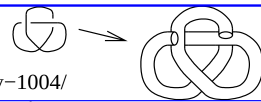
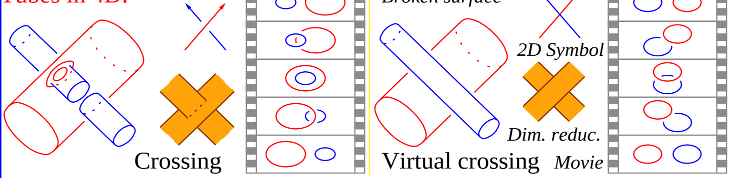
w–Knots from Z to A

Dror Bar–Natan, Luminy, April 2010
http://www.math.toronto.edu/~drorbn/Talks/Luminy–1004/

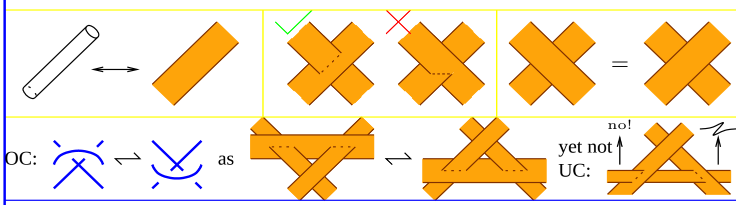


Abstract I will define w-knots, a class of knots wider than ordinary knots but weaker than virtual knots, and show that it is quite easy to construct a universal finite invariant Z of w-knots. In order to study Z we will introduce the "Euler Operator" and the "Infinitesimal Alexander Module", at the end finding a simple determinant formula for Z. With no doubt that formula computes the Alexander polynomial A, except I don't have a proof yet.

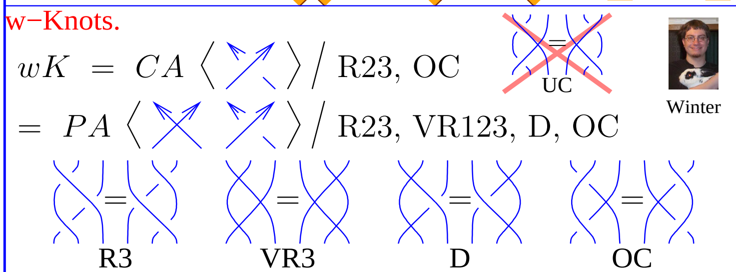
Tubes in 4D.



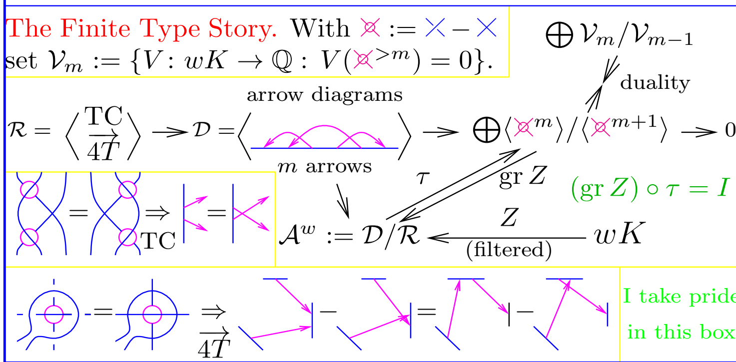
A Ribbon 2-Knot is a surface S embedded in $\mathbb{R}^4$ that bounds an immersed handlebody B, with only "ribbon singularities"; a ribbon singularity is a disk D of trasverse double points, whose preimages in B are a disk D_1 in the interior of B and a disk D_2 with $D_2 \cap \partial B = \partial D_2$, modulo isotopies of S alone.



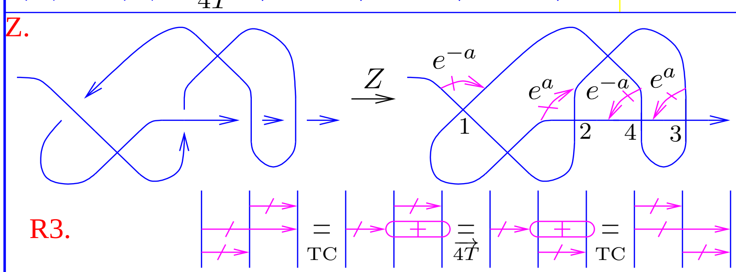
w–Knots.

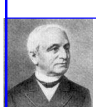


The Finite Type Story. With $\times := \times - \times$ set $\mathcal{V}_m := \{V : wK \to \mathbb{Q} : V(\times^{>m}) = 0\}$.



Z.





"God created the knots, all else in topology is the work of mortals."
Leopold Kronecker (modified)

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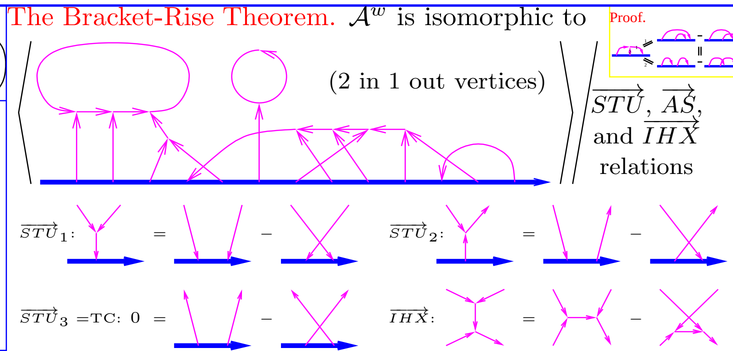
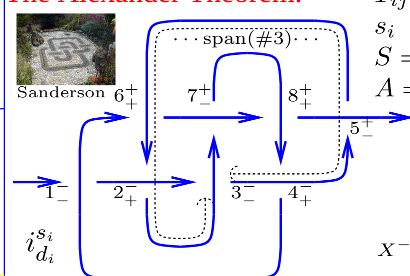
The Bracket-Rise Theorem. $\mathcal{A}^w$ is isomorphic to



Corollaries. (1) Related to Lie algebras! (2) Only wheels and isolated arrows persist.

Habiro - can you do better?

The Alexander Theorem.



$T_{ij} = |\mathrm{low}(\#j) \in \mathrm{span}(\#i)|$, $s_i = \mathrm{sign}(\#i)$, $d_i = \mathrm{dir}(\#i)$, $S = \mathrm{diag}(s_i d_i)$, $A = \det\left(I + T(I - X^{-S})\right)$.

$$T = \begin{pmatrix} 0&1&1&1&1&0&1&0 \\ 0&0&1&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&1&0 \\ 0&1&0&1&0&1&1&1 \\ 0&1&0&1&0&0&1&0 \\ 0&0&0&1&0&1&0&0 \\ 0&0&0&1&0&1&0&0 \end{pmatrix},$$

$X^{-S} = \mathrm{diag}(\frac{1}{X}, X, \frac{1}{X}, X, X, \frac{1}{X}, X, \frac{1}{X})$.

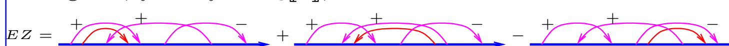
Conjecture. For u-knots, A is the Alexander polynomial.

Theorem. With $w : x^k \mapsto w_k =$ (the k-wheel),

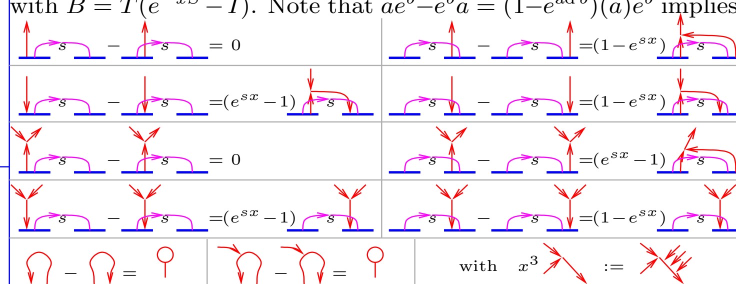
$$Z = N \exp_{\mathcal{A}^w}\left(-w\left(\log_{\mathbb{Q}[\![x]\!]} A(e^x)\right)\right)$$

mod $w_k w_l = w_{k+l}$, $Z = N \cdot A^{-1}(e^x)$

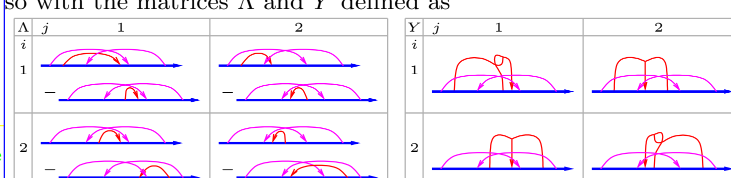
Proof Sketch. Let E be the Euler operator, "multiply anything by its degree", $f \mapsto xf'$ in $\mathbb{Q}[\![x]\!]$, so $Ee^x = xe^x$ and



We need to show that $Z^{-1}EZ = N' - \mathrm{tr}\left((I-B)^{-1}TSe^{-xS}\right)w_1$, with $B = T(e^{-xS} - I)$. Note that $ae^b - e^b a = (1 - e^{\mathrm{ad}\, b})(a)e^b$ implies



so with the matrices Λ and Y defined as



we have $EZ - N'' = \mathrm{tr}(S\Lambda)$, $\Lambda = -BY - Te^{-xS}w_1$, and $Y = BY + Te^{-xS}w_1$. The theorem follows.

So What? • Habiro-Shima did this already, but not quite. (HS: *Finite Type Invariants of Ribbon 2-Knots, II*, Top. and its Appl. **111** (2001).)
• New (?) formula for Alexander, new (?) "Infinitesimal Alexander Module". Related to Lescop's arXiv:1001.4474?
• An "ultimate Alexander invariant": local, composes well, behaves under cabling. Ought to also generalize the multi-variable Alexander polynomial and the theory of Milnor linking numbers.
• Tip of the Alekseev-Torossian-Kashiwara-Vergne iceberg (AT: *The Kashiwara-Vergne conjecture and Drinfeld's associators*, arXiv:0802.4300).
• Tip of the v-knots iceberg. May lead to other polynomial-time polynomial invariants. "A polynomial's worth a thousand exponentials".
Also see http://www.math.toronto.edu/~drorbn/papers/WKO/

Video and more at http://www.math.toronto.edu/~drorbn/Talks/Luminy-1004/