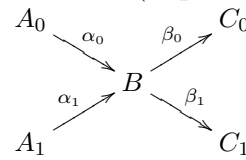
$$\dots \longrightarrow \frac{K_{p+1}}{\ker \delta_{p+1}} \xrightarrow{\bar{\delta}_{p+1}} \frac{K_p}{\ker \delta_p} \xrightarrow{\bar{\delta}_p} \frac{K_{p-1}}{\ker \delta_{p-1}} \longrightarrow \dots$$

is injective; i.e. if for all p , $\ker(\bar{\delta}_p \circ \bar{\delta}_{p+1}) = \ker \bar{\delta}_{p+1}$. A pair (K, I) is “2-injective” if its singularity tower is 2-injective.

Proposition 2. If (K, I) is 2-local and 2-injective, it is quadratic.

Proof. Staring at the 1-reduced sequence $\frac{I^{p+1}}{\ker \mu_{p+1}} \xrightarrow{\mu_{p+1}} \frac{I^p}{\ker \mu_p} \xrightarrow{\mu_p} \dots \longrightarrow K$, get $\frac{I^p}{\ker \mu_{p+1}} \simeq \frac{I^p/\ker \mu_p}{\mu(I^{p+1}/\ker \mu_{p+1})} \simeq \frac{I^p}{\mu(I^{p+1}) + \ker \mu_p}$. But $\frac{I^p}{\mu(I^{p+1})} \simeq (I/I^2)^{\otimes p}$, so the above is $(I/I^2)^{\otimes p} / \sum (I^{j-1} : \mathfrak{R}_2 : I^{p-j-1})$. But that’s the degree p piece of $q(K)$.

The X Lemma (inspired by [Hut]).

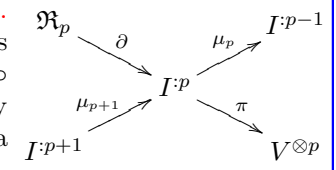


If the above diagram is Conway (\simeq) exact, then its two diagonals have the same “2-injectivity defect”. That is, if $A_0 \rightarrow B \rightarrow C_0$ and $A_1 \rightarrow B \rightarrow C_1$ are exact, then $\ker(\beta_1 \circ \alpha_0) / \ker \alpha_0 \simeq \ker(\beta_0 \circ \alpha_1) / \ker \alpha_1$.

Proof. $\frac{\ker(\beta_1 \circ \alpha_0)}{\ker \alpha_0} \xrightarrow[\alpha_0]{\sim} \ker \beta_1 \cap \text{im } \alpha_0 = \ker \beta_0 \cap \text{im } \alpha_1 \xleftarrow[\alpha_1]{\sim} \frac{\ker(\beta_0 \circ \alpha_1)}{\ker \alpha_1}$.

The Hutchings Criterion [Hut].

The singularity tower of (K, I) is 2-injective iff on the right, $\ker(\pi \circ \partial) = \ker(\partial)$. That is, iff every “diagrammatic syzygy” is also a “topological syzygy”.



Conclusion. We need to know that (K, I) is “syzygy complete” — that every diagrammatic syzygy is also a topological syzygy, that $\ker(\pi \circ \partial) = \ker(\partial)$.